On the sample mean of locally stationary long-memory processes

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Abstract

Some asymptotic statistical properties of the sample mean of a class locally stationary long-memory process are studied in this paper. Conditions for consistency are investigated and precise convergence rates of the variance of the sample mean are established for a class of time-varying long-memory parameter functions. A central limit theorem for the sample mean is also established. Furthermore, the calculation of the variance of the sample mean is illustrated through several numerical and simulation experiments.

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1. Introduction

This paper discusses the statistical properties of the sample mean of a class of locally stationary long-memory processes. The analysis of the sample mean is an essential part of the theory and application of stochastic processes. As pointed out by Parzen (1986a), the behavior of sample means, which needs to be understood by all applied statisticians and users of simulation methods, can be considered to be the most basic question of both classical and modern probability and statistics. The asymptotic behavior of the sample mean has been well established in the context of stationary processes, see for example Section 5.6.1 of Pourahmadi (2001). In particular, several authors have studied the problem for stationary long-memory models, including (Adenstedt, 1974; Samarov and Taqqu, 1988; Hosking 1996a,b), among others. In addition, the behavior of the sample mean has been examined in the context of short-memory locally stationary processes, see for example Dahlhaus (1996, 1997). However, to the best of our knowledge, no general asymptotic results are available yet in the context of long-memory locally stationary processes.

Locally stationary processes are becoming an important tool for analyzing non-stationary time series data. Many authors have suggested definitions for this type of processes, including Silverman (1957), Priestley (1965), Dahlhaus (1996) and Genton and Perrin (2004), among others. Furthermore, the theory of locally stationary processes has been recently extended to encompass non-stationary long-range dependent time series data, see for example Jensen and Witcher (2000) and Beran (2009). Long-memory time series have attracted a great deal attention in the last decades, see for example the monographs by Beran (1994) and Palma (2007). In particular, characterization of long-memory has been studied by Parzen (1992) and Hall (1997) while a large number of estimation methodologies have been proposed for this type of processes, see for example Geweke and Porter-Hudak (1983), Parzen (1986b), and Chapter 4 of Palma (2007) and references therein.

The parameter estimation of locally stationary long-memory (LSLM) processes has been studied by Jensen and Witcher (2000) and Beran (2009), among others. However, it seems that the estimation of the mean of such processes has received far less attention. In this article we study this problem in the context of a family of LSLM processes with a general class of time-varying long-memory parameters. We establish conditions to ensure the consistency of the sample mean, provide
precise convergence rates for its variance and show that a central limit theorem holds for this estimator of the mean. Apart from establishing these asymptotic results, this paper explores the finite sample calculation of the theoretical variance of the sample mean of a LSLM. These empirical studies show that in order to be precise, the use of the asymptotic formula for the variance of the sample mean requires a large sample size. Consequently, we propose alternative approximation formulas which work well for moderate sample sizes.

The remaining of this paper is structured as follows. Section 2 discusses a class of LSLM processes. Section 3 establishes the consistency of the sample mean of this family of LSLM models, provides convergence rates for the variance of this estimator and shows its asymptotic normality. Section 4 illustrates the use of the asymptotic formulas for the variance of the sample mean as well as finite sample approximations. Final remarks are given in Section 5.

2. A class of LSLM processes

A family of constant mean locally stationary processes is defined by the spectral representation

\[ Y_{t,T} = \mu + \int_{-\pi}^{\pi} A_0^T(\lambda) e^{ij\lambda} d\xi(\lambda), \]

for \( t=1, \ldots, T \), where \( A_0^T(\cdot) \) is a transfer function, \( \xi(\cdot) \) is an orthogonal increment process on \([-\pi, \pi]\) and there is a positive constant \( K \) and a \( 2\pi \)-periodic function \( A : (0,1) \times \mathbb{R} \to \mathbb{C} \) with \( A(u,-\lambda) = \overline{A(u,\lambda)} \) such that

\[ \sup_{t,\lambda} |A_0^T(\lambda) - A \left( \frac{t}{T}, \lambda \right) | \leq \frac{K}{T}, \]

for all \( T \). The transfer function \( A^T_1(\cdot) \) of this class of nonstationary processes changes smoothly over time so that they can be locally approximated by stationary processes. An example of this class of locally stationary processes is given by the infinite moving average expansion

\[ Y_{t,T} = \mu + \sigma \left( \frac{t}{T} \right) \sum_{j=0}^{\infty} \psi_j \left( \frac{t}{T} \right) \xi_{t-j}, \]

where \( \xi_t \) is a zero-mean and unit variance white noise and \( (\psi_j(u)) \) are coefficients satisfying \( \sum_{j=0}^{\infty} \psi_j(u)^2 < \infty \) for all \( u \in [0,1] \). In this case, the transfer function of process (3) is given by \( A^T_0(\lambda) = \sigma(t/T) \sum_{j=0}^{\infty} \psi_j(t/T) e^{-ij\lambda} = A(t/T, \lambda) \), so that condition (2) is satisfied. The model defined by (3) generalizes the usual Wold expansion for a linear stationary process allowing the coefficients of the infinite moving average expansion vary smoothly over time. A particular case of (3) is the generalized version of the fractional noise process described by

\[ Y_{t,T} = \mu + \sigma \left( \frac{t}{T} \right) \sum_{j=0}^{\infty} \eta_j \left( \frac{t}{T} \right) \xi_{t-j}, \]

