Steps into Categorification

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Introduction

A category $\mathbf{C}$ is a structure given by a class of *objects* (or *points*) $Ob(\mathbf{C})$, and for all $A, B \in Ob(\mathbf{C})$, a set of *morphisms* (or *arrows*) $Hom_C(A, B)$, endowed with a composition law

$$\circ : Hom_{\mathbf{C}}(A, B) \times Hom_{\mathbf{C}}(B, C) \to Hom_{\mathbf{C}}(A, C)$$

which is

- **Associative:** $(f \circ g) \circ h = f \circ (g \circ h)$
- **with Identities:** $\forall X \in \mathbf{C}$ (This is common notation for $X \in Obj(\mathbf{C})$) $\exists\, 1_X \in Hom(X, X)$ such that $f \circ 1_X = f$ $\forall f \in Hom(X, A)$ and $1_X \circ g = g$ $\forall g \in Hom(A, X)$

The definition might look a little jammed, but it can be easily imagined as points with arrows between them with a suitable composition. We fix some notation for some examples that we are going to encounter

Category	Objects	Morphisms
Set	Sets	Functions
FinSet	Finite sets	Functions
Cat	Categories	Functors
Grp	Groups	Group homomorphisms
Ab	Abelian groups	Group homomorphisms
FinAb	Finite Abelian groups	Group homomorphisms
Ring	Rings with unit	Ring homomorphisms f with $f(1) = 1$
R-mod	R-modules	Linear maps
R-fmod	Finitely generated R-mods	Linear maps
Vect	Finite dimensional vector spaces	Linear maps
$\mathbf{Vect}_\infty$	Vector spaces	Linear maps
Top	Topological spaces	Continuous maps

Categories are sometimes seen with some obscurity, specially among undergraduates, but there is nothing dark about them, and their existence can be used for multiple purposes. Here we are going to take a strange-sounding focus: Categories are well suited for *modeling*.

Modeling is the process through which one corresponds an entity with a mathematical object (which in our cases will be categories). The idea is that the modeled entity behaves like the modeling object, so by studying the mathematical object, we are studying, by correspondence, the entity that is being modeled. For example, "the temperature of an oven" can be modeled by $\mathbb{R}_{\geq 0}$, afterwards we can mix it with other modeled entities like time and heating capacity, and get useful conclusions like "when would my pie be ready". The power of mathematics lies in the fact that it is easier to study than the particular object, and that the same mathematics can be applied to different situations.

The modeling process is on the heart of mathematics and can be seen all around it, usually specified with a morphism of some kind. It can also be seen outside of mathematics: one can say for example that a metaphor is a (weak) form of modeling. In fact, mathematics first came into place by doing a modeling. We illustrate this with a parable.

It is told that long ago, not that far from here, shepherds negotiated with sheep. It was important to see who had more sheep, and they had a good way (or so they thought) to find this out: They placed 2 herds of sheep in line and match each sheep in one herd with a sheep in the other. That is, they looked for an bijection. Two herds would be isomorphic if they could match exactly one sheep of one herd with one sheep of the other.

One day, a special shepherd had a enlightening of mathematical genius, and invented *decategorification* [1]. She saw that one could take each heard and "count" it, setting an identification between it and an element in a set of "numbers", which at the time where just non-sense words like "one, two, three,...". Comparing the resulting numbers, she could show that two herds were isomorphic without aligning them, saving all the shepherds a lot of energy. And so the natural numbers emerged.

When identifying a category, for example **Top**, we are doing something similar to modeling. The category itself is just a bunch of points and arrows, but we establish a correspondence (a model) by saying that each point corresponds to a topological space, and each arrow corresponds to a continuous function. Now, the same category can be used to model a different (isomorphic) entity.

The visualization of categories as points with arrows is very fertile since it gets the concept of a category close to the concept of a directed graph, which is widely used in modeling. In fact, categories can be used to model entities outside of mathematics. We are not covering that in this document, since it is a thesis for a master in mathematics, but we are going to give references about this in the final comments.

In this document we want to show the power of categories as an enhancing tool in mathematics. We particularly focus on a special kind of modeling with categories called *categorification*, which makes a step from sets to categories.

Categorification it is commonly known as the process of corresponding an object in a concrete category (That is, a category in which each object is a set, like **Set**, **Grp**, and **Top**) with a category, such that the following correspondence is satisfied

Set theory	Category theory
Elements of a set	Objects of a category
Functions between sets	Functors between categories
Equations between elements	Isomorphisms between objects
Equations between funtions	Natural isomorphisms between functors

etc. (there might be more structure to consider). Let us illustrate this concept with a classic example.

As the reader probably knows, we can count finite sets, like $A \in \mathbf{Set}$, by taking their cardinality $|A|$, and actually, this process identifies isomorphism classes of finite sets. We call the isomorphism classes by $S(\mathbf{FinSet}) = \mathbf{FinSet}/\simeq$. We have that cardinality sets a bijection

$$|\cdot| : S(\mathbf{FinSet}) \xrightarrow{\simeq} \mathbb{N}$$

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[1] We will soon explain what this is.

So each isomorphism class of finite sets can be uniquely identified with a natural number. This is what the genius shepherd did in our parable. Now we can identify isomorphic sets simply by counting them.

There is more. We have that

$$|A \sqcup B| = |A| + |B| \quad |A \times B| = |A| \cdot |B|$$

So

$$|\cdot| : (S(\mathbf{FinSet}), \sqcup, \times) \xrightarrow{\simeq} (\mathbb{N}, +, \cdot)$$

Is an isomorphism. We say that **FinSet** is a categorification of $\mathbb{N}$. We also say that **FinSet** decategorifies into $\mathbb{N}$. The sum in $\mathbb{N}$ is categorified with the disjoint union of sets, and the multiplication in $\mathbb{N}$ is categorified with the product of sets.

This makes a lot of sense, since objects in $S(\mathbf{FinSet})$ can be considered as a finite number of indistinguishable points

The disjoint union and the product can be seen as

Which is pretty much what our brain understands that is doing when it performs addition and multiplication.

Now, note that in **FinSet**, the disjoint union and the product satisfy associativity, commutativity and distributivity only up to (natural) isomorphism (see the table above). By taking the isomorphism classes, $S(\mathbf{FinSet})$, we transform these isomorphisms into equalities. This makes natural to choose $S(\mathbf{FinSet})$ rather than **FinSet** to categorify.

At this point, we loosely [2] define categorification of an object A in a concrete category, as a category $\mathbf{C}$, together with a decategorifying function

$$\phi : \mathbf{C}/\simeq \to A$$

That is a *isomorphism* of some kind. The morphism kind depends on the structure we are categorifying.

In chapter 1 we are going to introduce the *Grothendieck group*, which is a good way to deal with inverses. This gives a better (slightly different) definition of categorification of objects which are Abelian groups (like rings, vector spaces and algebras). In this context, a categorification of an Abelian group A is a (Abelian or additive) category whose Grothendieck group is A. For instance, in our example, we will see that the Grothendieck group of **FinSet** is $\mathbb{Z}$. So we say that a categorification of $\mathbb{Z}$ is **FinSet**.

Another categorification of $(\mathbb{N}, +, \cdot)$ is done with finite dimensional vector spaces, since the dimension dim of a vector space classifies it up to isomorphism, and

$$\dim(V \oplus W) = \dim(V) + \dim(W), \quad \dim(V \otimes W) = \dim(V) \cdot \dim(W)$$

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[2] There is not yet an agreement on how categorification should be defined for all cases.

The dimension sets a decategorifying function

$$\dim : (S(\mathbf{Vect}), \oplus, \otimes) \xrightarrow{\simeq} (\mathbb{N}, +, \cdot)$$

Note that when we are taking dimension, we are able to easily identify isomorphic vector spaces (just like cardinality and sets), but in the process we lost a lot of information, reducing a vector space to a number. In fact, each time we decategorify we loose information, namely, the morphism spaces and all the information they carry. Categorification is an attempt to recover information. We will go over this and other ideas through this document.

We start by covering the necessary material we are going to use, particularly showing a tool of great use: The Grothendieck group. Then we go over the process called *odification* or *horizontal categorification*, which is regarding an object (like a group) as a one object category (for a group, it would be a groupoid), introducing in the way the concept of a 2-category. After this we enter deeply into categorification, showing many examples of this process, and then giving a specific definition well suited for unital associative algebras.

We wish to cover from simple "well known" concepts, like numbers and equality (showing that there is value in categorifying them), to more complicated objects, like Hecke algebras and the Heisenberg algebra.

Knowledge Requirements

It is assumed that the reader is familiar with the basic concepts of categories, to say functors, natural transformations, (co)product, initial and final objects. A good reference for this is [Alu09] chapters I.3, I.4, I.5 and VIII.1; and [Lan10]. Also the reader can check [LM13] sections 1.1, 1.2, 1.3 and 1.5 in which the necessary knowledge is well covered. Also we assume knowledge of a graduate course on algebra. Some knowledge on algebraic topology would be desirable but not necessary (The concepts of CW-complexes and fundamental groupoid are called upon without definition). Knowledge about group representations is required in the last subsection of the last section of the last chapter.

Some Conventions

- For a category $\mathbf{C}$, we call $S(\mathbf{C})$ or $\mathcal{S}(\mathbf{C})$ the isomorphism classes of the objects $\mathbf{C}/\simeq$.
- Unless otherwise stated, all categories considered will be essentially small (that is, a category in which $S(\mathbf{C})$ is a set).
- All rings considered have unit.
- All algebras we are going to work with are associative and unital.
- We take $\mathbb{N} = \mathbb{Z}_{\geq 0}$

This is mainly a monographic work based on [LM13], [BD98], [BD00], [Kho16], [Sav14], [Maz12], [Lib17], [EW13] and [LS11].

Acknowledgments: I want to give special thanks to my thesis director (tutor de tesis), Nicolas Libedinsky, for the many corrections and all I have learned from him. I also give thanks to the professors which I consider closer and/or important in my mathematical development: Jorge Soto, Jose Montero, Renato Lewin. I also thank Proyecto Anillo ACT 1415 PIA-Conicyt for its

support during my thesis. To the evaluation committee, for taking time to read and evaluate this thesis. To my family, close friends and girlfriend, for being a great company throughout my master, I love you. And to God, for giving me life everyday and vocation towards this beautiful discipline that is mathematics.

Let us begin.

1 Grothendieck Group

In the world of categories, specifically when talking about categorification, it gets tricky to categorify inverses. We will see an example of this when attempting to categorify numbers in chapter 3. Good for us that there is a tool that can save us the trouble, and still guarantee nice useful categorifications, that is the Grothendieck Group, which is essentially our best shot at adding inverses to a monoid. This tool has had its fame for quite some time now, particularly in the field known as *K-theory*. Basically, this field of mathematics is about taking a bunch of objects with nice structure (like isomorphism classes of vector bundles over a topological space with $\oplus$), simplifying them by taking the Grothendieck group, and then adding structure to these groups.

I bet many mathematicians can recall the time when they first encountered the negative numbers $\mathbb{Z}_{\leq 0}$, since there is a really weird feeling attached to it. They can not really be seen as clearly in the concrete world as the natural numbers $\mathbb{N}$. They are an abstraction, more of a shadow of $\mathbb{N}$, in the sense that $\mathbb{Z}$ comes into place when deeply looking at the so called "cancellation property" of $\mathbb{N}$

$$a + c = b + c \Rightarrow a = b \ \forall a, b, c \in \mathbb{N}$$

In other words, the action of "adding" of $\mathbb{N}$ over $\mathbb{N}$ is reversible. Note that this leaves the adding action a little loose, to say, if $f_c : n \mapsto n + c$ is reversible, it has an inverse function $f_c^{-1} : n + c \mapsto n$. Now, the inverse has 2 unpleasant features which can be fixed:

1. It does not have a clear rule like f_c.
2. Its domain is $\mathbb{N}_{\geq c}$.

To fix this, we note that the cancellation property hints the existence of an entity $-c$, beyond the natural numbers, which can be added to c to get 0.

Can we attach these entities to $\mathbb{N}$ in a way compatible with addition? The answer as you may know is yes. This can be done in several ways. As children (assuming the reader is not super gifted) we took this entities $\{-n : n \in \mathbb{N}\}$, and made disjoint union with $\mathbb{N}$. Then we declared that their action is opposite to the action of $\mathbb{N}$. If n adds, $-n$ will subtract. We got a little lucky that this worked so well.

As mathematicians we learned a more sophisticated method. We take $\mathbb{N} \times \mathbb{N} / \sim$, where $(a, b) \sim (c, d) \iff b + c = d + a$ and call $n = [(n, 0)]$, $-n = [(0, n)]$. The same can be done at a very abstract level, let us see how.

1.1 Grothendieck Group of an Abelian Monoid

Definition. *A* monoid *is a set M with an operation $+$ such that*

a) $a+(b+c)=(a+b)+c \quad \forall\, a,b,c \in M$

b) $\exists\, 0 \in M$ *such that* $a+0=0+a=a \quad \forall\, a \in M$

If $+$ is also commutative, M is called Abelian monoid

A morphism of monoids is a function $f: N \to M$ such that $f(n+m) = f(n)+f(m)$ and $f(0)=0$.

Monoids with these morphisms form a category which we will call ***Mon****.The category of Abelian monoids will be called* ***AbMon***

(That is, the axioms of an Abelian group minus the existence of inverses.)

Examples

1) Our dear $(\mathbb{N},+)$ is an Abelian monoid, and quite an important one! It is the initial object in **AbMon** (since every morphism is determined by its action on 1).

2) Given $(R,+,\cdot\,)$ a commutative ring with 1, $(R-\{0\},\cdot\,)$ is an Abelian monoid.

3) Given X a set, $(P(X),\cup)$ and $(P(X),\cap)$ are Abelian monoids.

4) Given a category $\mathbf{C}$, the set of endomorphisms $End(X)$ of an object $X \in \mathbf{C}$ is a monoid with composition as the operation. Actually, note that we can define a monoid as a category with one object! More on that latter.

5) For $\mathbf{V} = S(\mathbf{Vect})$ the isomorphism classes of finite dimensional vector spaces, we have that $(\mathbf{V},\oplus)$ is an Abelian monoid.

6) Given X topological space, $[X,\mathbb{N}] = \{f : X \to \mathbb{N}\}$ is a monoid with componentwise addition.

7) $\mathbb{N}[q,q^{-1}]$ the Laurent polynomials with coefficients in $\mathbb{N}$ is a monoid with the usual sum of polynomials.

We wish to complete the Abelian monoids, attaching (if possible) ghost inverses to get an Abelian group.

Proposition 1.0.1. *Let $(M,+)$ be an Abelian monoid, there exists a group $K(M)$, called the* Grothendieck Group *of M, together with a morphism $M \xrightarrow{i} K(M)$ that satisfies the following universal property:*

For all monoid morphisms $M \xrightarrow{f} A$ to an Abelian group A, there exist a unique group morphism $K(M) \xrightarrow{\bar{f}} A$ such that $\bar{f} \circ i = f$. That is, the following diagram commutes:

$$\begin{tikzcd}
K(M) \arrow[r, "\bar{f}"] & A \\
M \arrow[u, "i"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Note that if $f : N \to M$ is a monoid morphism, by the universal property, the function $N \xrightarrow{f} M \xrightarrow{i} K(M)$ lifts to a unique morphism of groups $\bar{f} : K(N) \to K(M)$. This makes $K : \mathbf{AbMon} \to \mathbf{Ab}$ a functor.

The universal property is *nice*, in the sense that it makes $Hom_{AbMon}(M, A)$ canonically isomorphic to $Hom_{Ab}(K(M), A)$ for any $A \in \mathbf{Ab}$, and therefore the functor K is left adjoint to the forgetful functor.

We present two ways to construct the Grothendieck Group of an Abelian monoid $(M, +)$:

i) $K(M) = M \times M/\sim$, where $(n_1, m_1) \sim (n_2, m_2) \iff \exists k \in M$ such that $n_1 + m_2 + k = n_2 + m_1 + k$. It is easy to check that this is an Abelian group and it satisfies the universal property taking $i(m) = [(m, 0)]$. A morphism to an Abelian group $f : M \to A$ is lifted uniquely to $\bar{f}([m, n]) = f(m) - f(n)$.

ii) We take $(Z(M), +_{Z(M)})$ the free Abelian group over M. We call $[m]$ the image of $m \in M$ on the free group. Take the quotient

$$K(M) = Z(M)/\langle\{[n + m] -_{Z(M)} ([n] +_{Z(M)} [m]) : n, m \in M\}\rangle$$

We abuse notation, and call again by $[m]$ the quotient of $[m] \in Z(M)$. Again this is an Abelian group and it satisfies the universal property taking $i(m) = [m]$. This time, the morphism f gets lifted to $\bar{f}([m]) = f(m)$ (defined in the base and then composed with the quotient morphism.)

The first construction is simpler to work with and it generalizes the step from $\mathbb{N}$ to $\mathbb{Z}$. The second construction has the advantage that can be generalized to semigroups (that is, monoids minus the identity element axiom.) Also, the second construction is very similar to the construction we will later do for additive and Abelian categories.

Examples

1) As it is expected from the introduction, the Grothendieck group of $\mathbb{N}$ is $\mathbb{Z}$.
2) If A is an Abelian group, then $K(A) = A$.
3) Note that since $X \cup A = X\ \forall A \in P(X)$, $[A] = [X]\ \forall A \in P(X)$, therefore $K(P(X), \cup) = \{0\}$. Same happens for $K(P(X), \cap)$, taking $\emptyset$.
4) $(\mathbb{Z}, \cdot)$ is not a monoid since it does not have identity (it would be 1, but sadly $1 \cdot 0 = 0$.) However, it is a semigroup and we can still make our second construction. But again, since $0 \cdot a = 0\ \forall a \in \mathbb{Z}$ we have that $K(\mathbb{Z}, \cdot) = 0$.
5) $K(\mathbb{Z} - \{0\}, \cdot) = \mathbb{Q} - \{0\}$.
6) $K([X, \mathbb{N}]) = [X, \mathbb{Z}]$.
7) $K(\mathbb{N}[q, q^{-1}]) = \mathbb{Z}[q, q^{-1}]$.

We wish to avoid cases like 3) and 4). Since our objective is to add inverses, we wish that i works as an injection. The main problem with this examples is the existence of an absorbing element, which makes the group trivial. A more subtle examination takes our attention to $Ker(i) = \{(a,b) \in M^2 \mid i(a) = i(b)\}$ considered as an equivalence relation (That is, $a \sim b \Leftrightarrow (a,b) \in Ker(i) \Leftrightarrow i(a) = i(b)$.) It is known from universal algebra (see for instance [BS81] chapter II) that $M/Ker(i) \simeq Im(i)$. In particular i is injective if and only if $Ker(i)$ is the trivial relation ($a \sim b \Leftrightarrow a = b$.)

Note that cancellation property makes $Ker(i)$ trivial. The converse is true.

Proposition 1.0.2. *Let M be a monoid and $K(M)$ its Grothendieck group, then*

a) Every element of $K(M)$ is of the form $[n] - [m]$ for $n, m \in M$

b) i is injective $\Leftrightarrow$ M has cancellation property

Proof. We use our second construction. To prove a), note that every element in $Z(M)$ is of the form $\sum_{i \in I} s_i[m_i] - \sum_{i \in J} r_i[n_i]$, with $m_i, n_i \in M$ and $s_i, r_i \in \mathbb{N}$.