for \( t=1, \ldots, T \), where \( \xi_t \) is a white noise sequence with zero mean and unit variance and the infinite moving average coefficients \( (\eta_j(u)) \) are given by

\[ \eta_j(u) = \frac{\Gamma(j+d(u))}{\Gamma(j+1) \Gamma(d(u))}, \]

where \( \Gamma(\cdot) \) is the Gamma function and \( d(\cdot) \) is a smoothly time-varying long-memory coefficient. For simplicity, the locally stationary fractional noise process (4) will be denoted as LSFN. Lemma A.1 provides a closed-form formula for calculating the covariance function \( \kappa_T(s,t) = \text{cov}(Y_{s,T}, Y_{t,T}) \) for a LSFN, which is useful for simulating this class of processes, see Section 4 for details. The class of LSFN models can be extended to the locally stationary ARFIMA processes, see Jensen and Witcher (2000) for details.

As an example, consider the locally stationary ARFIMA(0, d, 1) model defined by

\[ Y_{t,T} = \sigma \left( \frac{t}{T} \right) \left[ 1 - \theta \left( \frac{t}{T} \right) \right] B \left( 1 - B \right)^{-d(t/T)} \xi_t, \]

where \( \theta(\cdot) \) is a smoothly varying moving average coefficient satisfying \( |\theta(u)| < 1 \) for \( u \in [0,1] \). Similarly to Lemma A.1, it can be readily proved that the covariances \( \kappa_T(s,t) \) of the process (6) is given by

\[ \kappa_T(s,t) = \sigma \left( \frac{s}{T} \right) \sigma \left( \frac{t}{T} \right) \frac{\Gamma \left[ 1-d \left( \frac{s}{T} \right) \right] }{\Gamma \left[ 1-d \left( \frac{s}{T} \right) \right] } \left( \frac{t}{T} \right)^{s-t} \]

\[ \times \left[ 1 + \theta \left( \frac{s}{T} \right) \theta \left( \frac{t}{T} \right) \right] \left( \frac{t}{T} \right)^{s-t} \left( \frac{s}{T} \right)^{s-t} \left( \frac{t}{T} \right)^{s-t} \]

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for \( s, t=1, \ldots, T, s \geq t \).
3. Estimation of the mean

The problem of estimating the mean of the process \((1)\) is analyzed in this section. In particular, we study the behavior of the variance of the sample mean for a LSLM process satisfying some regularity assumptions. Before exploring that situation, recall that for a stationary long-memory process \(Y_t\) with long-memory parameter \(d\), the variance of the sample mean \(\bar{Y}_T = \frac{1}{T} \sum_{t=1}^T Y_t\) behaves like \(\text{Var}(\bar{Y}_T) \sim cT^{2d-1}\), as \(T \to \infty\). Given a sample \(\{Y_1, Y_2, \ldots, Y_T\}\) of the process \((1)\) we can estimate the mean of the process, \(\mu\), by using \(\hat{\mu}_T = \frac{1}{T} \sum_{t=1}^T Y_t\). In what follows, we study some asymptotic properties of the sample mean \(\hat{\mu}_T\) as an estimator of \(\mu\), under the following regularity conditions:

A1. The time-varying covariance function of the process \((1)\) satisfies

\[
K_T(s, t) \sim g \left( \frac{s}{T}, \frac{t}{T} \right) (s-t)^{d_0/T} + d(T)/T - 1,
\]

for large \(s-t > 0\), where \(0 < d(u) \leq d_0 < \frac{1}{2}\) for \(u \in [0, 1]\) and \(g\) is a \(C^1\) function over \([0, 1] \times [0, 1]\) with \(g(u, u) > 0\) for all \(u \in [0, 1]\).

A2. The function \(d(\cdot)\) reaches its maximum value, \(d_0\), at \(u_0\) with \(d'(u_0) < 0\) and continuous third derivative.

A3. There exists a positive constant \(K\) such that \(|\sigma(u)\sigma(u)| \leq Kd_{d_0-1}\), for all \(u \in [0, 1]\) and \(j \geq 1\).

Note that from (7) and by Stirling’s approximation, the elements \(K_T(s, t)\) for the locally stationary ARFIMA\((0, d, 1)\) process defined by (6) can be approximated by

\[
K_T(s, t) \sim \sigma \left( \frac{s}{T} \right) \sigma \left( \frac{t}{T} \right) \left[ 1 - \theta \left( \frac{s}{T} \right) \right] \left[ 1 - \theta \left( \frac{t}{T} \right) \right] \times \frac{\Gamma \left[ 1 - d \left( \frac{s}{T} \right) \right] - d \left( \frac{t}{T} \right)}{\Gamma \left[ 1 - d \left( \frac{s}{T} \right) \right] \Gamma \left[ d \left( \frac{s}{T} \right) \right] T - 1},
\]

for large \(s-t > 0\). Hence, as long as \(\sigma(u), d(u)\) and \(\theta(u)\) are continuously differentiable functions, this locally stationary ARFIMA process satisfies Assumption A1. The next theorem establishes the consistency of the estimator \(\hat{\mu}_T\).

Theorem 3.1 (Consistency). Assume that the process \(\{Y_t\}_{t=1}^T\) satisfies (1) and Assumption A1. Then, \(\hat{\mu}_T \to \mu\), in probability, as \(T \to \infty\).