When we take the quotient, since $[a+b] = [a] + [b]$, we get that

$$\sum_{i \in I} s_i[m_i] - \sum_{i \in J} r_i[n_i] = [\sum_{i \in I} s_i m_i] - [\sum_{i \in J} r_i n_i]$$

b) is a direct consequence of a). ☐

Example

We have that $\mathbf{V} = S(\mathbf{Vect})$ has the cancellation property (there is a good reason for taking just finite dimensional vector spaces, as we will see in 1.3), so it injects on its Grothendieck group. We "added inverses" $-V$ of (isomorphism classes of) vector spaces $V \in S(\mathbf{Vect})$ for the direct sum.

What kind of structure do these inverses $-V$ have? can we say more about them than just "there are added points"? For example, can they be regarded as vector spaces?

The answer to the third question is no. If $-V$ is a vector space, then $V \oplus -V \simeq 0 \Rightarrow \dim(V) + \dim(-V) = 0 \Rightarrow V \simeq -V \simeq 0$.

As a matter of fact, as we will see in next section, in a nontrivial category $\mathbf{C}$ with coproducts $\sqcup$ and an initial object i (like $(\mathbf{Vect}, \oplus, 0)$), for any objects $a, -a \in \mathbf{C}$, we have that $a \sqcup -a \simeq i \Rightarrow a \simeq -a \simeq i$.

We refine the question.

Can we inject **Vect** in a well behaved category in which we can describe inverses?

We do not have a complete answer to this question, but there is still more to say. We will continue this discussion in chapter 4, after going further into the abstraction rabbit hole.

Let us now take a look on the algebraic features of $K(\mathbf{V})$.

The dimension of a vector space defines a bijection

$$d : \mathbf{V} \to \mathbb{N}$$

And since $d(V \oplus W) = d(V) + d(W)$ and $d(0) = 0$, this bijection is an isomorphism of monoids. So we obtain

$$K(\mathbf{V}) \simeq \mathbb{Z}.$$

This may come as a disappointment after the discussion about inverses of vector spaces, was all the discussion at the end just about numbers?

That disappointment is a delusion, derived from a vice of abstract algebra: to think that two isomorphic objects are the same. Vector spaces and natural numbers are not the same, otherwise the Linear Algebra course would be composed just by one lecture. There is a lot of information that we are loosing when we calculate $K(\mathbf{V}) \simeq \mathbb{Z}$. Particularly, we are loosing all the information about matrices. We are forgetting the fact that vector spaces are arrays of elements of a field and just focusing in the size of the array.

This process of loosing information is done by decategorification. Categorification is the opposite process, that is, to gather information with a category that can decategorify into the object of study.

Getting a bit claustrophobic? Do not worry, we are going into more abstraction levels.

1.2 Grothendieck Group on a Category

A binary operation on a nonempty category $\mathbf{C}$ (a category with at least one object) is a bifunctor

$$\begin{array}{lll} * : & \mathbf{C} \times \mathbf{C} \to \mathbf{C} & \\ & (a,b) \mapsto a * b & \text{for } a, b \in \mathbf{C} \\ & (f,g) \mapsto f * g & \text{for } f, g \in Mor(\mathbf{C}) \end{array}$$

As examples we have the direct sum and tensor product on **Vect** and the disjoint union in **Set**.

We have the functorial properties $id_a * id_b = id_{a*b}$ and $(g \circ f) * (g' \circ f') = (g * g') \circ (f * f')$

But what is more interesting is that whenever f and g are isomorphisms we have that $f * g$ is isomorphism with $(f * g)^{-1} = g^{-1} * f^{-1}$. From this we get

Proposition 1.0.3. *For any objects $a, b, a', b' \in Ob(\boldsymbol{C})$*

$$a \simeq a', b \simeq b' \Rightarrow a * b \simeq a' * b'$$

Proof. $a \simeq a', b \simeq b' \Rightarrow \exists f : a \xrightarrow{\sim} a'$ and $g : b \xrightarrow{\sim} b'$ with inverses f^{-1} and g^{-1}. Then $f * g : a * b \to a' * b'$ is isomorphism with inverse $g^{-1} * f^{-1}$. ☐

This is of great value because it allows us to define the operation $*$ on the skeleton $S(\mathbf{C}) = Ob(\mathbf{C})/\simeq$. Categorification deals with isomorphism classes, therefore, working with $S(\mathbf{C})$ is a natural choice.

Since in this document we are restricting to essentially small categories, we have that $S(\mathbf{C}) = Ob(\mathbf{C})/\simeq$ is a set. Now we are ready to define the Grothendieck Group

Definition (Grothendieck Group)**.** *Let $\boldsymbol{C}$ be an essentially small category and $*$ a binary operation on $\boldsymbol{C}$. The Grothendieck Group $K(\boldsymbol{C}, *)$ is the quotient of the free Abelian group over $S(\boldsymbol{C})$ by the subgroup generated by*

$$\{[a * b] - ([a] + [b]) \mid a, b \in S(\boldsymbol{C})\}$$

Note that if $*$ is weakly commutative (that is, $a*b \simeq b*a \ \forall a, b \in \mathbf{C}$), weakly associative ($(a*b)*c \simeq a*(b*c) \ \forall a, b, c \in \mathbf{C}$), and has a weak unit (there is $e \in \mathbf{C}$ such that $a*e \simeq e*a \simeq a \ \forall a \in \mathbf{C}$) we recover the second construction we gave for the Grothendieck Group of an Abelian monoid, so there is no much to be alarmed here. In fact we already calculated

$$K(\mathbf{Vect}, \oplus) \simeq (\mathbb{Z}, +)$$

We can also consider the tensor product $\otimes$ of vector spaces. This is a weakly associative and commutative operation with weak unit on $\mathbf{Vect}_{>0}$ (Where $\mathbf{Vect}_{>0}$ is the category of nonzero finite dimensional vector spaces), so $(S(\mathbf{Vect}_{>0}), \otimes)$ is a monoid. Since $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$, $\dim : (\mathbf{V} - \{0\}, \otimes) \xrightarrow{\simeq} (\mathbb{N}_{>0}, \cdot)$ establishes an isomorphism of monoids. We therefore have

$$K((\mathbf{Vect}_{>0}, \otimes)) \simeq (\mathbb{Q}_{>0}, \cdot)$$

Here we have other example.

Example

Take **FinSet** the category of finite sets with functions as morphisms (again, there is a good reason for which we don't take **Set** and just the finite sets.) We know that the isomorphism classes are given by the cardinality, so

$$c : S(\mathbf{FinSet}) \to \mathbb{N}$$

With $c(X) = |X|$ is a bijection.

Moreover, we can take the disjoint union $X_1 \sqcup X_2 = \{(x, i) \mid i \in \{0, 1\}, x \in X_i\}$ as binary operation (note that $(f \sqcup g)(x, i)$ can be defined as $f(x)$ if $i = 0$ and $g(x)$ if $i = 1$.) We have that c is an isomorphism of monoids, from which we get

$$K(\mathbf{FinSet}, \sqcup) \simeq (\mathbb{Z}, +)$$

Again we obtain $(\mathbb{Z}, +)$, and (again) we say that **FinSet** is a categorification of $\mathbb{Z}$. Note that this captures really well our understanding of numbers when we were kids, that is, as equivalence classes of finite sets.

We may ask for the enlargement of sets into a structure with negative cardinality, since $c : S(\mathbf{FinSet}) \to \mathbb{Z}$ defines a morphism $\bar{c} : K(\mathbf{FinSet}) \to \mathbb{Z}$. In fact, it makes sense that negative dimensional vector spaces would be spanned over sets with negative cardinality.

Sadly, the high expectations that the question offers are doomed to fail (or at least is not that easy to answer.) Consider a category $\mathbf{C}$ with coproducts $\sqcup$ and initial object 0 (like **FinSet** with disjoint union and $\emptyset = 0$)

Say for that any object a there is $-a$, such that $a \sqcup -a \simeq 0$. Then, for any object y, there is only one morphism $a \sqcup -a \to y$, which is the same as morphisms $a \to y$ and $-a \to y$. We get that both a and $-a$ are initial, ie $a \simeq -a \simeq 0$.

That does not mean that we can not categorify the subset $\mathbb{Z}_{<0}$ of $\mathbb{Z}$ without seeing it as the Grothendieck group of $\mathbb{N}$. It just means that we have to be more careful. We will get to that on chapter 4.

Note that, as in the case of vector spaces, we lost a lot of information. Somewhere along the way we lost the morphisms and start working only with objects.This is weird since the key information of categories are the morphisms, but are useful for decategorification, since we get a simple object.

Categorification gathers information about morphisms.

So, for this example, it is not just about the numbers, it is about how the numbers interact with each other when regarded as (isomorphism classes of) sets.

Now the classical (engineer like) question. Beautiful as this is, does it have applications?

The answer is Yes of course, otherwise the theory would not be under development and I would not be writing this thesis. We have to wait a little though, there are still a couple of abstraction levels for the Grothendieck Group to show its full power.

1.3 Grothendieck Group of an Additive Category

An additive category is a category in which we can take direct sum $\oplus$ of objects, which is at the same time product and coproduct. This turns out to be a nice binary operation which can be used to define a Grothendieck Group. We will go over the main results for additive categories which will be of use to us, but the reader that wants to learn more about the subject and the proofs of our statements can read [Lan10] or [LM13].

Definition. *A category* $\boldsymbol{A}$ *is* additive *if*

i) Each $Hom(a, b)$ *is an Abelian group and the composition is bilinear*

ii) $\boldsymbol{A}$ *has a zero object*

iii) The product and coproduct exist

A category satisfying just i) is called preadditive. A preadditive category can be completed in a nice way into a additive category as shown in [LM13].

The canonical example is R-mod for a ring R, which includes **Vect** and **Ab**. We will get deeper into the examples after introducing Abelian categories.

An example of a category that is not (pre)additive is **Set** (for instance, $Hom(\{1, 2\}, \{1, 2\})$ can not satisfy i) (Exercise! do the tables.))

Proposition 1.0.4. *On an additive category products and coproducts are isomorphic.*

They are called biproducts and denoted $a \oplus b$, emulating the direct sum of R-mod. Now, for a biproduct $a \oplus b$ we have distinguished morphisms $i_a : a \to a \oplus b$, $i_b : b \to a \oplus b$, $p_a : a \oplus b \to a$, and $p_b : a \oplus b \to b$. It is not hard to see that this morphisms satisfy the equations

$$p_a i_a = id_a, \quad p_b i_b = id_b, \quad p_a i_b = 0_{b,a}, \quad p_b i_a = 0_{a,b}, \quad i_a p_a + i_b p_b = id_{a \oplus b} \quad (0)$$

Funny thing is that this equations characterize the biproduct:

Proposition 1.0.5. *For objects* a, b, c *on an additive category* $\boldsymbol{A}$ *we have*

$c \simeq a \oplus b \iff$ *there are morphisms* $i_a : a \to c$*,* $i_b : b \to c$*,* $p_a : c \to a$*, and* $p_b : c \to b$ *satisfying* (0)*.*

We can produce the direct sum of morphisms too:
for $f : a \to b$ and $g : c \to d$ we define $f \oplus g : a \oplus c \to b \oplus d$ as

$$f \oplus g = i_b f p_a + i_d g p_c$$

And with this we can define an operation

Proposition 1.0.6. *Given $\boldsymbol{A}$ an additive category, we have that*

$$\begin{aligned} \oplus : \quad & \boldsymbol{A} \times \boldsymbol{A} \to \boldsymbol{A} & \\ & (a,b) \mapsto a \oplus b & a, b \in Obj(\boldsymbol{A}) \\ & (f,g) \mapsto f \oplus g & f, g \in Mor(\boldsymbol{A}) \end{aligned}$$

Is a binary operation on $\boldsymbol{A}$.

We can also sum functors. Given two functors $F, G : \mathbf{A} \to \mathbf{B}$ between Abelian categories, we define their direct sum as the functor

$$\begin{aligned} F \oplus G : \quad & \mathbf{A} \to \mathbf{B} & \\ & a \mapsto F(a) \oplus G(a) & a \in Obj(\mathbf{A}) \\ & f \mapsto F(f) \oplus G(f) & f \in Mor(\mathbf{A}) \end{aligned}$$

Since $\oplus$ defines an operation, we can use the definition from last section and define

Definition. *The* split Grothendieck group *of an additive category $\boldsymbol{A}$, denoted $K^{\oplus}(\boldsymbol{A})$ is the Grothendieck Group of $\boldsymbol{A}$ with respect to the binary operation $\oplus$.*

That is, the quotient of the free Abelian group $\{[a] : a \in \boldsymbol{A}\}$ by the subgroup generated by $\{[a \oplus b] - [a] - [b] \mid a, b \in \boldsymbol{A}\}$

The split Grothendieck group has a universal property. For $\mathbf{A}$ and $\mathbf{B}$ additive categories, and A Abelian group, we say that

i) A map $\phi : Obj(\mathbf{A}) \to A$ is *additive* if $\phi(a \oplus b) = \phi(a) + \phi(b), \forall\, a, b \in Obj(\mathbf{A})$

ii) A functor $F : \mathbf{A} \to \mathbf{B}$ is *additive* if $F(a \oplus b) \simeq F(a) \oplus F(b), \forall\, a, b \in Obj(\mathbf{A})$

Universal Property. *Given $\boldsymbol{A}$ an additive category, let $[\cdot] : Ob(\boldsymbol{A}) \to K^{\oplus}(\boldsymbol{A})$ be the natural map $a \mapsto [a]$. Then for every additive map $\phi : Ob(\boldsymbol{C}) \to A$ to an abelian group A, there is a unique group homomorphism $\bar{\phi} : K^{\oplus}(\boldsymbol{A}) \to A$ such that $\bar{\phi} \circ [\cdot] = \phi$*

$$\begin{tikzcd}
K^{\oplus}(\boldsymbol{A}) \arrow[r, "\bar{\phi}"] & A \\
Obj(\boldsymbol{A}) \arrow[u, "{[\cdot]}"] \arrow[ur, "\phi"'] &
\end{tikzcd}$$

Note that if we have an additive functor $F : \mathbf{A} \to \mathbf{B}$, we get that $F \circ [\cdot] : \mathbf{A} \to K^{\oplus}(\mathbf{B})$ is additive, so we can lift it with the universal property to a group homomorphism which we will call $[F] : K^{\oplus}(\mathbf{A}) \to K^{\oplus}(\mathbf{B})$ (or simply by F if the context is clear. We will try not to do this.)

As an example, we may present $\mathbf{Vect}_\infty$, the category of vector spaces. This is an additive category, and we can in all right calculate its split Grothendieck group $K^\oplus(\mathbf{Vect}_\infty)$.

This group is trivial.

The reason is that for any vector space V there is a vector space of higher dimension W such that $V \oplus W \simeq W$, which implies that $[V] + [W] = [W]$ in $K^\oplus(\mathbf{Vect}_\infty)$, and so $[V] = [0]$. Since every element in $K^\oplus(\mathbf{Vect}_\infty)$ is of the form $[V_1] - [V_2]$ with $V_1, V_2 \in \mathbf{Vect}_\infty$ (the reader can make the proof of this, it is the same as what we did for Abelian monoids), we obtain that $K^\oplus(\mathbf{Vect}_\infty) \simeq \{0\}$.

The reader can copy this argument to obtain that $K(\mathbf{Set}, \sqcup) \simeq \{0\}$.

Let us go to Abelian categories.

1.4 Grothendieck Group of an Abelian Category

An Abelian category is a category on which we can comfortably do homology, that is we can unambigously talk about concepts like kernel, cokernel, complexes, exactness, etc.

They feel a lot like working in R-mod, and there is a mathematical reason for that (the Freyd-Mitchell theorem): if $\mathbf{C}$ is a essencially small Abelian category there is a ring R and a fully faithful functor $F : \mathbf{C} \to R$-mod. A good reference to study the subject is [Alu09] Chapter IX.

We start recalling the definitions of monomorphism and epimorphism. In a category $\mathbf{C}$ an arrow $f : A \to B$ is called

- *monomorphism* if for every parallel morphisms $g_1, g_2 : Z \to A$, we have that $f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$.
- *epimorphism* if for every parallel morphisms $h_1, h_2 : B \to Z$, we have that $h_1 \circ f = h_2 \circ f \Rightarrow h_1 = h_2$.

In categories like **Set**, **Grp**, **Vect**, etc., these correspond to the notions of injective and surjective function or homomorphism.

Through abstract algebra, injectivity is closely related to a famous object: the kernel. In categories like **Grp**, **Ring**, R-mod, etc. we think about the kernel $Ker(f)$ of a morphism $f : A \to B$ as a subobject of A. When going into semigroups and other objects with less algebraic structure this notion it is not enough, so we expand it and let the kernel be an equivalence relation ($a \sim b \Leftrightarrow f(a) = f(b)$.) This leads to an awesome theory, but it is still in the realm of **Set**. In a category we don't always have notions like "subobjects" or "quotients", we need to get into the "arrow-thinking" setting.

Fortunately for us, in additive categories the kernel is given by a universal property.

Definition (Kernel of a morphism)**.** *Let $f : A \to B$ be a morphism in an additive category $\boldsymbol{A}$. A morphism $i : K \to A$ is a kernel of f if $f \circ i = 0$, and any other morphism $g : Z \to A$ such that $f \circ g = 0$ factors uniquely through i, that is, $\exists!\, \bar{g} : Z \to K$ such that the following diagam commutes*

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "\bar{g}"'] & A \arrow[r, "f"] & B \\
K \arrow[ur, "i"'] & &
\end{tikzcd}$$

We call $i = Ker(f)$. In the categories that we mentioned (**Grp**, **Ring** and R-mod), i corresponds to the inclusion of the usual kernel into A. If you feel uncomfortable seeing the kernel as a morphism, don't worry, through our examples you might think of the kernel in the usual way.

The cokernel is the dual (reverse arrows) of the kernel

Definition (Cokernel of a morphism)**.** *Let $f : A \to B$ be a morphism in an additive category $\boldsymbol{A}$. A morphism $j : CoK \to A$ is a cokernel of f if $j \circ f = 0$, and any other morphism $h : Z \to A$ such that $h \circ f = 0$ factors uniquely through j, that is, $\exists!\, \bar{h} : Z \to CoK$ such that the following diagam commutes*

$$\begin{tikzcd}
Z & A \arrow[l, "h"'] \arrow[dl, "j"] & B \arrow[l, "f"'] \\
CoK \arrow[u, "\bar{h}"] & &
\end{tikzcd}$$

We call $j = Coker(f)$. In R-mod, if f is a monomorphism, the cokernel is isomorphic to the quotient B/A.

Now, the existence of (co)kernels is fundamental when talking about exact sequences, but it is not guaranteed in any additive category. Also, one can show ([Alu09] Lemma IX.1.4) that kernels are monomorphisms and cokernels are epimorphisms, but the inverse is not guaranteed.

We impose these conditions.

Definition (Abelian Category)**.** *An additive category is* Abelian *if*

i) Every morphism has a kernel and a cokernel.

ii) Every monomorphism is a kernel and every epimorphism is a cokernel.

As said before, in Abelian categories we can define exact sequences

Definition. *A sequence of morphisms*

$$\dots \xrightarrow{f_{i-1}} a_i \xrightarrow{f_i} a_{i+1} \xrightarrow{f_{i+1}} \dots$$

Is exact if $ker(f_{i+1}) = ker(Coker(f_i))$

Remark. We call $Im(f_i) = Ker(Coker(f_i))$. For the examples covered in this document, it corresponds to the usual image of f_i, so the reader might think the condition to be the usual $ker(f_{i+1}) = Im(f_i)$

A short exact sequence is an exact sequence of the form

$$\dots \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \dots$$

Which we write as

$$0 \to a \xrightarrow{f} b \xrightarrow{g} c \to 0$$

or simply as

$$0 \to a \to b \to c \to 0$$

When there is no need of specifying the morphisms. We abbreviate "short exact sequence" as s.e.s.

Note that if $b = a \oplus c$, we always have a short exact sequence

$$0 \to a \xrightarrow{i_a} a \oplus c \xrightarrow{p_c} c \to 0$$

So s.e.s. are a generalization of the sum $\oplus$.

Conversely, we say that a s.e.s. $0 \to a \xrightarrow{f} b \xrightarrow{g} c \to 0$ *splits* if it is isomorphic to $0 \to a \xrightarrow{i_a} a \oplus c \xrightarrow{p_c} c \to 0$, that is, if there are isomorphisms $a \xrightarrow{\simeq} a$, $b \xrightarrow{\simeq} a \oplus c$ and $c \xrightarrow{\simeq} c$ such that the following commutes

$$\begin{tikzcd}
0 \arrow[r, "0"] & a \arrow[r, "f"] \arrow[d, "\simeq"] & b \arrow[r, "g"] \arrow[d, "\simeq"] & c \arrow[r, "0"] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r, "0"] & a \arrow[r, "i_a"] & a \oplus c \arrow[r, "p_c"] & c \arrow[r, "0"] & 0
\end{tikzcd}$$

In particular, $b \simeq a \oplus c$

Now, not all the categories we wish to consider are Abelian. There is a weaker context in which we can define the Grothendieck group

Definition. *A* category with exact sequences $\boldsymbol{S}$ *is a full additive subcategory of an Abelian category closed under s.e.s., that is if*

$$0 \to a \to b \to c \to 0$$

is a s.e.s. and $a, c \in \boldsymbol{S}$*, then* $b \in \boldsymbol{S}$*.*

The nice morphism concept for this categories would have to respect s.e.s. Such a morphism is called an exact functor

Definition (Exact Functor)**.** *A functor* $F : \boldsymbol{C} \to \boldsymbol{D}$ *between categories with exact sequences (in particular between Abelian categories) is called exact if*

$$0 \to a \to b \to c \to 0 \text{ is a s.e.s.} \Rightarrow 0 \to F(a) \to F(b) \to F(c) \to 0 \text{ is a s.e.s.}$$

The Grothendieck group considers s.e.s. in its definition

Definition (Grothendieck Group)**.** *Given* $\boldsymbol{C}$ *a category with exact sequences, the Grothendieck Group of* $\boldsymbol{C}$*,* $K(\boldsymbol{C})$ *is the free Abelian group over* $S(\boldsymbol{C})$*, quotient by the (normal) subgroup generated by*

$$\{[b] - [a] - [c] \mid 0 \to a \to b \to c \to 0 \text{ is a s.e.s.}\}$$

The results in the rest of this section also hold for categories with exact sequences. We state them for Abelian categories.

Note that since $b = a \oplus c \Rightarrow 0 \xrightarrow{i_a} a \xrightarrow{p_b} b \to c \to 0$ is exact, we get the equation

$$[a \oplus b] = [a] + [b]$$

In the Grothendieck group. We also have that the group for which we are taking the quotient in the split case is subgroup of the not split case. So we get that

Proposition 1.0.7. $K(\boldsymbol{C}) \simeq K^{\oplus}(\boldsymbol{C})/N$

N is a measure of how many short exact sequences do not split.

The Grothendieck group satisfies an universal property. If $\mathbf{C}$ is an Abelian category, a map ϕ from $Ob(\mathbf{C})$ of an Abelian category to an Abelian group is additive if for every s.e.s. $0 \to a \to b \to c \to 0$, we have $\phi(b) = \phi(a) + \phi(c)$.

Universal Property. *Given $\boldsymbol{C}$ an Abelian category, and let $[\cdot] : Ob(\boldsymbol{C}) \to K(\boldsymbol{C})$ be the natural map $a \mapsto [a]$. Then for every additive map $\phi : Ob(\boldsymbol{C}) \to A$ to an Abelian group A, there is a unique group homomorphism $\bar{\phi} : K(\boldsymbol{C}) \to A$ such that $\bar{\phi} \circ [\cdot] = \phi$*

$$\begin{tikzcd}
K(\boldsymbol{C}) \arrow[r, "\bar{\phi}"] & A \\
Obj(\boldsymbol{C}) \arrow[u, "{[\cdot]}"] \arrow[ur, "\phi"'] &
\end{tikzcd}$$

Note that if we have an exact functor $F : \mathbf{C} \to \mathbf{D}$, we get that $F \circ [\cdot] : \mathbf{C} \to K(\mathbf{D})$ is additive, so we can lift it with the universal property to a group homomorphism which we will call $[F] : K(\mathbf{C}) \to K(\mathbf{D})$ (or simply by F if the context is clear. We will try not to do this.)

Before going into the examples, let us show a useful characterization of the classes in $K(\mathbf{C})$. The proof of this is on [LM13] Proposition 2.4.19.

Proposition 1.0.8. *For $\boldsymbol{C}$ Abelian category and $a, b \in \boldsymbol{C}$, the following are equivalent*

i) $[a] = [b]$ in $K(\boldsymbol{C})$

ii) There exist objects $c, u, v \in \boldsymbol{C}$ such that

$$0 \to u \to a \oplus c \to v \to 0 \text{ and } 0 \to u \to b \oplus c \to v \to 0$$

Are exact.

We will make two more definitions before going into the examples

Definition. *i) An object a in an additive category is said to be indecomposable if whenever $a = \bigoplus_{i \in I} b_i$, then $a \simeq b_{i_0}$ for some $i_0 \in I$ and $b_i \simeq 0$ for $i \neq i_0$*

ii) An object a in an Abelian category is said to be simple if any monomorphism $f : b \to a$ is either 0 or an isomorphism.

Examples

1. As said, R-mod is the canonical example of an Abelian category. Now, if we take this just like that, we have the same problem as we had for the example $\mathbf{Vect}_\infty$ of last section, and so R-mod is not a useful category to work with.

2. Instead, we work with R-fmod, the category of finitely generated modules over a ring R. This category is not Abelian if R is not left noetherian, since it is not close under ker, but it is a category with exact sequences. Now, the calculation of $K(R\text{-fmod})$ and $K^\oplus(R\text{-fmod})$ can get really complicated and it depends a lot on the ring. There are many known cases, for instance

    (a) If $R = \mathbb{K}$, we have vector spaces. Here every s.e.s. splits, so we have $K(\mathbf{Vect}) \simeq K^\oplus(\mathbf{Vect}) \simeq \mathbb{Z}$.

    (b) If $R = \mathbb{Z}$, we have the fundamental theorem of finitely generated Abelian groups, so every $\mathbb{Z}$-module A can be written uniquely (up to order) as $A = \mathbb{Z}^j \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$ for n_i powers of a prime. Therefore, in the split Grothendieck group we have that $[A] = j[\mathbb{Z}] + [\mathbb{Z}_{n_1}] + \cdots + [\mathbb{Z}_{n_k}]$. That says that $\{[\mathbb{Z}]\} \cup \{[\mathbb{Z}_{p^i}] \mid p \text{ prime and } i \in \mathbb{N}\}$ is a basis of $K^\oplus(\mathbb{Z} - \text{fmod}, \oplus)$ as a $\mathbb{Z}$-mod. Since $K^\oplus(\mathbb{Z} - \text{fmod})$ has infinite numerable basis, we obtain that $K^\oplus(\mathbb{Z} - \text{fmod}) \simeq \mathbb{Z}[x]$.

    (c) **The Krull-Schmidt Property**: Generalizing last example, there is a way to calculate $K^\oplus(\mathbf{A})$ in a wide number of examples. Let S be the set of equivalence classes of indecomposable objects of an additive category.

        We say that $\mathbf{A}$ satisfies the Krull-Schmidt property if

        i. Every nonzero object $a \in \mathbf{A}$ can be written as $a \simeq a_1 \oplus \cdots \oplus a_n$ for $a_i \in S$.

        ii. If a can also be written as $a = b_1 \oplus \cdots \oplus b_m$ for $b_i \in S$, then $n = m$ and for some $\sigma \in S_n$ we have that $b_i \simeq a_{\sigma(i)} \forall i$.

        Theorem: If $\mathbf{A}$ satisfies the Krull-Schmidt property, then S is basis of $K^\oplus(\mathbf{A})$.

    (d) To calculate $K(\mathbb{Z} - \text{fmod})$, note that the exact sequence $\mathbb{Z} \xrightarrow{n\cdot(\ )} \mathbb{Z} \xrightarrow{mod(n)} \mathbb{Z}_n$ makes $[\mathbb{Z}_n] = 0$ for all $n \in \mathbb{N}$, so we are left just with $\mathbb{Z}$ in the basis, $K(\mathbb{Z} - fmod) \simeq \mathbb{Z}$.

    (e) $K(\mathbf{FinAb}) \simeq \mathbb{Z}[x]$, as we can show using the following theorem.

    (f) **Devissage**: We can also calculate $K(\mathbf{C})$ for $\mathbf{C}$ Abelian (or with exact sequences) under certain conditions. Under these conditions, $K(\mathbf{C})$ is the free Abelian group over S, now with S being isomorphism classes of simple objects. On an Abelian category we define the length of an object inductively. We say that simple objects have length 1, and M is of length n if there is a s.e.s.

        $$0 \to M_1 \to M \to M_2 \to 0$$

        with M_1 of length $n-1$ and M_2 simple. Call $\mathbf{C}_{<\infty}$ The full subcategory of objects with finite length. The following theorem is proved in [LM13] section 2.5.10.

        Theorem: $K(\mathbf{C}_{<\infty})$ is freely generated by $S = \{[a] : a \text{ is simple }\}$.

3. There is a well studied (see next example to see why) simpler case, the category $Proj(R)$ of finitely generated projective modules over an unital ring R. Recall that a R-module P is projective iff every exact sequence $0 \to N \xrightarrow{i} M \xrightarrow{j} P \to 0$ splits iff is isomorphic to a free sumand of a free R-mod.

   Since every s.e.s. of projective modules splits, we have that $K^{\oplus}(Proj(R)) = K(Proj(R))$. The group $K(Proj(R)) = K_0(R)$ is object of study of the so called *Algebraic K-theory.* There are several examples in which we can calculate $K_0(R)$, we list the basic ones.

   (a) If $R = \mathbb{K}$ a field (a division ring also works), all modules are projective, and moreover, they are defined by their dimension. The dimension gives an isomorphism $(Proj\mathbb{K}, \oplus) \simeq (\mathbb{N}, +)$, therefore $K_0(\mathbb{K}) \simeq \mathbb{Z}$.

   (b) More generally, if R is a PID (this includes of course $\mathbb{Z}$), then every projective module over R is isomorphic to R^n for some n called the *rank.* This rank induces an isomorphism $K_0(R) \simeq \mathbb{Z}$.

   (c) If R is a local (i.e. if it has only one maximal ideal), every projective module is free, so $K_0(R) \simeq \mathbb{Z}$. This helps studying the Spec of a ring.

   (d) If R is a Dedekind domain (i.e. an integral domain in which every proper ideal factors into a product of prime ideals), $K_0(R) = Pic(R) \oplus \mathbb{Z}$ (where $Pic(R)$ is the Picard group of R.)

   K-theory actually goes further. One can define higher dimensional K-groups $K_1(R)$, $K_2(R)$, and so on, and relative K-groups $K_n(R, I)$ for $I \subset R$ ideal, and one can prove that there exists a long exact sequence $K_1(R, I) \to K_1(R) \to K_1(R/I) \to K_0(R, I) \to K_0(R) \to K_0(R/I)$. There is also a definition of K_1 for an Abelian category, but we will not get into that.

4. Given a compact topological space X, consider the category of (complex) vector bundles $Vect(X)$. This is a category with exact sequences, where $\oplus$ is the usual sum of vector bundles. This is actually a particular example of $Proj(R)$, since $Vect(X)$ is equivalent to $Proj(C(X))$, with $C(X)$ being the continuous functions from X to $\mathbb{C}$. Now, the Grothendieck group $K_0(X) = K(Vect(X))$ is an important object of study of the topological K-theory. The fact that X is a topological space actually changes the flavor of the theory to topology, but all the works with $K_n(X)$ and sequences remains algebraic.

5. The representations of finite groups form the category of $\mathbb{K}[G]$-modules. The Grothendieck group is called the *Representation Ring.* It is a ring with the usual sum $\oplus$, and the product given by the tensor product of representations. This forms a ring, as we will see on the next section.

1.5 Monoidal Category and the Grothendieck Ring

1.5.1 Monoidal Category

A monoidal category is a category is which we can talk about tensor products, which is a nice way to stick objects together. We require this tensor product to be associative and to have a unit, up to isomorphism. Again, an example to keep in mind is R-mod and its tensor product $\otimes$.

The name *monoidal category* comes from the fact that we are generalizing the concept of a monoid from a set to a category, in a sense that would (hopefully) be clear after this chapter and the next one. We may loosely say [3] that the concept of a monoidal category categorifies the concept of a monoid.

Since monoidal categories appear a lot in the next chapters, we will take some time studying them.

We want the definition to include a category $\mathcal{M}$ with an operation $\otimes$ that is (weakly) associative $x \otimes (y \otimes z) \simeq (x \otimes y) \otimes z$ and with a (weak) unit $1 \otimes x \simeq x \otimes 1 \simeq x$. Subtle problems arise from a definition just like that, to say, there may be many ways in which $x \otimes (y \otimes z)$ can be isomorphic to $(x \otimes y) \otimes z$. We want a unique, canonical, way (see [BD98] for a really nice discussion on this.) We need more structure.

Definition. *A* Monoidal Category $\langle \mathcal{M}, \otimes, 1, \alpha, \lambda, \rho \rangle$ *(or just* $(\mathcal{M}, \otimes, 1)$*, or even just* $\mathcal{M}$ *when the context is clear) is a category* $\mathcal{M}$ *equipped with a functor* $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$*, called tensor product, and a distinguished object* $1 \in Obj(\mathcal{M})$*, called the unit, such that*

- *i) There exist a natural isomorphism* $\alpha : ((-) \otimes (-)) \otimes (-) \xrightarrow{\simeq} (-) \otimes ((-) \otimes (-))$ *with components* $\alpha_{x,y,z} : (x \otimes y) \otimes z \overset{\simeq}{\to} x \otimes (y \otimes z)$*, called the associator. (So here, for example,* $((-) \otimes (-)) \otimes (-)) : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ *is the functor that takes* $(x, y, z) \mapsto (x \otimes y) \otimes z$ *and* $(f, g, h) \mapsto (f \otimes g) \otimes h$*.)*
- *ii) There exists a natural isomorphism* $\lambda : (1 \otimes (-)) \xrightarrow{\simeq} (-)$ *with components* $\lambda_x : 1 \otimes x \overset{\simeq}{\to} x$*, called left unitor.*
- *iii) There exists a natural isomorphism* $\rho : ((-) \otimes 1) \xrightarrow{\simeq} (-)$ *with components* $\rho_x : x \otimes 1 \overset{\simeq}{\to} x$*, called right unitor.*

That satisfy commutation of diagrams for the association

$$\begin{tikzcd}
((w \otimes x) \otimes y) \otimes z \arrow[r, "\alpha_{w,x,y} \otimes id_z"] \arrow[d, "\alpha_{w \otimes x, y, z}"'] & (w \otimes (x \otimes y)) \otimes z \arrow[r, "\alpha_{w, x \otimes y, z}"] & w \otimes ((x \otimes y) \otimes z) \arrow[d, "id_w \otimes \alpha_{x,y,z}"] \\
(w \otimes x) \otimes (y \otimes z) \arrow[rr, "\alpha_{w,x,y \otimes z}"'] & & w \otimes (x \otimes (y \otimes z))
\end{tikzcd}$$

and the unit

$$\begin{tikzcd}
(x \otimes 1) \otimes y \arrow[rr, "\alpha_{x,1,y}"] \arrow[dr, "\rho_x \otimes id_y"'] & & x \otimes (1 \otimes y) \arrow[dl, "id_x \otimes \lambda_y"] \\
& x \otimes y &
\end{tikzcd}$$

So, for objects $x_1 \dots x_n \in \mathcal{M}$, the object $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ is well defined modulo canonical isomorphism.

The monoidal category is called strict if the associators and unitors are identities.

---

[3] We have to talk loosely since a rigorous definition has not been made yet.

The definition looks huge and scary (although it is rather cute compared to the one of a triangular category), but monoidal categories are a pretty common thing. Here we have some examples

Examples

1) Any monoid can be endowed with the structure of a strict monoidal category. Given a monoid M, consider the underling category of the set M, that is, the category whose objects are elements of M and whose morphisms are only identities. Take the tensor product as the monoid operation and the unit as the monoid unit.

2) The category of Vector Spaces is a monoidal category with $\otimes$ as tensor product and $\mathbb{K}$ as unit.

3) Vector spaces, now with $\oplus$ as tensor product and $\{0\}$ as unit, also form a monoidal category.

4) Generalizing 2) and 3), we may take R-mod, for R any commutative ring.

5) **Set** is a monoidal category, taking $\times$ as tensor product, and the final object $\{*\}$ as the unit. It is also a monoidal category for $\sqcup$, taking now the empty set (initial object) as the unit.

6) Let **Cat** be the category with categories as objects and functors as morphisms. The usual product of categories $\times$ makes **Cat** into a monoidal category. The unit will be the final object: the category $\{*\}$ with one object and one morphism.

7) You might see where we are going with this. Any category with products $\sqcap$ and final object F is monoidal, taking $\sqcap$ as tensor product and F as unit (take projections as unital morphisms.) Dually, any category with coproducts and initial object is monoidal. These are not all the monoidal categories, as example 2) shows.

8) The category of endofunctors of a category $\mathbf{C}$ (objects are functors $F : \mathbf{C} \to \mathbf{C}$ and morphisms are natural transformations) is a strict monoidal category, taking composition of functors as tensor product, and the identity functor as unit.

9) Recall that a bimodule over two rings R and S is an Abelian group B with a left action of R and a right action of S. The actions make B a left R-mod and a right S-mod, and the actions have to be compatible $(rb)s = r(bs)$.

An (R,S)-bimodule M and an (S,T)-bimodule N can be tensorized on a natural way

$$M \otimes_S N$$

Giving a (R,T)-bimodule. This tensor product satisfies associativity (modulo canonical isomorphism.)

This is not a tensor category, because there is no operation $\otimes$ defined for all the bimodules. If we want to make this into a tensor category, we have to restrict to just (R,R)-bimodules. This is in fact a monoidal category taking R (with actions given by left and right product over R) as the unit.

Problem solved, we got a monoidal category, but... there is a loose end... we remain with the same feeling as when we construct the fundamental group and we feel there should exist a structure that encompasses the fundamental group "in all points at once".

Is there a bigger structure, that considers all (R, S)-bimodules, such that when looking at a singular component (a ring R) would give us the structure we just described?

In the next chapter, after introducing 2-categories, we successfully construct such a structure. It is the analog of the fundamental group(oid). The analogy is given by taking rings R as points in the topological space $x \in X$, (R, S)-bimodules as (homotopy classes of) paths between points, and $\otimes$ as concatenation. We also show more examples of this process. The many examples make this process have a name: it is called *horizontal categorification* or *oidification* (name coming from group*oid*).