Proof. The variance of the estimator \(\hat{\mu}_T\) is given by

\[
\text{Var}(\hat{\mu}_T) = \frac{1}{T^2} \sum_{s=1}^T \sum_{t=1}^T K_T(s, t) = \frac{1}{T^2} \left[ 2 \sum_{s=1}^T \sum_{t=1}^{s-1} K_T(s, t) + \sum_{s=1}^T \sum_{t=1}^{s-1} K_T(s, t) \right].
\]

Therefore,

\[
\text{Var}(\hat{\mu}_T) \sim \frac{2}{T^2} \sum_{s=1}^T \sum_{t=1}^{s-1} K_T(s, t),
\]

as \(T \to \infty\), where \(a_T \sim b_T\) means that \(a_T/b_T \to 1\), as \(T \to \infty\). Furthermore, by virtue of Assumption A1 we conclude that

\[
\text{Var}(\hat{\mu}_T) \sim \frac{2}{T^2} \sum_{s=1}^T \sum_{t=1}^{s-1} g \left( \frac{s}{T}, \frac{t}{T} \right) (s-t)^{d_0/T} + d(T)/T - 1.
\]

Since by Assumption A1 \(|g(x, y)|\) is bounded for all \((x, y) \in [0, 1] \times [0, 1]\) we have that

\[
\text{Var}(\hat{\mu}_T) \leq \frac{K}{T^2} \sum_{s=1}^T \sum_{t=1}^{s-1} (s-t)^{2d_0-1} \leq \frac{K}{T^{1-2d_0}} \sum_{s=1}^T \sum_{t=1}^{s-1} \left( \frac{s-t}{T} \right)^{2d_0-1} \frac{1}{T^2} \leq \frac{K'}{T^{1-2d_0}} \int_0^1 \int_0^x (x-y)^{2d_0-1} dy dx \leq \frac{K'}{T^{1-2d_0}}.
\]

where \(K'\) is a positive constant. Now, by Chebyshev’s inequality, for any \(\varepsilon > 0\) we have

\[
\mathbb{P}(|\hat{\mu}_T - \mu| > \varepsilon) \leq \frac{\text{Var}(\hat{\mu}_T)}{\varepsilon^2} \leq \frac{K'}{\varepsilon^2 T^{1-2d_0}}.
\]

Since \(0 < d_0 < \frac{1}{2}\), \(\mathbb{P}(|\hat{\mu}_T - \mu| > \varepsilon) \to 0\) as \(T \to \infty\), proving the result. \(\square\)
The next two results specify the convergence rate of the variance of the sample mean. Theorem 3.2 deals with a linear case while Theorem 3.3 focuses on a more general specification of \( d(u) \). Theorem 3.2 has been proved by Palma and Ferreira (2009), consequently the reader is referred to that paper for its proof. On the other hand, the proof of the new result reported in Theorem 3.3 is provided in detail.

**Theorem 3.2 (Linear case).** Assume that the process \( \{Y_t, T\} \) satisfies (1) and \( d(u) = au + bu \) with \( b > 0 \). Then, under Assumption A1 the estimator \( \hat{\mu}_T \) satisfies

\[
T^{1-2a-2b}(b \log T)^{2a+2b+1}\text{Var}(\hat{\mu}_T) \to g(1,1)\Gamma(2a+2b),
\]

as \( T \to \infty \).

**Theorem 3.3 (General case).** Assume that the process \( \{Y_t, T\} \) satisfies (1) and that Assumptions A1–A2 hold, then the variance of \( \hat{\mu}_T \) satisfies

\[
T^{1-2d_0}(\log T)^{d_0 + 1/2}\text{Var}(\hat{\mu}_T) \to V(u_0),
\]

as \( T \to \infty \) where

\[
V(u_0) = \begin{cases} 
2^{2d_0} \sqrt{\pi}g(u_0, u_0)\Gamma(d_0) & \text{if } u_0 \in (0,1), \\
2^{2d_0-1} \sqrt{\pi}g(u_0, u_0)\Gamma(d_0) & \text{if } u_0 = 0,1
\end{cases}
\]

(10)

**Proof.** From (9) we have that

\[
\text{Var}(\hat{\mu}_T) \sim 2 \int_0^1 \int_0^\infty g(x,y)(x-y)^{d_0 + dy - 1} T^{d_0 + dy - 1} \, dy \, dx.
\]

Thus, similarly to the proof of Lemma A.4, the asymptotic value of \( \text{Var}(\hat{\mu}_T) \) depends only on the evaluation of the double integral (11) in a neighborhood of \((x,y) = (u_0, u_0)\). Consequently, let us define for any \( \epsilon > 0 \) the set

\[
A_T = \{(x,y) : u_0-\epsilon \leq x,y \leq u_0+\epsilon, 1/T < x-y, |d'(x) - d'(u_0)| < \delta, |d''(y) - d''(u_0)| < \delta \},
\]

for some \( \delta > 0 \). This is a nonempty set since \( d'(\cdot) \) and \( g(\cdot,\cdot) \) are continuous functions in a neighborhood of \( u_0 \). Define \( C_T = T^{1-2d_0}(\log T)^{d_0 + 1/2} \). Then,

\[
\lim_{T \to \infty} C_T \text{Var}(\hat{\mu}_T) = \lim_{T \to \infty} 2C_T \int_0^1 \int_0^\infty g(x,y)(x-y)^{d_0 + dy - 1} T^{d_0 + dy - 1} \, dy \, dx
\]

\[
= \lim_{T \to \infty} 2C_T \int_{A_T} g(x,y)(x-y)^{d_0 + dy - 1} T^{d_0 + dy - 1} \, dy \, dx.
\]