From the example of the bimodules arises a set of questions: Can we make $(R_1, \ldots, R_n)$-modules? Is there a natural way to tensor them? (what does tensor mean?). Is there a (probably categorical in nature) structure such that $(R_1, \ldots, R_n)$-modules with the tensor product can be presented as an example of this structure? Is there an analog (or, can an analog be abstractly defined) of the fundamental group-oidoid...(n times)...oid?

The first question is pretty easy to answer, just take n actions of rings $R_1, \ldots, R_n$ and make all the actions commute pairwise. The follow up questions are much harder, for example, how would you tensor (R, S, T)-modules? Note that if we tensor 2 of such by sticking as in bimodules, using rings as glue, we would have to stick with one and a half ring... The tensor product here seems to be a ternary rather than a binary operation.

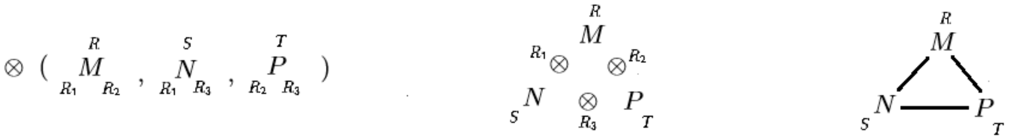


Figure 1: Conjecture for the tensor product of three trimodules

While for the (R_1, R_2, R_3, R_4)-modules there are two options. But following the 3-dimension case, it is more likely that we have a 4-ary operation here

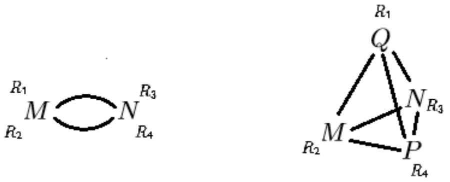


Figure 2: Possibilities for the tensor product of 4-modules

We do not have an answer to this questions (yet).

An interesting bunch of monoidal categories arise from the next construction.

Definition (Free Monoidal Category). *Given a category* $\boldsymbol{C}$*, the free monoidal category* $\Sigma(\boldsymbol{C})$ *over* $\boldsymbol{C}$ *is constructed as follows.*

i) Objects of $\Sigma(\boldsymbol{C})$ *are finite sequences* $A_1 \dots A_n$ *of objects* $A_i \in Ob(\boldsymbol{C})$*.*

ii) There are arrows between $A_1 \dots A_n$ *and* $B_1 \dots B_m$ *iff* $n = m$*. And in that case*

$$Hom_{\Sigma(\boldsymbol{C})}(A_1 \dots A_n, B_1 \dots B_n) = \{f_1 \dots f_n | f_i \in Hom_{\boldsymbol{C}}(A_i, B_i)\}$$

iii) Tensor product is given by concatenation

$$A_1 \dots A_n \otimes B_1 \dots B_m = A_1 \dots A_n B_1 \dots B_m$$

(and $f_1 \dots f_n \otimes g_1 \dots g_m = f_1 \dots f_n g_1 \dots g_m$*)*

A free monoid over a set S is categorified by a free monoidal category with one object per element of S. (note there are a huge amount of different categorifications here, that is because we are not putting any restrictions to the morphisms.)

Note that we are asking $\otimes$ to be a functor, so when we have a monoidal category, besides of being able to take tensor product of objects, we can also take tensor product of morphisms:

For any morphisms $f : A_1 \to B_1$ and $g : A_2 \to B_2$, there is a morphism $f \otimes g : A_1 \otimes A_2 \to B_1 \otimes B_2$ which composes componentwise

$$(f_1 \otimes g_1) \circ (f_2 \otimes g_2) = (f_1 \circ f_2) \otimes (g_1 \circ g_2)$$

This makes the tensor product a binary operation in the sense of the definition we gave in 1.2, and we can take the Grothendieck group. We saw that for (nonzero finite dimensional) vector spaces we get $(\mathbb{Q}_{>0}, \cdot)$.

The plan now is to enhance the Grothendieck group with a ring structure, using $\otimes$ as product (or a similar functor), so for example we can obtain $K(\mathbf{Vect}, \oplus, \otimes) \simeq (\mathbb{Z}, +, \cdot) \simeq K(\mathbf{Set}, \sqcup, \times)$.

1.5.2 Grothendieck Ring

On a monoidal category $(\mathbf{C}, \otimes, 1)$ we can define a product on the free Abelian group generated by $S(\mathbf{C})$

$$[A] \cdot [B] = [A \otimes B]$$

Extended by linearity (so for example $[A] \cdot ([B] + [C]) = [A] \cdot [B] + [A] \cdot [C]$.) This operation is associative and has identity $[1]$.

Recall that to get the split Grothendieck group we quotient by the subgroup $N^{\oplus}(\mathbf{C}) = \langle\{[A \oplus B] - [A] - [B] \mid A, B \in S(\mathbf{C})\}\rangle$, and by $N(\mathbf{C}) = \langle\{[C] - [A] - [B] \mid 0 \to A \to C \to B \to 0 \text{ is exact }\}\rangle$ for the Grothendieck group if $\mathbf{C}$ Abelian category. For the Grothendieck group to be a ring these have to be ideals.

This is not always the case, for example if we take $\mathbf{Vect}$ with $\oplus$ as tensor product. Suppose that $N(\mathbf{Vect})$ is an ideal of $(S(\mathbf{Vect}), \oplus, \oplus)$, then $\forall\, V, W \in \mathbf{Vect}$ we will have that $N(\mathbf{Vect}) \ni [V] \cdot ([V \oplus W] - [V] - [W]) = [V \oplus V \oplus W] - [V \oplus V] - [V \oplus W] = -[V]$. This would imply that $N(\mathbf{Vect}) = S(\mathbf{Vect})$, which can not be true since $\forall\, a \in N(\mathbf{Vect})$ we have that $\dim(a)=0$.

We will make two definitions, one for each case, for which we will get that N and $N^{\oplus}$ are ideals.

Definition.

1) *For $\boldsymbol{C}$, $\boldsymbol{D}$ and $\boldsymbol{E}$ preadditive categories, a functor $F : \boldsymbol{C} \times \boldsymbol{D} \to \boldsymbol{E}$ is said biadditive if*

   $F(a \oplus b, c) \simeq F(a, c) \oplus F(b, c)$ and $F(a, d \oplus c) \simeq F(a, d) \oplus F(a, c)$ for all $a, b \in \boldsymbol{C}$ and $c, d \in \boldsymbol{D}$.

2) *For $\boldsymbol{C}$, $\boldsymbol{D}$ and $\boldsymbol{E}$ Abelian categories, a functor $F : \boldsymbol{C} \times \boldsymbol{D} \to \boldsymbol{E}$ is biexact if $F(a, -)$ and $F(-, b)$ are exact $\forall a \in \boldsymbol{C}$ $b \in \boldsymbol{D}$.*

Proposition 1.0.9. *Given $(\boldsymbol{C}, \otimes, 1)$ a monoidal additive category, we have that*

1. *If $\otimes$ is biadditive, then $N^{\oplus}(\boldsymbol{C})$ is an ideal*
2. *If $\boldsymbol{C}$ is Abelian, and $\otimes$ is exact, then $N(\boldsymbol{C})$ is an ideal*

In both cases, we get that the corresponding Grothendieck group associated is a ring with the product $[A] \cdot [B] = [A \otimes B]$.

Proof. Take $r = \sum\limits_{i=1}^{n} a_i[A_i]$ an element in the free Abelian group over $S(\mathbf{C})$, and $x = \sum\limits_{j=1}^{m} b_j([B_j] - [C_j] - [D_j])$ in $N(\mathbf{C})$ (resp. $N^{\oplus}(\mathbf{C})$), and multiply $rx = \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{m} a_i b_j([A_i \otimes B_j] - [A_i \otimes C_j] - [A_i \otimes D_j])$. Now, if $\otimes$ is exact, then $0 \to A_i \otimes C_j \to A_i \otimes B_j \to A_i \otimes D_j \to 0$ is a s.e.s. $\forall\, i, j$, so $rx \in N(\mathbf{C})$ (resp. if $\otimes$ is biadditive, then $[B_j] = [C_j \oplus D_j] \Rightarrow [A_i][B_j] = [A_i][C_j \oplus D_j] \Rightarrow [A_i \otimes B_j] = [(A_i \otimes C_j) \oplus (A_i \otimes D_j)]$, $\forall\, i, j$ and so $rx \in N^{\oplus}(\mathbf{C})$).
The inclusion $xr \in N(\mathbf{C})$ (resp. $xr \in N^{\oplus}(\mathbf{C})$) is proved in the same way. ☐

Examples
Now we get $K(\mathbf{Vect}, \oplus, \otimes) \simeq (\mathbb{Z}, +, \cdot) \simeq K(\mathbf{Set}, \sqcup, \times)$, with (the extension of) dimension and cardinality being the corresponding isomorphisms.

2 Interlude

Now that we have introduced the reader to concepts like the Grothendieck group, we start going on the opposite direction. That is, as the Grothendieck group consists on simplifying the algebraic structure, categorification goes in the direction of enriching it.

Let us start this chapter discussing the beautiful concept of

2.1 Enriched Categories and Horizontal Categorification

We are going to enter a journey through oidification land. We are going to see (and enumerate) a way in which many of the basic algebraic structures can be regarded as categories. Moreover, we are going to oidificate, or horizontal categorificate, many of these. Oidification, as we explained in the last section, is the process in which a concept is realized to be equivalent to a certain type of category with a single object. The oid process continues with studying the category, now with many objects with the given structure. The classical example is the oidification of groups into groupoids. We are about to see some more examples and, again, go a little bit up in the abstraction level.

We start with the very basic

1) **Monoid**

   We saw that the endomorphisms of an object on a category is a monoid with composition. There is more to that.

   We can define a monoid to be a one-object category. Thus, we say that the horizontal categorification of a monoid is a category.

2) **Grupoid**

   A grupoid is a category in which all morphisms are isomorphisms.

   Besides the purely algebraic study, and of course the example of the fundamental groupoid, groupoids are useful in analysis. They can be enriched with a Haar measure and can be nicely integrated over compact domains. Although an analyst would probably see a groupoid as a set rather than as a category.

3) **Group**

   A group is a one object groupoid. In other words, the oidification of a group is a groupoid.

Just as the set of endomorphisms of an object in a category is a monoid, we have that the set of automorphisms of an object is a group. This explains the symmetry feeling that one gets when working with groups (see [Alu09] chapter II.1 for a nice discussion on this.)

To continue our journey, we have to make the following definition.

Definition (Enriched Category)**.** *Let* $(M, \otimes, \mathbb{1})$ *be a monoidal category. A category* $\boldsymbol{C}$ *is said to be enriched over* M *if*

i) $Hom_{\boldsymbol{C}}(X, Y) \in M$ *for all* $X, Y \in \boldsymbol{C}$*.*

ii) For every $a, b, c \in \boldsymbol{C}$ *there exist morphisms in* M
$\circ : Hom_{\boldsymbol{C}}(b, c) \otimes Hom_{\boldsymbol{C}}(a, b) \to Hom_{\boldsymbol{C}}(a, c)$ *and* $id : \mathbb{1} \to Hom_{\boldsymbol{C}}(a, a)$*, such that* $\circ$ *is associative and* id *is the unit for the operation* $\circ$*.*

A category enriched over M is also called a M-category.

Examples

i) Note that if we replace $(M, \otimes, \mathbb{1})$ by $(\mathbf{Set}, \times, \{*\})$ we obtain the definition of a category. This says that any category is already enriched over **Set**.

ii) If we take $(M, \otimes, \mathbb{1}) = (\mathbf{Ab}, \otimes, \mathbb{Z})$ and $\mathbf{C}$ to be an arbitrary category, we obtain the definition of a preadditive category. So a category is said to be preadditive if it is enriched over **Ab**.

iii) The category of all vector spaces over a field is enriched over itself, since $Hom(V, W)$ is a vector space, and composition is bilinear (hence defines a linear map $Hom(W, Z) \otimes Hom(V, W) \to Hom(V, Z)$)

iv) Take $\mathbb{R}^+$ to be the category such that $Obj(\mathbb{R}^+) = [0, \infty]$ and the morphism structure given by $\leq$ (that is, $Hom_{\mathbb{R}^+}(a, b)$ is $\{*\}$ if $a \leq b$ and $\emptyset$ otherwise.)

This category is monoidal taking $+$ as tensor product and 0 as unit object.

If we take a category enriched over $(\mathbb{R}^+, +, 0)$ we obtain a generalized metric space (one in which we are not asking $d(a, b) = d(b, a)$ nor $d(a, b) = 0 \Rightarrow a = b$) by taking $d(a, b) = Hom_{\mathbb{R}+}(a, b)$

We continue our journey.

4) **Rings**

A ring is the set $Hom_{\mathbf{C}}(X, X)$, where $\mathbf{C}$ a category with one object X enriched over **Ab**. Product is given by composition and sum is given by the Abelian structure of $Hom_{\mathbf{C}}(X, X)$.

So, a Ring is a PreAdditive Category with one object.

Equivalently, the horizontal categorification of a ring is a preadditive category.

We can generalize this

5) **Unital Associative Algebras**

Given a commutative ring R, a unital associative R-algebra is $Hom_{\mathbf{C}}(X, X)$, where $\mathbf{C}$ a category with one object X enriched over R-mod.

A category enriched over R-mod is called R-linear Category. A unital associative algebra over R is a R-linear category with one object.

For examples 4) and 5), rings and algebras morphisms are given by functors that respect the structure in which we are enriching. These are called *M-Functors.*

Definition. *Let M be a monoidal category. A M-Functor F between M-categories **C** and **D** consists on*

i) A function $F : Ob(\boldsymbol{C}) \rightarrow Ob(\boldsymbol{D})$ *on the objects*

ii) An M-morphism $F : Hom_{\boldsymbol{C}}(a, b) \rightarrow Hom_{\boldsymbol{D}}(F(a), F(b))$

That satisfies that the M-composition map commutes with the action of F on morphisms.

So for example, a functor between categories is the same as a **Set**-functor. Also, as we said, for examples 4) and 5) we have that **Ab**-functors and R-mod functors correspond to morphisms of rings and algebras respectively.

A weirder example would be for $\mathbb{R}^+$-categories. In this case we obtain that the condition $F : Hom_{\mathbf{C}}(a, b) \rightarrow Hom_{\mathbf{D}}(F(a), F(b))$ translates to $d(F(a), F(b)) \leq d(a, b)$. Such functions between generalized metric spaces are called *non-expanding functions.*

Now the fun part is that we can define M-natural transformations, which will be in our context arrows between morphisms. This arrows are transformations which satisfy the naturality square.

So, for example, if we have a vector space V regarded as a $\mathbb{K}$-linear category, an isomorphism of functors is just a change of basis.

We continue our journey.

6) **Group Actions over a Set**

   Consider a group G as a one-object groupoid. What is a functor $F : G \to \mathbf{Set}$?

   The one object of the category is sent into some set A, and the morphisms (ie, elements of the group) are sent into bijections of A in a way compatible with the composition (ie, group operation).

   So we get that a group action over a set is a functor $F : G \to \mathbf{Set}$

This is amazing! we use the category **Set**, and obtain the representations of a group. We leave to the reader the exercise of overdriving the imagination with what would happen if we change the category **Set**. We give one example of this.

7) **Representations of Groups, Monoids, Algebras**

   If $\mathbf{C}$ is a Group, Monoid or Algebra regarded as a one-object category, a representation of $\mathbf{C}$ is a functor $F : \mathbf{C} \to \mathbf{Vect}_{\mathbb{K}}$ (where $\mathbf{Vect}_{\mathbb{K}}$ is the category of vector spaces over $\mathbb{K}$.

   What are natural transformations between this functors $F : \mathbf{C} \to \mathbf{Vect}_{\mathbb{K}}$? the reader can check these are just the morphisms between representations.

The power of categories as a tool for defining and studying objects in algebra starts to arise. Note that this starts to be particularly interesting when we consider natural transformations, which allow us to work with arrows between arrows. Many authors call this raising one dimension, and enlightens processes that were not apparent on a first look.

Now, there is power in the sense of modeling with categories that goes beyond abstract algebra. We give 2 examples

8) **Directed Graphs**

   Consider $\mathbf{C}$ the category with two objects E and V (which we can call *edges* and *vertices*, just for fun...), and two parallel morphisms $s, t : E \to V$ (which we can call *source* and *target*) together with the identity morphisms.

   A directed graph is a functor $F : \mathbf{C} \to \mathbf{Set}$. The edges, vertices, sources and targets of the graph are given by $F(E)$, $F(V)$, $F(s)$ and $F(t)$ respectively.

Again, this puts the mind on overdrive. What happens if we change the category **Set**? What happens if we take 3 morphisms instead of 2? What happens if we change the number of objects?

This definitions are really easy to make, and get to concepts relatively close to the concept of a Graph. As an instance, if we change **Set** with **Vect**, we obtain representations of quivers. Although these could be merely abstract objects, without a concrete object which can be modeled by it.

This is not completely bad, and we can make abstract definitions (that are not necessarily related to a known object) which can turn out to be useful, such as

9) **Simplicial Set**

   Let Δ denote the simplex category, whose objects are ordered sets $[n] = \{1, \ldots, n\}$ and whose morphisms are order preserving functions between these.

   A simplicial set is a contravariant functor $F : \Delta \to \mathbf{Set}$.

This is cool since it gives a categorical model to topological spaces that can be build from simplices and their incidence relations. Most of classical results for CW-complexes have analogous results for simplicial sets, which is nice since this has applications on algebraic geometry where CW complexes don't naturally exist. Simplicial Sets are also a higher dimensional generalization of directed multigraphs, and can be used to define the so called Quasi-Categories. We are not covering features of higher dimensional category theory, although it might be good to cover the next definition.

2.2 2-Categories

So far we have been avoiding to mention the enrichment over an important example of monoidal category: **Cat**. A category enriched over **Cat** gives a particularly interesting concept.

Definition. *A* 2-category *is a category enriched over* $(\boldsymbol{Cat}, \times, \{*\})$

Let us unravel this definition:

A 2-category $\mathbf{C}$ is given by

i) A set of objects $Ob(\mathbf{C})$

ii) $\forall A, B \in \mathbf{C}$, a category $Hom(A, B)$ of morphisms, equipped with

   - A composition functor $Hom(B, C) \times Hom(A, B) \to Hom(A, C)$
   - An identity functor $\{*\} \to Hom(A, A)$ $\forall A \in Obj(\mathbf{C})$

   Satisfying associativity and unit diagrams respectively.

Note that this corresponds to the horizontal categorification of monoidal categories (so a monoidal category corresponds to a one object 2-category, with $\otimes$ as composition.)

Let us unravel the information:

First, note that $Hom(A, B)$ is a category. That means that the objects are morphisms

$$A \xrightarrow{f} B$$

And there are morphisms between these morphisms

$$\begin{tikzcd} B & A \arrow[l, bend right=50, "g"', ""{name=U, below}] \arrow[l, bend left=50, "f", ""{name=D, above}] \arrow[Rightarrow, from=D, to=U, "\alpha"'] \end{tikzcd}$$

These are usually called 2-morphisms. They also receive the suggestive name of 2-cells (whereas objects A, B, C,etc are called 0-cells (points) and arrows $A \xrightarrow{f} B$ are called 1-cells (segments). Compare this with the example of Simplicial Sets.)

As a category, the (2-)morphisms in $Hom(A, B)$ have composition

$$\begin{tikzcd} B & A \arrow[l, bend right=60, "h"', ""{name=H, below}] \arrow[l, "g"' near end, ""{name=G1, above}, ""{name=G2, below}] \arrow[l, bend left=60, "f", ""{name=F, above}] \arrow[Rightarrow, from=G2, to=H, "\beta"'] \arrow[Rightarrow, from=F, to=G1, "\alpha"'] \end{tikzcd} \longmapsto \begin{tikzcd} B & A \arrow[l, bend right=50, "h"', ""{name=U, below}] \arrow[l, bend left=50, "f", ""{name=D, above}] \arrow[Rightarrow, from=D, to=U, "\beta\alpha"'] \end{tikzcd}$$

This is called vertical composition, and it is denoted $\alpha\beta$ or $\alpha \circ_1 \beta$.

Also, for each morphism $A \xrightarrow{f} B$ we have a 2-morphism 1_f which is unit for $\circ_1$.

Now, we also have that the composition is a functor. This means that besides the usual evaluation on objects $(B \xrightarrow{f} C, A \xrightarrow{g} B) \mapsto A \xrightarrow{f \circ g} C$, we have an evaluation on morphisms

This is called the horizontal composition and is denoted $\alpha * \beta$ or $\alpha \circ_0 \beta$.

$$\begin{tikzcd} C & B \arrow[l, bend right=50, "g'"', ""{name=U, below}] \arrow[l, bend left=50, "f'", ""{name=D, above}] \arrow[Rightarrow, from=D, to=U, "\beta"'] \end{tikzcd} , \begin{tikzcd} B & A \arrow[l, bend right=50, "g"', ""{name=U, below}] \arrow[l, bend left=50, "f", ""{name=D, above}] \arrow[Rightarrow, from=D, to=U, "\alpha"'] \end{tikzcd} \longmapsto \begin{tikzcd} C & A \arrow[l, bend right=50, "g'g"', ""{name=U, below}] \arrow[l, bend left=50, "f'f", ""{name=D, above}] \arrow[Rightarrow, from=D, to=U, "\beta*\alpha"'] \end{tikzcd}$$

Note that 1_f works as an unit for $\circ_0$

We have that the functoriality of the composition implies that

i) $1_f \circ_0 1_g = 1_{f \circ g}$

ii) $(\delta \circ_1 \gamma) \circ_0 (\beta \circ_1 \alpha) = (\delta \circ_0 \beta) \circ_1 (\gamma \circ_0 \alpha)$

The second property is called interchange law, picturely it says that it does not matter the order in which we compose the following diagram.

$$\begin{tikzcd}
C & B \arrow[l] \arrow[l, bend right=60, "g'"'] \arrow[l, bend left=60, "f'"] \arrow[l, phantom, shift right=3, "\delta\Uparrow"] \arrow[l, phantom, shift left=3, "\gamma\Uparrow"] & A \arrow[l] \arrow[l, bend right=60, "g"'] \arrow[l, bend left=60, "f"] \arrow[l, phantom, shift right=3, "\beta\Uparrow"] \arrow[l, phantom, shift left=3, "\alpha\Uparrow"]
\end{tikzcd}$$

Figure 3: interchange law.

These are all the features contained on a 2-category. There is a nice pictorial way to represent objects, morphisms and 2-morphisms, the so called diagrammatic algebra. We are not going into that, but the interest reader can check out [Lau11] for a nice exposition on this. Let us check out some examples of 2-categories.

Examples

1) The canonical example of a 2-category is **Cat**. Objects are categories, 1-morphisms are functors and 2-morphisms are natural transformations.

2) As we said, a 2-category with one object is a monoidal category, with the composition functor serving as tensor product.

3) The BiMod 2-category consists on Rings as objects, (R, S)-bimodules as 1-morphisms, and morphisms of (R, S)-bimodules as 2-morphisms. The composition functor here is given by tensor product (so $\circ$ is tensor product of bimodules, $\circ_0$ is tensor product of bimodule morphisms, and $\circ_1$ is the usual composition of (R, S)-morphisms.) This is the oidification of the monoidal category of (R, R)-bimodules showed in the last chapter. The analogy we talked about can now be made explicit.

4) The fundamental groupoid 2-category: Given a topological space X, there is a 2-category having the points $x \in X$ as objects, the paths from x to y as 1-morphisms and the homotopies of paths as 2-morphisms. The composition functor is the concatenation of paths (doing one path after the other), and the vertical composition corresponds to stick homotopies.

   The isomorphism classes of objects here represent the path connected components, and two 1-morphisms f and g are isomorphic iff they are homotopic. We can say that when we decategorify the fundamental groupoid 2-category we get points connected by paths, but we forget *how* they are connected.

5) The 2-category of homotopies, in which objects are topological spaces, 1-morphisms are continuous maps, and 2-morphisms are classes of homotopy. Composition is given by the usual composition of continuous functions, and vertical composition is given by the usual concatenation of paths.

3 Categorification of Hecke Algebras: Soergel Bimodules

This is the first sophisticated example that we are going to present. Categorification in the abstract can get really ethereal, so we are presenting this example before defining (algebraic) categorification, since it can show concretely what we are going to talk about later.

We are going to go from the rustic, the symmetries in the plane, going up in levels of abstraction to the categorical level, the soergel bimodules.

Of course we are going to outline the main features of the subject, although the author recommends for a really good introduction to Soergel Bimodules [Lib17], which highly influenced this chapter.

3.1 Coxeter Groups: This is about symmetries

Hecke algebras are quantum deformations of Coxeter groups, which were discovered by Harold Coxeter, a particularly interesting geometer that worked with polytopes and higher dimensional geometry. His work inspired some of M. Escher's work.

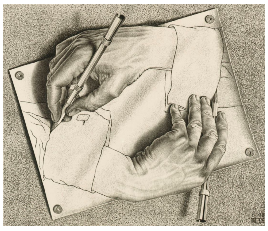


Figure 4: A picture by M. Escher

What does this have to do with Coxeter groups? Well, Coxeter groups arise naturally from the symmetries of polygons and n-dimensional polyhedra, particularly considering reflexions.

Definition. *Given a set* S*, a* Coxeter group *over* S *is the group given by the presentation*

$$\langle s \in S \mid (sr)^{m_{s,r}} = e \ \forall s, r \in S \rangle$$

Where $(m_{s,r})_{s,r\in S}$ *is a symmetric matrix with* 1*'s on the diagonal and values* $m_{s,r} \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$*.*

The pair (W, S) is called *Coxeter system*

Note that the 1's on the diagonal mean that $s^2 = 1 \ \forall s \in S$, and that $m_{s,r} = \infty$ just means there is no relation between s and r.

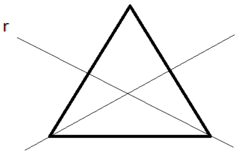
The elements of S are called simple reflections (because they commonly act as reflections, lets check some examples to make this clear.)

Examples

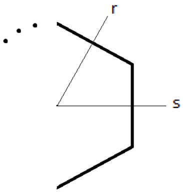
1. The group of symmetries of an equilateral triangle is a Coxeter group

$$D_6 = \rangle \{r, s \mid s^2 = r^2 = 1 \text{ and } rsr = srs\} \rangle$$

(that is, $m_{s,r} = 3$). The elements r and s can be seen graphically as



2. More generally, the group of symmetries of a regular n-agon is a Coxeter group
$$D_{2n} = \langle \{r, s \mid s^2 = r^2 = 1 \text{ and } (rs)^n = 1\} \rangle$$
Now, the elements r and s are 2 "consecutive" reflexions



3. If we "take the limit" as $n \to \infty$ in D_{2n} we get $U_2 = \langle r, s \mid r^2 = s^2 = 1\rangle$. This, rather than the symmetries of S^1, will be the symmetries of $(\mathbb{R}, \mathbb{Z})$, the figure obtained by taking the limit on n of the n-polygons but preserving the length of each side (see [Lib17] for more details). Moreover, we can define $U_n = \langle \{s_1, \dots, s_n \mid s_i^2 = 1 \ \forall i\} \rangle$. This is called the universal Coxeter group of rank n.

4. The group S_n of symmetries of an n-simplex (also of a set $\{1, \dots, n\}$) is also a Coxeter group, by considering the generators $s_i = (i, i+1)$, the simple transpositions.

5. Weyl groups are an important example of Coxeter groups. They are used in representations of Lie groups.

Bruhat Order

An element $x \in W$ may have many writings (for instance the element $x_0 = rsrsr = rssrs = s$ in D_6.) An *expression* of x is a string of elements $(s_1, \dots, s_n)$ of S such that $x = s_1 \dots s_n$. A *reduced expression* of x is an expression of minimal length. We call $l(x)$ the length of a reduced expression.

We can get from a reduced expression of an element $x \in W$ to any other reduced expression by using braid relations (that is $rsr.. = srs..$, with each side of the equality having $m_{s,r}$ simple reflexions.)

The Bruhat Order is a partial order in the Coxeter groups. We say that $x \leq y$ if (a reduced expression of) x is substring of (a reduced expression of) x, for instance

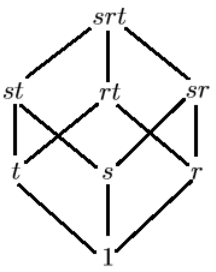


Figure 5: Bruhat order in U_2 for $[1, rst]$

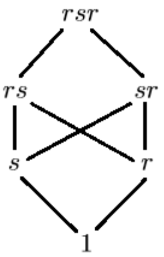


Figure 6: Bruhat order for D_6

3.2 Hecke Algebras

The Hecke algebra of a Coxeter group W is a quantum deformation of its group algebra.

Definition. *The* Hecke Algebra $H(W, S)$ *(or simply H) of the Coxeter system (W, S) is the $\mathbb{Z}[v, v^{-1}]$-algebra with generators $\{h_s\}_{s \in S}$ and relations*

i) $h_s^2 = (v^{-1} - v)h_s + 1$

ii) $h_s h_r h_s ... = h_r h_s h_r ...$ *(with the number of terms in each side of the equation being $m_{s,r}$)*

If we take $v = 1$, this is just the group algebra of W.

For a reduced expression $(s_1, \ldots, s_n)$ of $x \in W$, consider $h_{(s_1,\ldots,s_n)} = h_{s_1} \ldots h_{s_n}$. We have that for any other reduced expression $(r_1, \ldots, r_n)$, $h_{(s_1,\ldots,s_n)} = h_{(r_1,\ldots,r_n)}$. We call this simply h_x, and define $h_{id} = 1$.

Proposition 3.0.1. *The set $\{h_x\}_{x \in W}$ is a basis of $H(W, S)$ as a $\mathbb{Z}[v, v^{-1}]$-algebra.*

Which is good for starters, but we are going to find a better basis, more suitable for representation theory.

First, note that since $h_s(h_s + v - v^{-1}) = 1$, we have that h_x is invertible. Define the involution $d : H \to H$ as the ring homomorphism with $d(v) = v^{-1}$ and $d(h_x) = h_{x^{-1}}^{-1}$.

Self-dual elements are those invariant by d, such as $h_s + v$. We can find a basis of nice self-dual elements.

Proposition 3.0.2. *For all $x \in W$ there is an unique self-dual element $b_x \in H$ such that*

$$b_x = h_x + \sum_{y<x} v q_{y,x}(v) h_y$$

With $q_{y,x} \in \mathbb{Z}[v]$

The polynomials $p_{y,x}(v) = v^{l(x)-l(y)+1} q_{y,x}(v)$ are called *Kazhdan-Lusztig polynomials*. The elements $\{b_x\}_{x\in W}$ are called the *Kazhdan-Lusztig basis*. They are a hard combinatorial Headache! (I can say it from own experience), for instance, it was for many years thought that we will never find out if the conjecture "all the coefficients in $p_{y,x}$ are positive" was true, it just was a too ridiculously hard combinatorial problem. That is, until recently. The answer was found as an easy corollary in the development of a more abstract theory, the categorification of the Hecke algebras!

Such can be the power of categorification.

Did we name $\{b_x\}_{x\in W}$ Kazhdan-Lusztig *basis*? Well, they are.

Proposition 3.0.3. *The set $\{b_x\}_{x\in W}$ is a basis of H as a $\mathbb{Z}[v, v^{-1}]$-algebra.*

Example

1. In D_6, we have
   - $b_1 = h_1 = 1$
   - $b_s = h_s + v$
   - $b_r = h_r + v$

   To get b_{sr} lets multiply $b_r b_s = (h_s + v)(h_r + v) = h_{sr} + vh_s + vh_r + v^2$. It is in the desired form. Same happens for b_{rs}

   - $b_{sr} = b_s b_r$
   - $b_{rs} = b_r b_s$

   And we leave to the reader to check the calculation of

   - $b_{srs} = b_s b_r b_s - b_s$

3.3 Interlude: The Graded Setting and Categorification of $\mathbb{Z}[v, v^{-1}]$

Recall that a $\mathbb{N}$-graded ring R is a ring that can be decomposed (as an Abelian group) as $R = \bigoplus_{i\in\mathbb{N}} A_i$, and where $A_i A_j \subset A_{i+j}$. (Moreover, for G a monoid, a G-graded ring is a ring decomposable into $R = \bigoplus_{i\in G} A_i$, where $A_i A_j \subset A_{ij}$)

So every element in R can be written uniquely as a sum $x = a_{i_1} + \cdots + a_{i_n}$, with $a_{i_k} \in A_{i_k}$. The elements $a \in A_n$ are called homogeneous of degree n.

Examples

1) $\mathbb{R}[x]$ is a $\mathbb{N}$-graded ring with $A_i = \{a_i x^i \mid a_i \in \mathbb{R}\}$.

2) A really important example is the ring of polynomials $R[x_1, \ldots, x_n]$ over an integral domain R, here A_i are the polynomials of degree i.

Graded rings form a category called **GRing**, with morphisms being ring homomorphisms $f : R \to S$ such that $f(R_i) \subseteq S_i$.

Note that a ring A_0 is a ring on its own. Conversely, any ring R can be given graded structure taking $A_0 = R$ and $A_i = 0$ for $i \neq 0$ (this is called the trivial grading.) In some sense, "graded rings" extend the concept of a ring.

In this document, when we say *graded* we are going to mean $(\mathbb{Z}, +)$-graded.

Graded rings can easily be generalized to graded modules. A *graded module* M over a graded ring R is an R-module that can be decomposed into submodules $M = \bigoplus_{i\in\mathbb{Z}} M_i$, such that $R_i M_j \subset M_{i+j}$. They form the category **R-gMod**, with morphism being linear maps $f : M \to N$ such that $f(M_i) \subseteq N_i$.

Examples

1) If $M = R$ is regarded as module over itself, we recover the definition of a graded ring.

2) A common choice is to take R with the trivial grading. This is kind of boring, because we are not taking into account the group structure of $\mathbb{Z}$ (a grading over any countable set would give us equivalent definitions.) But since we still have that $f(M_i) \subseteq N_i$, this gives us a nice way to stick modules, different than $\oplus$ and $\otimes$.

3) For R integral domain with the trivial degree, the Laurent polynomials $R[x, x^{-1}]$ form a graded R-module. Note that we can multiply polynomials here... this is in fact an example of

4) A graded R-algebra A is a graded ring over a (graded) ring. These are really common, but we are not getting into that here.

5) We can graduate the symmetric algebra of a vector space V (over a field $\mathbb{K}$). Recall that the symmetric algebra

$$S(V) = T(V)/\mathcal{I}([V,V]) = \bigoplus_{n\in\mathbb{N}} V^{\otimes n} / \langle \{v \otimes w - w \otimes v \mid v, w \in V\} \rangle$$

is the free commutative (unital and associative) algebra over V, with morphisms $f : V \to A$ to an algebra being lifted by $\bar{f}(v \otimes w) = f(v) \cdot f(w)$. We graduate it by taking $S(V) = \bigoplus\limits_{n\in\mathbb{N}} S^n(V)$, where

$$S^n(V) = \{\lambda v_1 \otimes \cdots \otimes v_n \mid \lambda \in \mathbb{K},\, v_1, \dots, v_n \in V\}$$

R-gMod is in fact a monoidal additive category, defining

- The sum on **R-gMod** by components $(M \oplus N)_i = M_i \oplus N_i$.
- The tensor product on **R-gMod** by $(M \otimes N)_n = \bigoplus\limits_{i+j=n} M_i \otimes N_j$.

The proof of these follow from the additivity and monoidalness of R-mod. Also, it is easy to see from the construction that

$$M \otimes (N_1 \oplus N_2) \simeq (M \otimes N_1) \oplus (M \otimes N_2) \tag{1}$$

This will give structure of a ring to the split Grothendieck group. Let us define one more important construction.

Definition (Shift Construction)**.** *Let R be a graded ring. For $i \in \mathbb{Z}$, a shift in i is an endofunctor*

$$Sh_i : \boldsymbol{R\text{-}gMod} \to \boldsymbol{R\text{-}gMod}$$

With $Sh_i(M)_j = M_{j+i}$ (that is, we moved all the components i spaces to the left)

Observe that since $\oplus$ is defined componentwise, we get that

$$Sh_i(M \oplus N) \simeq Sh_i(M) \oplus Sh_i(N), \tag{2}$$

so Sh_i will induce an endomorphism in the split Grothendieck group.

Also, we have that $(Sh_i(M \otimes N))_n = \bigoplus\limits_{j+k=n+i} M_k \otimes N_j = \bigoplus\limits_{j+k=n} M_{k+i} \otimes N_j = (Sh_i(M) \otimes N)_n$, and therefore

$$Sh_i(M \otimes N) = Sh_i(M) \otimes N \tag{3}$$

Now we give a categorification of $\mathbb{Z}[v, v^{-1}]$. Consider the category $\mathbf{GVect}_o$ of graded vector spaces $V = \bigoplus\limits_{n \in \mathbb{Z}} V_n$ (so $R = \mathbb{K}$ a field with the trivial degree) such that all the components are finite dimensional and finitely many V_n's are nonzero.

The *graded dimension* of $V = \bigoplus\limits_{n \in \mathbb{Z}} V_n \in \mathbf{GVect}_o$ is defined to be the element of $\mathbb{N}[v, v^{-1}]$

$$gdim(V) = \sum_{n \in \mathbb{Z}} dim(V_n) v^n$$

It is not hard to see that $V \simeq_{\mathbf{GVect}_o} W \Leftrightarrow gdim(V) = gdim(W)$, moreover, $gdim$ sets a bijection $S(\mathbf{GVect}_o) \simeq \mathbb{N}[v, v^{-1}]$.

From (2) follows that the split Grothendieck group $K^{\oplus}(\mathbf{GVect}_o)$ is a ring.

Moreover, it is easy to see that $gdim(V \oplus W) = gdim(V) + gdim(W)$ and $gdim(V \otimes W) = gdim(V) \cdot gdim(W)$. So the function induced in the Grothendieck groups

$$gdim : K^{\oplus}(\mathbf{GVect}_o), \oplus, \otimes) \xrightarrow{\simeq} (\mathbb{Z}[v, v^{-1}], +, \cdot)$$

sets a categorification.