Since \( 1 < (x-y)T \) we have that

\[
\lim_{T \to \infty} C_T \int_{A_T} g(x,y)(x-y)^{d_0 + dy - 1} \, dy \, dx
\]

\[
\leq [g(u_0, u_0) + \delta] \lim_{T \to \infty} C_T \int_{A_T} (x-y)^{2d_0 + \delta'(u_0) - \delta} (x-u_0)^2 + (y-u_0)^2)^{1/2} \, dy \, dx.
\]

Hence, by virtue of Lemma A.4 with \( u_0 \in (0,1) \), we conclude that

\[
\lim_{T \to \infty} C_T \int_{A_T} g(x,y)(x-y)^{d_0 + dy - 1} \, dy \, dx \leq \frac{2^{2d_0-1} \sqrt{\pi} \Gamma(d_0)[g(u_0, u_0) + \delta]}{[\delta - d'(u_0)]^{d_0 + 1/2}}.
\]

By an analogous argument, we can also conclude that

\[
\lim_{T \to \infty} C_T \int_{A_T} g(x,y)(x-y)^{d_0 + dy - 1} \, dy \, dx
\]

\[
\geq \frac{2^{2d_0-1} \sqrt{\pi} \Gamma(d_0)[g(u_0, u_0) - \delta]}{[\delta - d'(u_0)]^{d_0 + 1/2}}.
\]

(12)
Now, since $\varepsilon$ and $\delta$ can be chosen arbitrarily small, we have that
\[
\lim_{T \to \infty} CT \int_{-\infty}^{\infty} g(x,y)((x-y)T)^{d(x) + d(y)-1} \, dy \, dx = \frac{2^{2d_0-1}}{\sqrt{\pi g(u_0,u_0) \Gamma(d_0)}} \frac{1}{[-d' \left( u_0 \right)]^{d_0+1/2}}.
\]
A similar argument yields the result for $u_0=0, 1$. \hfill \Box

The next theorem establishes the asymptotic normality of $\hat{\mu}_T$. Observe that we have added the assumption that the input noise $\{\varepsilon_t\}$ in the generalized Wold expansion (3) is a sequence of independent identically distributed random variables. As noted by Hosking (1996a, p. 264), this assumption seems to be essential for the existence of a central limit theorem for the sample mean.

**Theorem 3.4 (Normality).** Assume that the process $\{Y_{t,T}\}$ satisfies (3) where $\{\varepsilon_t\}$ is a sequence of independent identically distributed random variables. If Assumptions A1–A3 hold, then
\[
T^{1/2} \left( \log T \right)^{d_0+1/2} (\hat{\mu}_T - \mu) \rightarrow N(0, V(u_0)),
\]
as $T \to \infty$, where $V(u_0)$ is given by (10).

**Proof.** This result can be proved by an adaptation of Theorem 18.6.5 by Ibragimov and Linnik (1971), as corrected by Hosking (1996b). Without loss of generality, assume that $\mu = 0$ and define $S_T = Y_{1,T} + \cdots + Y_{T,T}$. Then, we can write
\[
S_T = \sum_{k = -\infty}^{T} c_{k,T} \varepsilon_k,
\]
where the coefficients $\{c_{k,T}\}$ are given by
\[
c_{k,T} = \sum_{j = \max\{1,k\}}^{T} \sigma \left( \frac{j}{T} \right) \psi_{j-k} \left( \frac{j}{T} \right).
\]
Let $\sigma_T^2 = \text{Var}(S_T)$. As pointed out by Hosking (1996b, p. 3), the key aspect of Ibragimov and Linnik’s proof is showing that $c_{k,T}/\sigma_T$ converges to zero uniformly as $T \to \infty$. In what follows, we prove that this is indeed the case for the class of locally stationary processes under study. First, observe that from Assumption A3 we may conclude that
\[
|c_{k,T}| \leq KT^{d_0},
\]
for all $k \leq T$. On the other hand, note that $\sigma_T^2 = T^2 \text{Var}(\hat{\mu}_T)$. Hence, by (12) we have that
\[
\frac{C_T}{\sigma_T^2} \sigma_T^2 \geq \frac{2^{2d_0} \sqrt{\pi} \Gamma(d_0)[g(u_0,u_0) - \delta]}{[-d'(u_0)]^{d_0+1/2}}
\]
for large $T$, where $C_T$ is defined in the proof of Theorem 3.3. Since $d'(u_0) < 0$, $g(u_0,u_0) > 0$, $\Gamma(u_0) > 0$ for any $u_0 \in [0,1]$ and $\delta$ can be chosen arbitrarily small, there exists a constant $K > 0$ such that
\[
\frac{C_T}{\sigma_T^2} \sigma_T^2 \geq K,
\]
for large $T$. Hence
\[
\frac{1}{\sigma_T} \leq K \sqrt{\frac{C_T}{T}}.
\]
Now, by combining (13) and (14) we conclude that
\[
\frac{|c_{k,T}|}{\sigma_T} \leq K \frac{(\log T)^{d_0/2 + 1/4}}{\sqrt{T}},
\]
which tends to zero uniformly as $T \to \infty$. \hfill \Box

### 4. Numerical and simulation studies

This section discusses the calculation of the variance of the sample mean of LSLM processes, assessing the accuracy of the asymptotic formula provided by Theorem 3.3 and comparing the sample variance obtained from several simulations to their theoretical counterparts. These calculations are illustrated with a LSFN process with quadratic long-memory function.