Note that for $n \in \mathbb{Z}$, the action of v^n

$$\begin{aligned} \cdot_{v_n} : \quad \mathbb{Z}[v, v^{-1}] &\to \mathbb{Z}[v, v^{-1}] \\ p(v) &\mapsto v^n p(v) \end{aligned}$$

Is categorified with the shift functor $Sh_n : \mathbf{GVect}_o \to \mathbf{GVect}_o$, that is, for all $n \in \mathbb{N}$, have the commutative diagram

$$\begin{tikzcd}
K^{\oplus}(\mathbf{GVect}_o) \arrow[r, "Sh_n"] \arrow[d, "gdim"'] & K^{\oplus}(\mathbf{GVect}_o) \arrow[d, "gdim"] \\
\mathbb{Z}[v, v^{-1}] \arrow[r, "\cdot_{v^n}"] & \mathbb{Z}[v, v^{-1}]
\end{tikzcd}$$

(Where the morphisms Sh_n are the induced on the Grothendieck group.)

Why did we do all this interlude? Well, first of all because it is a nice example of categorification, but mainly because now we are going to categorify a $\mathbb{Z}[v, v^{-1}]$-algebra!

Let R be a graded ring and **C** be a subcategory of **R-gMod** closed under Sh_n, $\oplus$ and $\otimes$. We get that

Proposition 3.0.4. *The split Grothendieck group $K^{\oplus}(\boldsymbol{C})$ is a $\mathbb{Z}[v, v^{-1}]$-algebra, with the action given by*

$$\left(\sum_{i=-n}^{n} a_i v^i\right) \cdot [M] = \sum_{i=-n}^{n} a_i [Sh_i(M)]$$

Proof. The proof follows straightforward from what we have. $K^{\oplus}(\mathbf{C})$ is a ring because of (1). The module axioms come from (2) and the fact that $Sh_i(Sh_j(M)) = Sh_{i+j}(M)$. The algebra axiom comes from (3). ☐

We are now ready to continue.

3.4 Soergel Bimodules

Soergel bimodules give a categorification of a Hecke algebra. Since the objective of this section is to present an example we are not going to cover the proofs of the theorems involved, but we refer the reader who wants to go deep into this subject to [EW13].

Through this section, let (W, S) be a Coxeter system with $|S| < \infty$. We start defining a particular representation of W to work with, the *geometric representation*. Although actually Soergel's theory works for a wider bunch of representations, called reflexion faithful representations.

The geometric representation

Consider $\mathfrak{h}$ the vector space over $\mathbb{R}$ with basis $\{\hat{\alpha}_s\}_{s\in S}$

$$\mathfrak{h} = \bigoplus_{s\in S} \mathbb{R}\hat{\alpha}_s$$

We define $\{\alpha_s\}_{s\in S} \subset \mathfrak{h}^*$ by $\alpha_s(\hat{\alpha}_s) = -2\cos(\frac{\pi}{m_{s,r}})$ (by convention $\frac{\pi}{\infty} = 0$)

Now, W acts on $\mathfrak{h}$ by extending the following action of $s \in S$

$$s \cdot v = v - \alpha_s(v)\hat{\alpha}_s,$$

giving us the geometric representation.

Let R be the symmetric algebra on $\mathfrak{h}^*$

$$S(\mathfrak{h}^*) = \bigoplus_{m\geq 0} S^m(\mathfrak{h}^*),$$

which we think of as polynomial functions on $\mathfrak{h}$.

Consider the $\mathbb{Z}$-graduation of R given by $R = \bigoplus_{i\in\mathbb{Z}} R_i$, with $R_{2i+1} = 0$ and $R_{2i} = S^i(\mathfrak{h}^*)$ for $i \in \mathbb{N}$ and $R_i = 0$ if $i < 0$

So R is a graded $\mathbb{R}$-algebra with $deg(\mathfrak{h}^*) = 2$. The group W acts on $\mathfrak{h}^*$ via $s \cdot \gamma = \gamma - \gamma(\hat{\alpha}_s)\alpha_s$. We are going to consider the extension of this action to R. Call R^s the subring of R fixed by the action of s. Our framework is the category **R-gBim** of graded R-bimodules. To follow common notation, we will call (n) the grading shift, so $M(n) = Sh_n(M)$. Also, we will call $\otimes$ the tensor product over R and $\otimes_s$ the tensor product over R^s.

We are going to work with some particular bimodules, called Bott-Samelson bimodules. For $s \in S$, call B_s the R-bimodule $R \otimes_s R(1)$.

The idea is that B_s will categorify the element b_s of the Hecke algebra. In order to construct B_w (to categorify b_w) we need to do some work. First, for an expression $\underline{w} = (s_1, \ldots, s_n)$ $(s_i \in S)$, define

$$B_{\underline{w}} = B_{s_1} \otimes \cdots \otimes B_{s_n}$$

These are called *Bott-Samelson bimodules.* Note that $B_{\underline{w}} \simeq R \otimes_{s_1} \cdots \otimes_{s_n} R\,(n)$. We call $\mathbb{BS}$Bim the full subcategory of **R-gBim** with objects as grading shifts of finite direct sums of Bott-Samelson bimodules. We need to refine this category a little bit.

Let $\mathbb{S}$Bim be the Karoubi envelope of $\mathbb{BS}$Bim. This is, the indecomposable objects of $\mathbb{S}$Bim are the indecomposable direct summands of shifts of Bott-Samelson bimodules.

Definition. *The objects of* $\mathbb{S}Bim$ *are called* Soergel Bimodules*.*

The following proposition is outlined in [EW13] section 3.4.

Proposition 3.0.5. *For all $w \in W$ there exist a unique (up to isomorphism) indecomposable R-bimodule B_w which appears as a direct summand of $B_{\underline{w}}$, for any reduced expression $\underline{w}$ of w. B_w does not appear in the summand decomposition of $B_{\underline{y}}$ for any $y < w$ ($\underline{y}$ reduced expression of y.)*

$\{B_w \mid w \in W\}$ constitutes a complete set of non-isomorphic indecomposable Soergel bimodules. Suggestive... could $\mathbb{S}$Bim satisfy the Krull-Smith property (recall the examples in section 1.4) with $\{B_w \mid w \in W\}$? Well, the name B_w and what we said early suggest that the answer is positive. But we have more, we are ready to state the categorification theorem. Consider the Grothendieck group $K^{\oplus}(\mathbb{S}Bim)$ as a $\mathbb{Z}[v, v^{-1}]$-algebra like in the last section (the results of last section apply to bimodules)

Theorem 3.1 (Soergel's categorification theorem)**.** *There is a unique isomorphism of* $\mathbb{Z}[v, v^{-1}]$*-algebras*

$$\begin{aligned} \epsilon : \quad H(W,S) &\xrightarrow{\simeq} K^{\oplus}(\mathbb{S}Bim) \\ b_w &\mapsto [B_w] \end{aligned} .$$

What about morphisms? One of the great advantages of categorifying some object is to make morphisms appear, so it would be really helpful if we can treat them with ease.

The reader might think that with such a bulky construction, morphisms in Soergel bimodules should be really complicated to treat with.

This is not the case. There is actually a nice combinatorial-like structure, called the light leaves, that help us treat pretty easily the morphisms between Soergel bimodules. A good exposition on this can be found in [Lib17].

Also, in [Lib17] section 4, Libedinsky shows the explicit isomorphisms, when taking $W = D_6$, for the following

- $B_{sr} \simeq B_s \otimes B_r$
- $B_{rs} \simeq B_r \otimes B_s$
- $B_{srs} \oplus B_s \simeq B_s \otimes B_r \otimes B_s$

These categorify the equations that we had for b_x, $x \in W$, in the Coxeter group D_6. Observe that for the third equation we had to consider $b_{srs} + b_s = b_s b_r b_s$ instead of $b_{srs} = b_s b_r b_s - b_s$ to be categorified. In general is difficult to work with inverses, and in an additive category it can not be nicely done. This is where the Grothendieck comes to aid.

Having this and other examples presented, we now enter into the treatment of categorification.

4 Categorification

The concept of categorification is not universally defined (that is, a definition that works in all contexts in which the word is applied, it is mathematically rigorous and everybody agrees with it.) In part because it is pretty new, and in part because it applies to many different contexts. We are going to give the idea of some categorifications without defining well what categorification is, and then we are going to give a precise specific definition that applies to a good number of situations, which we can call algebraic categorification (since it is well suited for algebras.)

Categorification is better defined with the inverse process, decategorification, which is a invariant of the isomorphism classes on a category, giving an bijection into the set in which the category is decategorified. An example is the invariant dim, the dimension of a vector space, or the Euler characteristic χ (we will soon get deeper into this one) of compact topological spaces.

Loosely speaking, a categorification of an object A is a category $\mathbf{C}$, and a decategorifying *morphism*

$$\phi : S(\mathbf{C}) \to A$$

As we have seen, a categorification is not unique. For example, for $(\mathbb{N}, +, \cdot)$ we gave two categorifications: $(\mathbf{FinSet}, \sqcup, \times)$ and $(\mathbf{Vect}, \oplus, \otimes)$.

Here, defining what a *morphism* is can get really tricky. It depends on the structure of A. For example, if A is an Abelian group, we wish that $\mathbf{C}$ has an Abelian operation like the biproduct $\oplus$ that one has on additive categories. If A is a ring, we also want a compatible product structure, like $\otimes$ for example. If A is an algebra, we also want the analog of an action of a ring to be defined on $\mathbf{C}$, this will be done by a functor.

There are many objects in mathematics aside from abstract algebra. Although the algebra alone captures a high variety of objects, since for example, given a combinatorial object (a graph, symmetries of some discrete object, knots, etc), we can usually assign an algebraic structure to it: A group of symmetries, an algebra with relations defined according to the combinatorial object, invariant polynomials, etc.

Still, can you imagine what would we need in a category that categorifies a topological space? Or a dynamical system? Or a (nonalgebraic) Riemman surface? Or a partial differential equation? We leave to the reader's imagination what would be needed to categorify his/her favorite mathematical objects.

So categorification replaces sets with categories, equalities with isomorphisms, functions with functors, etc. You can say it is going one step in the categorical abstraction (like we did when going from categories to 2-categories.)

A good example on this is the one we just saw: Soergel bimodules categorify Hecke Algebras. There, equations like $b_{srs} + b_s = b_s b_r b_s$ in the hecke algebra get lifted to isomorphisms $B_{srs} \oplus B_s \simeq B_s \otimes B_r \otimes B_s$ in the Soergel bimodules.

We already have many other examples of categorification.

- $(\mathbb{N}, +, \cdot)$ is categorified by $(\mathbf{FinSet}, \sqcup, \times)$
- $(\mathbb{N}, +, \cdot)$ is also categorified by $(\mathbf{Vect}, \oplus, \otimes)$

As said (and soon will be shown), it gets really challenging categorifying additive inverses, so we rather identify the Grothendieck group with the decategorified object, thus

- $K(\mathbf{Vect}, \oplus, \otimes) \simeq K(\mathbf{FinSet}, \sqcup, \times) \simeq (\mathbb{Z}, +, \cdot)$
- $K(\mathbf{Ab}_{<\infty}, \oplus) \simeq \mathbb{Z}[x]$
- $K^{\oplus}(\mathbf{GVect}_o, \oplus, \otimes) \simeq (\mathbb{Z}[v, v^{-1}], +, \cdot)$

Among the others examples we saw in chapter 1, also

- A monoidal category categorifies a monoid. In particular, a free monoidal category over a category with n objects is the categorification of a free monoid over a set with n elements.
- The categories enriched over $(\mathbb{R}_{\geq 0}, +)$ categorify generalized metric spaces.
- The K-theory of a topological space is the decategorification of the vector bundles over it, as well as the K-theory of a ring is the decategorification of the projective modules over it.
- In the same fashion, the burnside ring is the decategorification of the representations of a group.

(Although the last two are not that interesting since we want to go in the opposite direction, they reflect nice aspects of the relation between the object and it is categorification. In particular it makes evident the loss of information, and the fact that there is a deep relation within the decategorified and categorified objects.)

Karate was born in the Okinawa islands, derived from chinese martial arts with the development of master Kanryo Higaonna. Great masters of this martial art all come from Okinawa. What is their secret? They go through many hours of practicing, and the practice is specially focus on *the basics*. Basics are the key in many disciplines, and being as pyramidal as it is, Mathematics are not an exception. A nice feature about categorification is that it allows you to rethink basic concepts in mathematics, things that we take for granted because we think we know them while in fact knowledge is most likely infinite. Let us take a little survey through some basic concepts while at the same time viewing some important facts and simple illustrating examples of categorification.

4.1 Equality

Equality is one of the most basic concepts of mathematics and a hard philosophical issue (are two things ever equal? Was Plato doing equivalence classes when he talked about the theory of forms?). We are not going to get (much) philosophical but we are still discussing equality since it is in the heart of categorification.

First of all, it is not as easy as it seems to tell when two objects are the same, take for instance this two figures

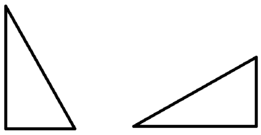


Figure 7: A triangle and its rotation.

Would you say these triangles are equal? Sure they have the same measures and angles, but they are faced differently. Even if we face them in the same direction

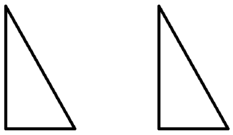


Figure 8: A triangle and its translation.

One can argue that they are still in different positions.

The only true equality in this world is the trivial one

$$x = x$$

But this one is absolutely boring and useless. Interesting equations are those which relate sides that are not obviously related. To regard two (different) objects as the same, there is a mathematical well known trick: take quotient by an equivalence relation. We carefully say then that two objects are equal up to equivalence relation. So for example triangles in figure 7 are equal up to rotation and translation (this is called congruence), and triangles in figure 8 are equal up to translation. (By the way, figures in $\mathbb{R}^2$ which are invariant under rotation and translation form euclidian geometry.) We eventually stop being careful and say that the objects are equal.

Recall that an equivalence relation is

i) Reflexive: $a \sim a$

ii) Transitive: $a \sim b$ and $b \sim c \Rightarrow a \sim c$

iii) Symmetric: $a \sim b \Rightarrow b \sim a$

Compare these with the axioms to define a category. We can pair up i) with the existence of identity morphism 1_a, and ii) with the composition of morphisms. To have a parallel for iii), we need a morphism coming back from b to a. In this loose comparison any morphism would work, but this is too grotesque. Sensitizing this we get to a isomorphism, ie, we want that when we go back we could say that nothing has past. So, this comparison gets us to a groupoid.

What is for real, the mathematical community already made a process like categorification, when they replaced true values of equations by sets, acknowledging that there may be many ways in which something might be true, and in particular, that there may be many ways in which things can be the same.

This is really important. We usually see an equality as a yes-no question, but there can actually be many ways in which an equation is true.

Let us explain what we mean with an example: Take the Klein Group $Kl = \{1, a, b, c\}$. We easily agree that equality for groups should be the isomorphism classes. Now, the Klein group is equal to itself in the obvious way $(1, a, b, c) = (1, a, b, c)$ (ie, by identity), but also, since a and b have the same role, interchanging a and b would get us to the *same* group $(1, a, b, c) = (1, b, a, c)$.

So now, when we say that the Klein group is equal to itself (in the sense of isomorphism classes), we have 6 ways in which this is true. So it is legitimate to ask *how* the Klein group is equal to itself. We get here to the concept of symmetry, which has the name of automorphism group $Aut(X)$ in category theory.

In this *how* is where categorification makes information appear. Take for example the fundamental groupoid. In the decategorified version we just have the path connected components, this is whether 2 points are connected, a yes-no question. But in the groupoid we have the information of *how* we get from a point to another. Moreover, it enhances the path-connected components with a groupoid structure. This is a lot of extra information.

So we have equations like $V \otimes (W \otimes Z) \simeq (V \otimes W) \otimes Z$, which can be equal in several ways. Now, in the case of the associativity of the tensor product, the isomorphism is *canonical.* That is, there is a natural isomorphism underling. This gives us an specific way to go from $V \otimes (W \otimes Z)$ to $(V \otimes W) \otimes Z$, and puts our minds to rest. Another example of canonical appearance is in the disjoint union. We have many different ways to construct it, but all are canonically isomorphic, with the isomorphism given by the universal property. With this in hand, we have a perfect dictionary between any two constructions, and so we can talk peacefully about *the* disjoint union of two sets.

Now, the Klein group is also equal to different sets satisfying the same group relations, for example it can be regarded as $\mathbb{Z}_2 \times \mathbb{Z}_2$, so $(1, a, b, c) = ((0, 0), (1, 0), (0, 1), (1, 1))$. We have that a more accurate structure to enclose the equality type of the Klein group is the groupoid consisting of all of the groups isomorphic to Kl.

So we made a step from a set with equalities to a groupoid by considering isomorphisms. Is there an structure to which we can apply the same process and produce a general category instead of a groupoid? Questions like that are called negative thinking (name coming from the concept of n-categories.) The answer to this particular question is yes. The structure to consider is a Poset. We are not going into this. We are moving on.

4.2 Numbers

The reader might be by now tired of the examples $|\cdot| : (S(\mathbf{FinSet}), \sqcup, \times) \overset{\cong}{\to} (\mathbb{N}, +, \cdot)$, and the example $dim : (S(\mathbf{Vect}), \oplus, \otimes) \overset{\cong}{\to} (\mathbb{N}, +, \cdot)$ as categorifications of the natural numbers.

To categorify $\mathbb{Z}$, we used the trick of passing through the Grothendieck group, which is nice, but leaves a loose end. We loose modeling. Let us clear this with an example:

Say we categorify $(\mathbb{N}, +, \cdot)$ with $(\mathbf{FinSet}, \sqcup, \times)$. We are *modeling* numbers with sets. Now, when we take the Grothendieck group and say that **FinSet** categorifies $\mathbb{Z}$ we are loosing the feature of *modeling*, because, for example, which set is modeling -1? We need to enlarge the category.

Of course we can always enlarge **FinSet** adding one object per set, with only identity morphisms, and say that those added objects model negative numbers. This is really unsatisfying.

A more satisfying enlargement would be again adding one object $-A$ for each set A and add morphisms $Hom(-A, -B) = \{-f \ : \ f \in Hom(A, B)\}$ composed as $(-f) \circ (-g) = -(f \circ g)$. Morphisms in the negative part do not interact with the positive part, i.e. $Hom(-A, B) = Hom(A, -B) = \emptyset$. We define the cardinality as $|-A| = -|A|$.

This is much better, but it still has problems: first of all we did not really know how this negative sets $-A$ *are*. For instance, are them composed by elements?

Moreover, as seen in 1.2, in this category $\sqcup$ is not a coproduct.

Analog problems occur when considering **Vect**. We ask the question:

Would there be a non artificial category with nice structure (like an Abelian or additive category) which categorifies the negative numbers $\mathbb{Z}_{<0}$? Or in other words, can we model subtraction with a nice and known category?

We can.

4.2.1 Euler Characteristic, Homology and Categorification of $\mathbb{Z}$

The Euler characteristic χ was first defined for polyhedra P as the alternating sum

$$\chi(P) = \sharp\{\text{Vertices of } P\} - \sharp\{\text{Edges of } P\} + \sharp\{\text{Faces of } P\}$$

And can be used to classify the platonic solids (among other theorems). It is an invariant of the homotopy type of a polyhedra. For example, all the platonic solids, and all convex polyhedra for that matter (but not just these), have the same Euler characteristic as the sphere $\chi(S^2) = 2$.

The sphere? But... wasn't it defined for polyhedra? Well, the Euler characteristic can be easily generalized to CW-Complexes X as the alternating sum

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i k_i(X)$$

Where $k_i(X)$ is the number of cells of dimension i of the complex. So

$$\chi : \text{CW-Complexes} \to \mathbb{Z}$$

Is a surjection onto $\mathbb{Z}$. (In order to obtain $-n$ ($n \geq 0$) just take two points joined with $n + 2$ laces.)

Euler characteristic χ is well defined, that is, it does not depend on the choice of the simplex structure. It is a topological invariant, $X \simeq Y \Rightarrow \chi(X) = \chi(Y)$. Moreover, it is an homotopy invariant: if X and Y have the same homotopy type, then $\chi(X) = \chi(Y)$

Now, there is a very suggesting property of the Euler characteristic

$$\chi(X \sqcup Y) = \chi(X) + \chi(Y) \text{ and } \chi(X \times Y) = \chi(X) \cdot \chi(Y)$$

It is a morphism of some kind!

Can the Euler invariant CW-complexes of dimension ≤ 2 be a categorification of $\mathbb{Z}$ with χ as decategoring function? Can it succeed where **Set** and **Vect** failed?

As a way to represent numbers it makes some sense, if you are a homotopical thinker, since for example n points have Euler characteristic n

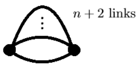
$\chi = 3$

And when we link points we make a homotopic components

$\chi = 2$

$\chi = 1$

We leave the exercise of making sense to



$\chi = -n$

To the imagination of the reader.

This could be used as a graphical representation of $\mathbb{Z}$, but as a categorification it has limitations. Remember that the power of categorification is to take morphisms into account. The natural category to use here would be a subcategory of **Top**, but how does the Euler characteristic act on a continuous functions? It is not clear at all.

Fortunately, there is a categorification of the Euler characteristic that fixes this problem, and greatly enhances the amount of spaces in which we can define χ. This is done considering the boundaries of the simplices. We are going to give some details about this, without engaging into proofs and examples. For a nice complete exposition on this see [Hat].

Proposition 4.0.1 (Details of This)**.** *Euler characteristic can be categorified*

Proof. We denote by $[v_0, \ldots, v_n]$ the simplex with vertices $v_0, \ldots, v_n$. A face of $[v_0, \ldots, v_n]$ is denoted $[v_0, \ldots, \hat{v_i}, \ldots, v_n] = [v_0, \ldots, v_{i-1}, v_{i-1}, \ldots, v_n]$ (we consider orientation, for instance $[v_0, v_1]$ has opposite orientation that $[v_1, v_0]$.)

The standard n-simplex is denoted $\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1}_{\geq 0} : \sum\limits_{i=0}^{n} t_i = 1\}$

To construct homology of a CW-complex X, take the free R-mod (for R a commutative ring, common choices are $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. Keep in mind $R = \mathbb{Q}$ for this document) over the n dimensional simplices of X, $\Delta_n(X) = \langle\{[v_0, \dots, v_n]\}\rangle$

The boundary operator $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ is defined in the basis as the alternating sum of faces $\partial_n([v_0, \dots, v_n]) = \sum\limits_{i=0}^{n}(-1)^i[v_0, \dots, \hat{v_i}, \dots, v_n]$. We abuse notation saying $\partial = \partial_n$.

Since $\partial^2 = 0$, $Im(\partial) \subset Ker(\partial)$, and we define the *simplicial homology group* (actually it is a R-module) as

$$H_n(X) = Ker(\partial_n)/Im(\partial_{n+1})$$

This can be generalized. For a topological space X, consider the free R-mod $C_n(X)$ with basis $\{\sigma : \Delta^n \to X : \sigma \text{ is continuous }\}$. Now the boundary is defined in the basis as $\partial_n(\sigma) = \sum\limits_{i=0}^{n}(-1)^i\sigma|_{[v_0,\dots,\hat{v_i},\dots,v_n]}$. Again $\partial^2 = 0$, so we define the *singular homology groups*

$$H_n(X) = Ker(\partial_n)/Im(\partial_{n+1})$$

We have that H_n is a covariant functor, for $f : X \to Y$ continuous we define $f' : C_n(X) \to C_n(Y)$ on the basis as $f'(\sigma) = f \circ \sigma$. Since $f'\partial = \partial f'$, we can define $f_* : H_n(X) \to H_n(Y)$ as $f_*([a]) = [f'(a)]$.

We define the *total homology group* as

$$H(X) = \bigoplus_{n \geq 0} H_n(X)$$

This is a graded vector space. We make H into a functor by adding the morphisms f_*. H will categorify χ □

For R field, we take $\chi(X)$ to be

$$\chi(X) = \sum_{i \geq 0}(-1)^i dim(H_n(X))$$

For those spaces X in which the sum converges. We leave as an exercise to check that for a CW-complex $dim(H_n(X))$ is the number of n-simplices.

H is a categorification since it lifts the function

$$\chi : \text{CW-Complexes} \to \mathbb{Z}$$

to a functor

$$H : \text{CW-Complexes} \to \mathbf{Ab}.$$

Moreover, H can be defined for any topological space.

$$H : \mathbf{Top} \to \mathbf{Ab}.$$

Now H has multiple advantages over χ. First of all, it is a collection of groups (or R-modules), not just a number. More importantly, it is a functor, it has a specific action on continuous functions $f : X \to Y$ that behaves nicely. The reader familiar with algebraic topology probably knows the big amount of information that the morphisms $f_* : H_n(X) \to H_n(Y)$ carry (they lead to exact sequences theorems that relate (sub)spaces.)

Also, note that we can define H for any commutative ring, and easily consider any topological spaces X, not just those nice spaces in which the sum converges, we "can freely work with infinite". Also, if we consider the dual of the homology groups, called *cohomology groups*, we can enhance them with a natural ring structure. Again, the reader might agree that this is a huge amount of extra information.

Moreover, Homology and Cohomology can be generalized to (co)homology theories, in contexts out of topological spaces. We just need to be able to construct complexes, that is, a chain $\ldots \overset{\partial}{\to} A_i \overset{\partial}{\to} A_{i-1} \overset{\partial}{\to} \ldots$ of morphisms of R-modules with $\partial^2 = 0$, we can again set $H_n(X) = Ker(\partial_n)/Im(\partial_{n+1})$ and $H = \bigoplus_{n \geq 0} H_n$ and make a functor. This is huge.

Such can be the power of categorification.

And what about $\mathbb{Z}$? at some point we left it behind.

Remember our talk about vector spaces of negative dimension? Well, there is a nice setting for the categorification of $\mathbb{Z}$ using "vector spaces", with a construction that categorifies the subtraction. We have to enlarge the category to the category $\mathbf{C}$ whose objects are complexes of vector spaces $V = \ldots \overset{\partial}{\to} V_i \overset{\partial}{\to} V_{i-1} \overset{\partial}{\to} \ldots$ such that the total homology group has finite dimension (so χ of an object in this category is well defined.) The morphisms are homomorphisms of complexes ($f\partial = \partial f$), mod out by chain-homotopic morphisms. A chain homotopy between f and g is a set of morphisms P of the form

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & V_{i-1} \arrow[r, "\partial"] \arrow[dl, "P"] & V_i \arrow[r, "\partial"] \arrow[dl, "P"] & V_{i+1} \arrow[r, "\partial"] \arrow[dl, "P"] & \cdots \\
\cdots \arrow[r, "\partial"] & W_{i-1} \arrow[r, "\partial"] & W_i \arrow[r, "\partial"] & W_{i+1} \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

such that $P\partial + \partial P = f - g$. Its existence implies that $f_* = g_*$ on H.

This forms a *Triangulated Category.* One can consider its Grothendieck group $K(\mathbf{C})$ (see [LM13] section 2.6). With the operation induced by tensor product of complexes, $K(\mathbf{C})$ becomes a ring. Here the Euler characteristic induces an isomorphism

$$\chi : K(\mathbf{C}) \xrightarrow{\simeq} \mathbb{Z}$$

The subtraction can be considered as cone map complexes: Given a morphism of complexes $f : V \to W$, we can form the complex $Cone(f)$ with components $V_{n-1} \oplus W_n$ and differentials $-\partial_V + \partial_W + f$. We have that

$$\chi(Cone(f)) = \chi(W) - \chi(V).$$

So we have the table

Structure	Elements and Operations	Categorification
$\mathbb{Z}_{\geq 0}$	$n, m \in \mathbb{Z}_{\geq 0}$ $n + m$ $n \cdot m$	Vector Spaces V, W Direct sum $V \oplus W$ Tensor product $V \otimes W$
$\mathbb{Z}$	$n, m \in \mathbb{Z}$ $n - m$	Complexes of vector spaces V, W Cone of a map $f : W \to V$
$\mathbb{Q}$	$\frac{n}{m}$	Open problem

Now to model a negative number $-n$ we can take $Cone(f : 0 \to V)$, with $\chi(V) = n$.

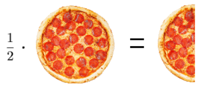
And our mind can rest in peace (at least for now).

The following step would be to put $\mathbb{Q}$ in this context, categorifying division. This is an open problem. Although we can say something about $\mathbb{Q}_{\geq 0}$, changing the environment from vector spaces to... sets!

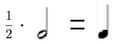
4.2.2 About $\mathbb{Q}_{\geq 0}$ and a little bit about $\mathbb{R}_{\geq 0}$

$\mathbb{Q}$ can be constructed in the usual way by localizing $\mathbb{Z}$. Now, this is a really abstract construction, not suitable for a (normal) child or a super concrete person. Actually, we can go further in concreteness considering that $\mathbb{Q}_{\geq 0}$ can be seen in the real world by the common monoid action of cutting something into *equal* pieces. For example, it is action on segments

Or the action on a pizza.



Or the action on a musical note.



Here, the natural numbers n have the action of repeating the *same* object n times, while 0 has the action of disappearing the object.

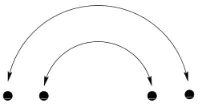
We can extend the action to negative numbers, -1 can be seen as making a phantom double of the object, which collapses into nothing when put together with the object. This action can be seen naturally for example when acting on money (as debt).

Real numbers in this manner are more esoteric and deal with infinity. They naturally appear in geometry as lengths and can extend the notion of proportion (there is actually a construction that makes this specific, we are not getting into it, but the interested reader can see [Art04])

Now, we agree that $\frac{1}{2}$ cuts into *equal* parts. But, for example, one can argue that when we cut the pizza in half, the slices of pizza are not equal. Let us relax equality to the equivalence relation of isomorphism. Recall that a category in which every morphism is isomorphism is called a groupoid.

We are going to work with groupoids.

Let us start this discussion by noticing that the action of $\frac{1}{n}$ on $m \cdot n$ points (a set S) can be seen as an action of an n-element group like $G = \mathbb{Z}_n$



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We call S/G the orbits of the action. In this case, we have the formula for orbits $|S/G| = \frac{|S|}{|G|}$, which is a division, and it can be seen in the picture if we fold S in half.

Now, this has the problem that the formula $|S/G| = \frac{|S|}{|G|}$ only works for free actions. Also, for example in figure 9 the point in the middle should count as half of a point, but how do we regard it as half?

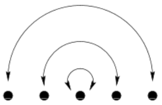

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Figure 9: Chopping a point into two.

This question can be answered with some child-like (innocent) imagination. Suppose that figure 9 is a picture of a category (add one identity per point). Now, the middle point has two automorphisms, that is, two ways of looking at itself and be regarded as *the same*: the identity, and the isomorphism drawn (observe that we are counting the inverse and the isomorphism as one, since a "way of looking at itself" is really the pair (f, f^{-1})). Since there are 2 ways of looking at itself, we can say that the point is cut in half (it is like when you talk to yourself, who are you talking with? are you two persons being the same?), in the sense of ways of being the same. (So, in our example of last section of the Klein group Kl, $(1, a, b, c)$ would be 1/6 of Kl)

Now, to get from where we were (groups acting on sets) to the category, we consider the groupoid $S//G$, called the weak quotient. The objects in this category are the elements of S, and we say that there is an arrow $g : s \to s'$ iff $g \cdot s = s'$.

We define the cardinality of the groupoid $S//G$ to be

$$|S//G| = \sum_{x \in S(S//G)} \frac{1}{Aut(x)}$$

Now we get that $|S//G| = \frac{|S|}{|G|}$. So $|\cdot|$ sets a surjection from this groupoids coming from group actions to $\mathbb{Q}_{\geq 0}$. Can these be a categorification for $\mathbb{Q}_{\geq 0}$? Certainly, but it will be good to get out of the group action frame.

Note that we can get out of the frame of group actions and define the cardinality for any category $\mathbf{C}$ with finite skeleton and finite number of automorphisms for each object as

$$|\mathbf{C}| = \sum_{x \in S(\mathbf{C})} \frac{1}{Aut(x)}$$

Now, it is easy to check that for these categories

$$|\mathbf{C} \sqcup \mathbf{D}| = |\mathbf{C}| + |\mathbf{D}| \text{ and } |\mathbf{C} \times \mathbf{D}| = |\mathbf{C}| \cdot |\mathbf{D}|$$

Where $\times$ and $\sqcup$ are the product and coproduct of categories respectively. So this sets a categorification of $\mathbb{Q}_{\geq 0}$. Moreover (although getting out a little from the spirit of this section), we can take the Grothendieck group (as in 1.2) and obtain a categorification for $\mathbb{Q}$.

Note that for a set regarded as category (one object per element and only identity morphisms), the function $|\cdot|$ is just the cardinality of the set. So we are extending the notion of cardinality to take values in $\mathbb{Q}_{\geq 0}$.

But there is more (oh, so much more...). Note also that we can make the definition of $|\cdot|$ whenever the sum makes sense. Categories in which the sum make sense are called *tame* categories. Now, the thought that non invertible morphisms are not considered in the formula of $|\cdot|$ might cross your head. In fact, we could be all the time talking just about groupoids, but it is nice to consider other categories. For example **FinSet** is a tame category. Its cardinality is

$$|\mathbf{FinSet}| = \sum_{n \in \mathbb{N}} \frac{1}{n!} = e$$

The Euler number!

So, as we see, we got out of the rational numbers into the real numbers $\mathbb{R}_{\geq 0}$. Now for tame categories we also have

$$|\mathbf{C} \sqcup \mathbf{D}| = |\mathbf{C}| + |\mathbf{D}| \text{ and } |\mathbf{C} \times \mathbf{D}| = |\mathbf{C}| \cdot |\mathbf{D}|$$

So tame categories offer a categorification of $\mathbb{R}_{\geq 0}$.

This is nice. We went one level on the categorical stairway, from sets to categories (recall that any category can be considered as enriched over $(\mathbf{Set}, \times, \{*\})$), and we made sense to numbers beyond natural numbers, extending the usual sense of counting $|\cdot|$

There is actually much more to tell in this realm we are getting into. It actually goes up into the terrain of quantum physics. We are not covering more here, since it is not really about categorification (I think...) but I highly recommend that if you enjoy this section, you read [BD00].

4.3 Categorification of (Representations of) Algebras

We can make a specific definition of categorification in the context of algebras. Specifically, we are going to categorify modules over algebras. As we saw, the horizontal categorification of an $\mathbb{R}$-algebra is an $\mathbb{R}$-linear category. So a nice setting in which to categorify algebras will be on additive or Abelian categories.

We are going to use the Grothendieck group. Although for now this is just an Abelian group (or in some cases a ring), we are going to attach more structure to it in order to get where we want (modules over algebras). We start by extending the scalars.

Throughout this section, let R be a unital commutative ring.

Definition (Categorification of a Module)**.** *Let V be a R-module. An R-categorification of V is a pair $(\boldsymbol{C}, \phi)$, where $\boldsymbol{C}$ is an Abelian category (resp. additive category* [4]*) and*

$$\phi : V \xrightarrow{\simeq} R \otimes_{\mathbb{Z}} K(\boldsymbol{C})$$

Is an isomorphism of R-modules (resp. $\phi : V \xrightarrow{\simeq} R \otimes_{\mathbb{Z}} K^{\oplus}(\boldsymbol{C})$)

We ask ϕ to be an isomorphism of R-algebras when $K(\mathbf{C})$ (resp. $K^{\oplus}(\mathbf{C})$) is a ring.

Examples

1) A trivial R-categorification of R as R-algebra is accomplished by taking $\mathbf{C} = \mathbf{Vect}$.

2) Take the algebra $D = \mathbb{C}[x]/(x^2)$ of *dual numbers*. Let $\mathbf{C} = D$-mod. Take the monomorphism $\phi : \mathbb{Z} \to K(\mathbf{C})$ such that $\phi(1) = [D]$. When we tensor with $\mathbb{Q}$ we induce an isomorphism

$$\bar{\phi} : \mathbb{Q} \xrightarrow{\simeq} \mathbb{Q} \otimes K(\mathbf{C})$$

So as you can see, there can be multiple R-categorifications. In fact they form a category with the following morphisms

Definition. *Let V be an R-module. A morphism between two R-categorifications $(\boldsymbol{C}, \phi)$ and $(\boldsymbol{D}, \psi)$ via Abelian (resp. additive) is an exact (resp. additive) functor $F : \boldsymbol{C} \to \boldsymbol{D}$ such that the following diagram commutes*

$$\begin{tikzcd}
R \otimes_{\mathbb{Z}} K(\boldsymbol{C}) \arrow[rr, "id \otimes {[F]}"] & & R \otimes_{\mathbb{Z}} K(\boldsymbol{D}) \\
& V \arrow[ul, "\phi"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

(resp. with $K^{\oplus}$)

Now we are going to work with modules over algebras. Let B be a unital associative algebra and let M be a B-module. We want to lift the action of B on M. It is enough to lift the action of a generating set. So let us fix $\{b_i\}_{i \in I}$ a set of generators of B, and call b_i^M the endomorphism

$$\begin{aligned} b_i^M : \quad & M \to M \\ & m \mapsto b_i \cdot m \end{aligned}$$

---

[4] With this we can consider categories that are additive but not Abelian, or regard an Abelian category as additive.

Definition (Naive Categorification)**.** *A naive categorification of* $(B, \{b_i\}_{i\in I}, M)$ *is a tuple* $(\mathcal{M}, \phi, \{F_i\}_{i\in I})$ *such that* $(\mathcal{M}, \phi)$ *is an R-categorification of* M*, and* $\forall i \in I$*,* $F_i : \mathcal{M} \to \mathcal{M}$ *is an exact (resp. additive) functor such that the following diagram commutes*

$$\begin{tikzcd}
R \otimes K(\mathcal{M}) \arrow[r, "id \otimes [F_i]"] \arrow[d, "\phi"'] & R \otimes K(\mathcal{M}) \arrow[d, "\phi"] \\
M \arrow[r, "b_i^M"] & M
\end{tikzcd}$$

(resp. $K^{\oplus}$*)*

This is ok for starters, but it does not capture the relations between the b_i's. A better categorification would take the products of the b_i's into account. A naive categorification that takes this into account is called a *weak categorification.*

Since we categorify the action of the b_i's with functors, it is natural that the product would be categorified with composition of functors. Sadly there is not (yet) a universal (accepted by all the authors) way to do this, and it depends on the context. There is a list of things to keep in mind when doing a functorial interpretation of the relations between the b_i's:

- Equalities can be interpreted as isomorphisms of functors.
- Addition in B can be interpreted as direct sum of functors.
- Negative coefficients (we don't want those since we don't have a subtraction of functors) can be made positive by moving terms of side in the equalities (like we did in the example in section 3.4.)
- In cases when the algebra is equipped with an anti involution, and we want to categorify it as well, a the common way to do this is via adjoint functors.

This will be clearer with a couple of examples, but first we give an example of a good definition of a weak categorification for certain modules.

Let A be a ring which is free as an Abelian group and suppose that there is a basis $\{a_i\}_{i\in I}$ of A (as a $\mathbb{Z}$-algebra) such that $a_i a_j = \sum_{k\in I} c_k^{i,j} a_k$ with $c_k^{i,j} \in \mathbb{N}$. Let M be a module over A.