Consider the following illustrative example consisting of a LSFN process defined by (4) and (5) with time-varying long-memory parameter given by
\[
d(u) = \hat{d}(2 + u - 2u^2),
\]
for $u \in [0,1]$. The coefficients in the quadratic polynomial above ensure that $0 < d(u) < \frac{1}{2}$. Furthermore, as depicted in Fig. 1, the function (15) has a maximum value $d_0=0.25$ reached at $u_0=0.25$. A realization of this process with 2000 observation is
shown in Fig. 2. The samples of this LSFN process used in these simulations are generated by means of the innovation algorithm, see for example Brockwell and Davis (1991, p. 172). In this implementation, the covariance function of the process, \(k_T(s,t)\), is given by Lemma A.1.

The following two tables report a set of simulation and numerical experiments that illustrate the calculation of the variance of the sample mean. We consider LSFN models with time-varying parameter specified by (15) and different sample sizes. Observe that calculating the exact value of the variance of the sample mean is a demanding computational task, especially for large sample sizes. Consequently, in this section we also discuss several approximated methods for calculating the variance of the mean.

The exact value of the variance of the sample mean is given by (8). On the other hand, an approximation of \(\text{Var}(\hat{\mu}_T)\) is provided by expression (11). For simplicity, this formula will be denoted as \textit{Approximation 1}. Another approximation of the variance of the sample mean is given by formula (20) in Lemma A.5 evaluated at \(u=0.25\). This formula will be denoted as \textit{Approximation 2}. Finally, we can approximate the value of the variance of the sample mean by the asymptotic expression provided by Theorem 3.3,

\[
\text{Var}(\hat{\mu}_T) \sim \frac{1}{2} \sqrt{\pi g(u_0, u_0)} \Gamma(d_0) T^{2d_0-1} (\beta \log T)^{-d_0-1/2}
\]

with \(\beta = 2/17\). For simplicity, this expression will be denoted as \textit{Asymptotic} formula.

\textbf{Table 1}  

\begin{tabular}{|c|c|c|c|}
\hline
\textbf{Method} & \textbf{Sample size} \\
\hline
 & \textbf{T=1000} & \textbf{T=2000} & \textbf{T=4000} \\
\hline
\textit{Exact} & 0.02145846 & 0.01475074 & 0.01014985 \\
\textit{Sample} & 0.01879493 & 0.01329861 & 0.00976621 \\
\textit{Approximation 1} & 0.02283783 & 0.01566236 & 0.01075406 \\
\textit{Approximation 2} & 0.01477839 & 0.01033301 & 0.00644014 \\
\textit{Asymptotic} & 0.04735846 & 0.03116997 & 0.02064407 \\
\hline
\end{tabular}

\textbf{Fig. 1.} Time varying long-memory function \(d(u) = a + b u - cu^2\), \(u \in [0,1]\) with \(a=4/17\), \(b=2/17\) and \(c=4/17\).

\textbf{Fig. 2.} Simulated LSFN process with \(d(u) = a + b u - cu^2\), \(u \in [0,1]\) with \(a=4/17\), \(b=2/17\) and \(c=4/17\).

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\hline
\end{tabular}

\textbf{W. Palma / Journal of Statistical Planning and Inference 140 (2010) 3764–3774}
Table 2
Estimation of the mean: ratio of variances.

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<th>log T=100</th>
<th>log T=500</th>
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</tbody>
</table>

The first row of the table provides the exact values of the variance of \( \hat{m}^T \) given by (8). The second row corresponds to the average of this value over 1000 repetitions. The third and fourth rows correspond to the sample mean variances obtained from Approximation 1 and Approximation 2. The fifth column shows the approximated values of the variance of \( \hat{m}^T \) provided by the Asymptotic formula (16). From this table, note that the sample mean variances from the simulations (second row) and Approximation 1 (third row) are relatively close to their theoretical counterparts displayed in the first row. On the other hand, Approximation 2 and the Asymptotic formula seems to be far off from the exact value for these three sample sizes. Thus, for these sample sizes, the Asymptotic formula is not very useful for calculating the variance of \( \hat{m}^T \).

In order to evaluate the accuracy of the asymptotic formula for larger sample sizes, Table 2 reports the variance ratios between the Approximation 1 and Approximation 2 to the Asymptotic formula. Due to the large sample sizes involved in this table, in these experiments we have not calculated the exact variance of \( \hat{m}^T \) nor the sample values. From this table, the Asymptotic formula seems to produce accurate values for very large sample sizes.

5. Final remarks

In this paper we have investigated the asymptotic behavior of the sample mean of a class of LSLM processes with a general specification for the time-varying long-memory parameter. As evidenced by Theorems 3.2 and 3.3, the asymptotic behavior of the variance of the sample mean of a LSLM process is more complex than its stationary long-memory counterpart. Despite this, as established in Theorem 3.4, a central limit theorem still hold for this estimate of the mean.

Acknowledgement

I would like to thank the Editor and two anonymous referees for their constructive comments which led to substantial improvements. This research was partially supported by Fondecyt Grant 1085239.

Appendix A

Lemma A.1. The covariances \( \kappa_T(s,t) \) of the process (4) is given by

\[
\kappa_T(s,t) = \sigma \left( \frac{s}{T} \right) \sigma \left( \frac{t}{T} \right) \sum_{j=0}^{\infty} \eta_{s-t+j} \left( \frac{s}{T} \right) \eta_j \left( \frac{t}{T} \right) \frac{\Gamma \left[1-d\left( \frac{s}{T} \right)\right] - \Gamma \left[1-d\left( \frac{t}{T} \right)\right]}{\Gamma \left[1-d\left( \frac{s}{T} \right)\right] \Gamma \left[d\left( \frac{s}{T} \right)\right] \Gamma \left[1-d\left( \frac{t}{T} \right)\right] \Gamma \left[d\left( \frac{t}{T} \right)\right]}. \]

for \( s, t = 1, \ldots, T, s \geq t \).

Proof. By definition,

\[
\kappa_T(s,t) = \sigma \left( \frac{s}{T} \right) \sigma \left( \frac{t}{T} \right) \sum_{j=0}^{\infty} \frac{\eta_{s-t+j} \left( \frac{s}{T} \right) \eta_j \left( \frac{t}{T} \right)}{\Gamma \left[s-t+j+1\right] \Gamma \left[j+1\right]} \frac{\Gamma \left[1-d\left( \frac{s}{T} \right)\right] - \Gamma \left[1-d\left( \frac{t}{T} \right)\right]}{\Gamma \left[1-d\left( \frac{s}{T} \right)\right] \Gamma \left[d\left( \frac{s}{T} \right)\right] \Gamma \left[1-d\left( \frac{t}{T} \right)\right] \Gamma \left[d\left( \frac{t}{T} \right)\right]}. \]

\[
= \sigma \left( \frac{s}{T} \right) \sigma \left( \frac{t}{T} \right) \sum_{j=0}^{\infty} \frac{\Gamma \left[s-t+j+1\right] \Gamma \left[1-d\left( \frac{s}{T} \right)\right] - \Gamma \left[1-d\left( \frac{t}{T} \right)\right] \Gamma \left[d\left( \frac{s}{T} \right)\right] \Gamma \left[1-d\left( \frac{t}{T} \right)\right] \Gamma \left[d\left( \frac{t}{T} \right)\right]}{\Gamma \left[s-t+j+1\right] \Gamma \left[j+1\right] \Gamma \left[1-d\left( \frac{s}{T} \right)\right] \Gamma \left[d\left( \frac{s}{T} \right)\right] \Gamma \left[1-d\left( \frac{t}{T} \right)\right] \Gamma \left[d\left( \frac{t}{T} \right)\right]} \right.
\]

\[
\times \left. \sum_{j=0}^{\infty} \frac{\Gamma \left[s-t+j+1\right] \Gamma \left[1-d\left( \frac{s}{T} \right)\right] - \Gamma \left[1-d\left( \frac{t}{T} \right)\right] \Gamma \left[d\left( \frac{s}{T} \right)\right] \Gamma \left[1-d\left( \frac{t}{T} \right)\right] \Gamma \left[d\left( \frac{t}{T} \right)\right]}{\Gamma \left[s-t+j+1\right] \Gamma \left[j+1\right] \Gamma \left[1-d\left( \frac{s}{T} \right)\right] \Gamma \left[d\left( \frac{s}{T} \right)\right] \Gamma \left[1-d\left( \frac{t}{T} \right)\right] \Gamma \left[d\left( \frac{t}{T} \right)\right]} \right]. \]
Therefore, by an application of the hypergeometric function $F(a,b;c,z)$ with $z=1$ we get

$$K_T(s,t) = \sigma(s,T) \sigma(t,T) \frac{\Gamma\left(s-t + \frac{z}{T}\right) \Gamma\left(s-t + \frac{z}{T}; s-t+1,1\right)}{\Gamma\left(\frac{z}{T}\right) \Gamma\left(s-t+1\right)}.$$  

Now, by Gradshteyn and Ryzhik (2000, Eq. 9.122) the result is obtained.  

**Lemma A.2.** Let $g : \mathbb{R} \to \mathbb{R}$ be a $C^1(\mathbb{R})$ function such that $g(0) \neq 0$ and let $\varphi : [0,1] \to \mathbb{R}$ be a continuous function such that $\varphi(u) > -1$ for all $u \in [0,1]$. Then,

$$I_n = (\sqrt{n})^{\varphi(u) + 1} \int_0^u x^{\varphi(u)} g(x) \exp(-nx^2) \, dx - \frac{1}{2} g(0) I\left[\frac{\varphi(u) + 1}{2}\right],$$

as $n \to \infty$, for any $u \in (0,1)$.

**Proof.** Since the function $g \in C^1(\mathbb{R})$, then by Taylor’s theorem we can write $g(x) = g(0) + g'(\xi_x) x$, for positive $x$, for some $0 \leq \xi_x \leq x$. Hence, we can write

$$I_n = g(0)(\sqrt{n})^{\varphi(u) + 1} \int_0^u x^{\varphi(u)} \exp(-nx^2) \, dx$$

$$+ (\sqrt{n})^{\varphi(u) + 1} \int_0^u x^{\varphi(u) + 1} g(\xi_x) \exp(-nx^2) \, dx.$$  

The first integral in the expression above can be written as

$$g(0)(\sqrt{n})^{\varphi(u) + 1} \int_0^u x^{\varphi(u)} \exp(-nx^2) \, dx = g(0) \int_0^{u/\sqrt{n}} y^{\varphi(u)} \exp(-y^2) \, dy.$$  

Consequently,

$$g(0)(\sqrt{n})^{\varphi(u) + 1} \int_0^u x^{\varphi(u)} \exp(-nx^2) \, dx$$

$$\to \frac{1}{2} g(0) \int_0^{\infty} y^{\varphi(u)} \exp(-y^2) \, dy = g(0) \frac{1}{2} I\left[\frac{\varphi(u) + 1}{2}\right].$$

as $n \to \infty$. On the other hand, the second integral in (17) converges to zero since

$$\left|\int_0^u x^{\varphi(u) + 1} g(\xi_x) \exp(-nx^2) \, dx\right|$$

$$\leq K(\sqrt{n})^{\varphi(u) + 1} \int_0^u x^{\varphi(u)} \exp(-nx^2) \, dx \leq K\int_0^{u/\sqrt{n}} x^{\varphi(u)} \exp(-y^2) \, dy \to 0,$$

as $n \to \infty$.  

**Lemma A.3.** Let $g : \mathbb{R} \to \mathbb{R}$ be a $C^1(\mathbb{R})$ function and define the integral $I_n$ as

$$I_n = \sqrt{n} \int_0^1 \exp(-c_n(x-x_0)^2) g(x) \, dx,$$

where $\{c_n\}$ is a sequence of positive real numbers such that $c_n/n \to 1$, as $n \to \infty$. Then,

$$I_n \to \sqrt{\pi} g(x_0) I_{0,1}(x_0) + \frac{\sqrt{\pi}}{2} g(x_0) I_{1,1}(x_0),$$

as $n \to \infty$.

**Proof.** Consider first the case $x_0 \in (0,1)$. Since $g \in C^1(\mathbb{R})$, we can write $g(x) = g(x_0) + g'(\xi_x)(x-x_0)$, for some $\xi_x$ between $x_0$ and $x$. Consequently,

$$I_n = g(x_0) \sqrt{n} \int_0^1 \exp(-c_n(x-x_0)^2) \, dx$$

$$+ \sqrt{n} \int_0^1 g'(\xi_x)(x-x_0) \exp(-c_n(x-x_0)^2) \, dx.$$
Lemma A.4. Define the double integral

\[ I_1 = 2 \int_0^1 \int_0^x g(x, y) |(x - y)|^{2d_0 - 1 - 2\beta(u - u_0)^2 + (y - u_0)^2} \, dy \, dx. \]

where \( d_0 < \frac{1}{2} \), \( \beta > 0 \), \( u_0 \in [0, 1] \), and \( g \) continuous with \( g(u_0, u_0) > 0 \). Then,

\[ T^{1 - 2d_0} / \beta \log T \, T^{d_0 + 1/2} I_T \to \begin{cases} 
1/2 \sqrt{\pi} g(u_0, u_0) I'(d_0) & \text{if } u_0 \in (0, 1), \\
1/4 \sqrt{\pi} g(u_0, u_0) I'(d_0) & \text{if } u_0 = 0, 1,
\end{cases} \]

as \( T \to \infty \).

Proof. By means of the variable transformation \( u = x + y \) and \( v = x - y \), we can write

\[ 2d_0 - 1 - 2\beta(u - u_0)^2 + (y - u_0)^2 = z(u) - \beta v^2, \]

where \( z(u) = 2d_0 - 1 - \beta(u - 2u_0)^2 \). Thus,

\[ I_T = \int_0^1 \int_0^u \hat{g}(u, v) z^{(u)} \beta^2 T^{z(u) - \beta v^2} \, dv \, du \\
+ \int_1^2 \int_0^2 \hat{g}(u, v) z^{(u)} \beta^2 T^{z(u) - \beta v^2} \, dv \, du, \]

where \( z(u) = 2d_0 - 1 - \beta(u - 2u_0)^2 \). Thus,

\[ I_T = \int_0^1 \int_0^u \hat{g}(u, v) z^{(u)} \beta^2 T^{z(u) - \beta v^2} \, dv \, du \\
+ \int_1^2 \int_0^2 \hat{g}(u, v) z^{(u)} \beta^2 T^{z(u) - \beta v^2} \, dv \, du, \]
where \( \tilde{g}(u,v) = g((u+v)/2,(u-v)/2) \). Therefore,

\[
T^{1-2b}(\beta \log T)^{\alpha_0+1/2} I_T \\
\sim \int_0^1 T^{-\beta(u-2u_0)^2} (\sqrt{\beta \log T})^{1+\beta(u-2u_0)^2} h_1(u) \, du \\
+ \int_1^2 T^{-\beta(u-2u_0)^2} (\sqrt{\beta \log T})^{1+\beta(u-2u_0)^2} h_1(2-u) \, du,
\]

where

\[
h_1(u) = (\sqrt{\beta \log T})^{\alpha(u)+1} \int_0^u \tilde{g}(u,v)v^{\alpha(u)-\beta u^2} T^{-\beta u^2} \, dv.
\]

Now, an application of Lemma A.2 yields,

\[
T^{1-2b}(\beta \log T)^{\alpha_0+1/2} I_T \\
\sim \int_0^1 T^{-\beta(u-2u_0)^2} (\sqrt{\beta \log T})^{1+\beta(u-2u_0)^2} h(u) \, du \\
+ \int_1^2 T^{-\beta(u-2u_0)^2} (\sqrt{\beta \log T})^{1+\beta(u-2u_0)^2} h(u) \, du,
\]

where

\[
h(u) = \frac{1}{2} \tilde{g}(u,0) \Gamma \left[ \frac{\alpha(u)+1}{2} \right].
\]

On the other hand,

\[
\int_0^1 T^{-\beta(u-2u_0)^2} (\sqrt{\beta \log T})^{1+\beta(u-2u_0)^2} h(u) \, du \\
= \sqrt{n} \int_0^1 \exp[c_n(u-2u_0)^2] h(u) \, du,
\]

where \( n = \beta \log T \) and \( c_n = n - \beta \log \sqrt{n} \). Since \( c_n/n \to 1 \) as \( n \to \infty \) and \( c_n > 0 \), by Lemma A.3 we conclude that

\[
\int_0^1 T^{-\beta(u-2u_0)^2} (\sqrt{\beta \log T})^{1+\beta(u-2u_0)^2} h(u) \, du \\
\to \sqrt{n} \log(2u_0) I_{[0,1)}(2u_0) + \frac{\log(2u_0) I_{[0,1)}(2u_0)}{2}.
\]

as \( T \to \infty \), where \( I_A \) is the indicator function of \( A \). An analogous argument leads to

\[
\int_1^2 T^{-\beta(u-2u_0)^2} (\sqrt{\beta \log T})^{1+\beta(u-2u_0)^2} h(u) \, du \\
\to \sqrt{n} \log(2u_0) I_{[0,1)}(2u_0-1) + \frac{\log(2u_0) I_{[0,1)}(2u_0-1)}{2}.
\]

as \( T \to \infty \). Now, by observing that

\[
h(2u_0) = \frac{1}{2} \tilde{g}(2u_0,0) \Gamma \left[ \frac{\alpha(2u_0)+1}{2} \right] = \frac{1}{2} g(u_0,u_0) \Gamma(d_0),
\]

the result is proved. \( \Box \)

**Lemma A.5.** Assume that the process \( \{ Y_{t,T} \} \) satisfies (1) and (15), and that Assumption A1 holds. Then the variance of \( \hat{\mu}_T \) satisfies

\[
\text{Var}(\hat{\mu}_T) \sim \frac{1}{2} \int_0^1 T^{\alpha(u)} (\sqrt{\beta \log T})^{\alpha(u)+1} g(u, u/2) \Gamma \left[ \frac{\alpha(u)+1}{2}, \beta(\log T)u^2 \right] \, du,
\]

where \( \beta = -c/2 \) and \( \gamma(x,a) \) corresponds to the incomplete Gamma function

\[
\gamma(x,a) = \int_0^x t^{a-1} \exp(-t) \, dt.
\]
Proof. From expression (18) we have that

\[
\text{Var}(\hat{\mu}_T) \sim \frac{T^{2d_\alpha-1}}{(\beta \log T)^{d_\alpha+1/2}} \int_0^1 T^{-\beta(u-2\alpha)^2} (\sqrt{\beta \log T})^{1+\beta(u-2\alpha)^2} h_1(u) \, du
\]

\[
\sim \int_0^1 T^{\gamma(u)} (\sqrt{\beta \log T})^{-2(u)\gamma-1} h_1(u) \, du.
\]

But, from (19) with \( n = \beta \log T \) we can write

\[
h_1(u) = (\sqrt{n})^{2(u)\gamma+1} \int_0^u \tilde{g}(u,v) \nu^{2(u)\gamma} \exp(-nv^2) \, dv.
\]

Now, by a similar argument leading to (17) we have that

\[
h_1(u) \sim (\sqrt{n})^{2(u)\gamma+1} \int_0^u \nu^{2(u)\gamma} \exp(-nv^2) \, dv
\]

\[
+ (\sqrt{n})^{2(u)\gamma+1} \int_0^u \tilde{g}(u,v) \nu^{2(u)\gamma} \exp(-nv^2) \, dv,
\]

for some \( \tilde{\varsigma}_v \in [0,u] \). But, analogously to the proof of Lemma A.2, the second integral in the expression above is negligible for large \( n \). Thus,

\[
h_1(u) \sim (\sqrt{n})^{2(u)\gamma+1} \tilde{g}(u,0) \int_0^u \nu^{2(u)\gamma} \exp(-nv^2) \, dv.
\]

Now, by the change of variable \( x = nv^2 \) we conclude that

\[
h_1(u) \sim \frac{1}{2} \tilde{g}(u,0) \int_0^{\mu^2} x^{2(u+1)/2-1} \exp(-x) \, dx = \frac{1}{2} \tilde{g}(u,0) \gamma \left[ \frac{2(u+1)}{2} \right] \beta(\log T) u^2.
\]

Finally, by replacing this expression in (21), the result follows. \( \square \)

References