Definition. *A weak Abelian categorification of* $(M, A, \{a_i\}_{i\in I})$ *is a naive categorification* $(\boldsymbol{C}, \phi, \{F_i\}_{i\in I})$ *(considering* A *as an algebra over itself) such that there are natural isomorphisms*

$$F_i \circ F_j \simeq \bigoplus_{k\in I} F_k^{\oplus c_k^{i,j}} \quad \forall\, i, j \in I$$

This definition could be made easily because we assumed a really strong hypothesis: that all the $c_k^{i,j}$ are positive. This will not be always the case, and in fact, the existence of a basis that behaves like that can be a hint of the existence of a nice weak categorification. Let us check out the promised examples.

Examples

1) Consider the $\mathbb{C}$-algebra $B = \mathbb{C}[b]/(b^2 - 2b)$. B has generating set $\{b\}$, and relations for the generators $b^2 = 2b$ (all coeficients made positive). Let $M = \mathbb{C}$ be the B-module with $b \cdot m = 0 \; \forall m \in M$. We categorify M with $\mathcal{M} = \mathbf{Vect}$ (finite dimensional vector spaces) and the action of b with $F = 0$ ($F(V) = 0$ for all V.) We have that $\mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{M}) \simeq \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z} \simeq \mathbb{C}$. We take as categorification function the identity

$$\begin{aligned} \phi : \quad \mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{M}) &\to \mathbb{C} \\ z \otimes [1] &\mapsto z \end{aligned}$$

Where $[1]$ is the class of a vector space of dimension 1. We have that F lifts the action of b since

$$\phi \circ (id \otimes [F])(z \otimes [1]) = 0 = b \cdot \phi(z \otimes [1]).$$

Finally, since $F = 0$ we get that $F \circ F = F \oplus F$, which lifts the relation $b^2 = b$.

2) Let $B = \mathbb{C}[b]/(b^2 - 2b)$ as before, but now consider the B-module $N = \mathbb{C}$ with action given by $b\dot{n} = 2n$. As before we take $\mathcal{M} = \mathbf{Vect}$ and the categorification function $\phi(z \otimes [1]) = z$. Although this time we lift the action of b with the functor $G = id \oplus id$ (so $G(V) = V \oplus V$) as the following calculation shows

$$\phi \circ (id \otimes [G])(z \otimes [1]) = \phi(z \otimes [1 \oplus 1]) = \phi(2z \otimes [1]) = 2z = b \otimes z.$$

Now, since $(G \circ G)(V) = V \oplus V \oplus V \oplus V = (G \oplus G)(V)$ (and same for linear maps) we have that $G \circ G = G \oplus G$, which lifts the relation $b^2 = 2b$.

Note that this two weak categorifications have a really similar recipe. We took sums of identity to lift the action of b. There is a setting that envelopes these two cases, and generalizes the categorification of algebras. The stone in the shoe of weak categorification is that there are too many restrictions in the work with functors. We wish for example that we can subtract them. This calls for a Grothendieck group. And since we are working with functors, it is natural to think that to get where we want to go we have to take one step up in the categorical stairway and work with 2-categories.

This new enhanced categorification will be called *strong categorification*, and it will categorify R-linear categories. Recall that R-linear categories are the oidification of unital associative R-algebras, so to apply strong categorification to an algebra, just consider one object R-linear categories in the definition of strong categorification.

Definition. *For R a commutative ring, a 2-category $\mathcal{C}$ is called*

i) Additive if it is enriched over the category of additive categories.

ii) Abelian if it is enriched over the category of Abelian categories.

iii) R-linear if it is enriched over the category of R-linear categories.

We can take the Grothendieck group of such categories

Definition (Grothendieck Group)**.**

1. *The split Grothendieck group of an additive 2-category $\mathcal{C}$ is the category $K^{\oplus}(\mathcal{C})$ whose objects are the objects of $\mathcal{C}$ and whose morphisms are $Hom_{K^{\oplus}(\mathcal{C})}(A,B) = K^{\oplus}(Hom_{\mathcal{C}}(A,B))$*
2. *The Grothendieck group of an Abelian 2-category $\mathcal{C}$ is the category $K(\mathcal{C})$ whose objects are the objects of $\mathcal{C}$ and whose morphisms are $Hom_{K(\mathcal{C})}(A,B) = K(Hom_{\mathcal{C}}(A,B))$.*

Note that both $K(\mathcal{C})$ and $K^{\oplus}(\mathcal{C})$ are preaddivite categories. The composition here is defined by

$$[F] \circ [G] = [F \circ_0 G]$$

We denote by $R \otimes_{\mathbb{Z}} K(\mathcal{C})$ (resp. $R \otimes_{\mathbb{Z}} K^{\oplus}(\mathcal{C})$) the category form by tensoring with R each morphism set (so the objects remain the same, and $Hom_{R\otimes_{\mathbb{Z}}K(\mathcal{C})}(A,B) = R \otimes_{\mathbb{Z}} Hom_{K(\mathcal{C})}(A,B)$.) This forms an R-linear category, which is what we want to categorify, so now we can define strong categorification

Definition (Strong Categorification)**.** *Given R a commutative ring and $\boldsymbol{C}$ an R-linear category. A strong categorification of $\boldsymbol{C}$ is a pair $(\mathcal{C}, \Phi)$ where either*

i) $\mathcal{C}$ is an additive 2-category and $\Phi : R \otimes_{\mathbb{Z}} K^{\oplus}(\mathcal{C}) \to \boldsymbol{C}$ is an isomorphism, or

ii) $\mathcal{C}$ is an Abelian 2-category and $\Phi : R \otimes_{\mathbb{Z}} K(\mathcal{C}) \to \boldsymbol{C}$ is an isomorphism.

Example

1) We are going to strongly categorify $B = \mathbb{C}[b]/(b^2 - b)$. Let $D = \mathbb{C}[x]/(x^2)$ be the algebra of dual numbers, and consider $\mathcal{C}$ the 2-category with one object $I = D$-mod, and $Hom_{\mathcal{C}}(I,I)$ being the full (additive) subcategory of endofunctors of D-mod consisting in the functors that are isomorphic to direct sums of copies of id_I and $F = - \otimes_D D \otimes_{\mathbb{C}} D$ ($F(M) = M \otimes_D D \otimes_{\mathbb{C}} D$), so

$$End_{\mathcal{C}}(I) = \{G \in End(D\text{-mod}) \mid G \simeq F^{\oplus n} \oplus id^{\oplus m}\}$$

Vertical and horizontal compositions in this 2-category are given by direct sums and composition of functors respectively. When we take the Grothendieck group, $[F]$ and $[id]$ form a basis of $\mathbb{C} \otimes K(End_{\mathcal{C}}(I))$. The idea here is that F categorifies b. A simple calculation shows that $F \circ F \simeq F \oplus F$ (which lifts $b^2 = b$.) So we take the categorifying isomorphism as

$$\begin{aligned} \Phi : \quad & \mathbb{C} \otimes K(End_{\mathcal{C}}(I)) \to B \\ & \Phi([id]) = 1 \quad \Phi([F]) = b \end{aligned}$$

Note that when we are (strongly) categorifying algebras we use a one object 2-category $\{*\}$, which is the same as a monoidal category! So we can use monoidal additive (or Abelian) categories as the prototypical construction to categorify algebras

An example of this happened when we categorify hecke algebras with the additive monoidal category of Soergel bimodules.

Recall that the composition of 1-morphisms corresponds to the tensor $\otimes$, and this in turn will categorify the algebra product (as in the case of Soergel bimodules). The additive structure gives the biproduct $\oplus$ which will categorify the sum. The action of the base ring in this setting is considered when taking the tensor $R \otimes_{\mathbb{Z}} K(End(*))$ with the Grothendieck group. Note that Soergel bimodule categorification has the advantage here that it considers an specific action of the element v of the base ring $\mathbb{Z}[v, v^{-1}]$ on the objects of the category (which is as you might remember the sheaf action.)

Now we will go over a specific example, really important in the work of quantum physics, that has been around for some years: the Heisenberg Algebra.

4.4 The Heisenberg Algebra

4.4.1 The Uncertainty Principle

If the reader has had some interest in quantum physics, $\{\text{he}, \text{ she}\}$ has probably heard about Heisenberg's uncertainty principle, which states that one can not measure at the same time the velocity and the position of a particle with absolute precision.

This principle is in the heart of quantum physics, and it has a precise mathematical modeling. Roughly it goes like this [5]:

Observables in the world, like position q or momentum p, are modeled as elements of a C^*-algebra A [6], and measures are modeled as operators $w : A \to \mathbb{C}$. One can make many measures to get a more refined approximation to the real value. This refinement is measured by a statistical notion $\Delta_w(a) = w((a - w(a))^2)$ for $a \in A$ (that is, if $|\Delta_w(a)|$ is smaller, a is measured with more precision.)

In classical physics is assumed that we can make $\Delta_w(a)$ as small as we want by making many observations. But when looking at really tiny objects like electrons, the wave longitud of the light of the measurement equipement λ comes into place, and this makes $\Delta q \geq \frac{1}{4\pi}\Delta\lambda$ and $\Delta p = \frac{h}{2\pi\Delta\lambda}$, and therefore

$$|\Delta_w(q)\Delta_w(p)| \geq \frac{h}{2} \quad \forall w$$

Where h is a constant known as the plank constant. Now, $|\Delta_w(q)\Delta_w(p)| \geq \frac{1}{2}|w(pq - qp)|$, so this phenomena is modeled by considering $pq - qp = 1$ (so $|w(pq - qp)| = 1$). We get an algebra over $\mathbb{C}$, called the *Heisenberg algebra*, with generators p and q and relation $pq - qp = 1$.

Observe that p and q do not commute. Here we gasp over the main difference between classic and quantum world. The classical world is commutative, whereas the quantum world is not. This makes the quantum world encode much more information that the classical world.

4.4.2 The Heisenberg Algebra and the Fock Representation

We are going to work with the infinite Heisenberg algebra.

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[5] The uninterested reader can skip ahead to the next subsection.

[6] That is an algebra A over a field with a norm that makes it a Banach algebra, and an involution $a \mapsto a^*$ such that $\|xx^*\| = \|x\|^2$

Definition. *The* Heisenberg algebra $\mathfrak{h}$ *in infinitely many variables is the unital associative* $\mathbb{C}$*-algebra with generators* $\{p_n, q_n\}_{n\in\mathbb{Z}_{>0}}$ *and relations*

$$p_n p_m = p_m p_n, \quad q_n q_m = q_m q_n, \quad p_n q_m = q_m p_n + 1\delta_{m,n} \quad n, m \in \mathbb{Z}_{>0} \tag{1}$$

We present two more presentations of this algebra that will come of use. First, if we take $q_n = p_{-n}$ for $n \in \mathbb{Z}_{>0}$, we obtain generators p_n $n \in \mathbb{Z} \setminus \{0\}$ and relations

$$p_n p_m = p_m p_n + n\delta_{n,-m}, \quad n, m \in \mathbb{Z} \setminus \{0\} \tag{2}$$

A less obvious equivalence (shown in [Kho14]) is given by generators $\{a_n, b_n\}_{n\in\mathbb{Z}_{>0}}$ and relations

$$a_n a_m = a_m a_n, \quad b_n b_m = b_m b_n, \quad a_n b_m = b_m a_n + b_{m-1} a_{n-1} \quad n, m \in \mathbb{Z}_{>0} \tag{3}$$

We are going to work with a special module over $\mathfrak{h}$, called *the Fock representation.*

As a vector space it is *Sym*, the $\mathbb{C}$-algebra of symmetric functions over infinitely many variables $\{x_i\}_{i\in\mathbb{Z}_{>0}}$, which can be presented as

$$Sym \simeq \mathbb{C}[P_1, P_2, \dots].$$

The polynomial algebra over the power-sum symmetric functions $P_n = \sum_{i=1}^{\infty} x_i^n$.

This is the algebra is generated as a vector space by the Schur functions [Mac Sym I.3]. It has an inner product $< \cdot, \cdot >$: $Sym \times Sym \to \mathbb{C}$ made by declaring the Schur functions to be orthogonal.

Take the presentation 2 of $\mathfrak{h}$. $\mathfrak{h}$ acts on *Sym* as follows. For $i \in \mathbb{Z}_{>0}$,

- The generator p_{-i} acts by multiplication by the power sum P_i.
- The generator p_i acts by derivation over the power sum times i (that is, $p_i \cdot \alpha = i\frac{\partial}{\partial P_i}\alpha$.)

It turns out that these actions are adjoint to each other, respect to $< \cdot, \cdot >$.

We denote the Fock representation by $\mathcal{F}$. An alternative construction of $\mathcal{F}$ is made by considering $\mathfrak{h}^+$ the subalgebra of $\mathfrak{h}$ generated by $\{p_i\}_{i\in\mathbb{Z}_{>0}}$ (considering $\mathfrak{h}$ with presentation 2). Let V_0 be the trivial representation of $\mathfrak{h}^+$. Then $\mathcal{F}$ is the induced representation

$$\mathcal{F} \simeq \mathfrak{h} \otimes_{\mathfrak{h}^+} V_0$$

4.4.3 Categorification of the Fock representation

The following facts are proved in [Sav14] section 4.4.

Consider the group algebra of the symmetric group $\mathbb{C}[S_n]$, and by convention take $\mathbb{C}[S_0] = \mathbb{C}$. We call

$$K(S_n) = \mathbb{C} \otimes_{\mathbb{Z}} K^{\oplus}(\mathbb{C}[S_n]\text{-mod}_{f.g.})$$

$K(S_n)$ is a vector space with dimension equal to the number of indecomposable $\mathbb{C}[S_n]$-modules, which is equal to the number of partitions of n. There is an isomorphism of $\mathfrak{h}$-modules

$$\mathcal{F} \simeq \bigoplus_{n\in\mathbb{N}} K(S_n).$$

The action of $\mathfrak{h}$ that gives rise to this isomorphism is given as follows.

The natural embeddings $\mathbb{C}[S_n]\otimes\mathbb{C}[S_m]\to\mathbb{C}[S_{n+m}]$ (given by $(\sigma,\tau)\mapsto\sigma\sqcup\tau$) give rise to induction and restriction functors

$$Ind^{S_{n+m}}_{S_n,S_m}:\mathbb{C}[S_n]\text{-mod}_{f.g.}\times\mathbb{C}[S_m]\text{-mod}_{f.g.}\to\mathbb{C}[S_{n+m}]\text{-mod}_{f.g.}$$

$$Res^{S_{n+m}}_{S_n,S_m}:\mathbb{C}[S_{n+m}]\text{-mod}_{f.g.}\to\mathbb{C}[S_n]\text{-mod}_{f.g.}\times\mathbb{C}[S_m]\text{-mod}_{f.g.}$$

Which are adjoint to each other. For any M representation of S_m, the induction functor gives rise to a functor $Ind_n(M)$.

$$\begin{aligned}Ind_n(M):\quad &\mathbb{C}[S_n]\text{-mod}_{f.g.}\to\mathbb{C}[S_{n+m}]\text{-mod}_{f.g.}\\ &N\mapsto Ind^{S_{n+m}}_{S_n,S_m}(N,M)\end{aligned}$$

Which has an adjoint functor

$$\begin{aligned}Res_n(M):\quad &\mathbb{C}[S_{n+m}]\text{-mod}_{f.g.}\to\mathbb{C}[S_n]\text{-mod}_{f.g.}\\ &N\mapsto Hom(M,Res^{S_{n+m}}_{S_n,S_m}(N))\end{aligned}$$

These are additive functors, and hence induce linear maps in the tensored split Grothendieck groups, which we call $K(Ind_n(M))$ and $K(Res_n(M))$. Let V_n and W_n be the trivial and sign representation of S_n, respectively. Set the functors

$$\hat{a}_n=\bigoplus_{m\in\mathbb{N}}K(Res_m(V_n))$$

and

$$\hat{b}_n=\bigoplus_{m\in\mathbb{N}}K(Ind_m(W_n))$$

We have that $\hat{a}_n$ and $\hat{b}_n$ lift the actions of a_n and b_n on Sym. Moreover, they satisfy the relations

$$\hat{a}_n\circ\hat{a}_m=\hat{a}_m\circ\hat{a}_n,\quad\hat{b}_n\circ\hat{b}_m=\hat{b}_m\circ\hat{b}_n,\quad\hat{a}_n\circ\hat{b}_m=\hat{b}_m\circ\hat{a}_n\oplus\hat{b}_{m-1}\circ\hat{a}_{n-1}\quad n,m\in\mathbb{Z}_{>0}$$

which are the relations for the presentation 3 of $\mathfrak{h}$. So this construction gives a weak categorification of $\mathcal{F}$.

Some Final Comments

There is much about categorification which we did not cover. For example, there is a special category, called the category $\mathcal{O}$, which serves as a common tool for many algebraic categorifications. Full exposition about this category can be found in [Maz12].

It is also common to consider the Grothendieck group of *triangulated categories*. These can be used to deal nicely with subtraction since it can be commonly represented using a shift, as we did in section 4.2.1.

We also did not cover the topic of diagrammatic algebra. It turns out there is a way to compose and tensor morphism in a monoidal category in a graphical manner. Moreover, this can be done for a 2-category, and hence it is used when strongly categorifying. The graphical presentation simplifies the calculations by simplifying the visualization of morphisms, but moreover, the topology of the diagrams can be used to prove deep theorems about the categories involved. A good exposition about diagrammatic algebra, together with the categorification of the quantum group for $Sl(2, \mathbb{R})$ can be found in [Lau11]. A graphical categorification (using diagrammatic algebra) for the Hecke algebras is exposed in [EW13], and a graphical categorification for the Heisenberg algebra is outlined in [LS11] (and also [Kho14].)

In the bibliography there are references which the author considers a good read about categorification. Besides the ones already mentioned, in [BD98], J. Baez and J. Dolan introduce the role of n-categories into categorification. The reader would find some [7] other examples of categorification following [Dia14], [MK15], [FR11] and [Kho99]. This last one is particularly important in the history of the subject.

It was also a objective of this document to show categories as a nice way to *model*, although we only covered the modeling of other mathematical objects. There are actually uses of categories to model other areas outside mathematics. Reference of these are, for example, [Fon16], [And+13], [Ben91]. This sounds weird since we are used to think about categories as complicated objects with structure preserving morphisms, but if we think of them as directed graphs (or undirected if we have a groupoid) with an associative structure for the arrows, the scene becomes simpler.

As final thoughts for this document, the author will dare to sketch a product of his own imagination on how can categories be used to model entities outside mathematics. This should be taken at this point just as a rough introduction. We do not intend to give any formal meaning to our words.

There is a weird feeling that some people have when driving a car, the feeling that the driven car is an entity representing the driver. In this setting, other cars get considered as other entities, which *the driver* is not allowed to touch. The same feeling can occur when playing a video game, when the player gets identified with the character in the game.

This identification can be explain because of some associations the brain does. Seeing through the mirrors of the car is like seeing through the eyes. Where as turning the wheel left or right is like the function of the brain that allows you to turn left or right. The acceleration pedal associates with the function that allows you to advance (slower or faster), and so on.

This associations hint the existence of a *morphism*, that is, an identification that preserves structure. In this case the first structure is related to the way the brain processes work, and the second structure is related to the functionality of the car, or, more precisely (or rather more useful to the concept we want to introduce) the functionallity of driving.

Now, the processes on the brain are processes, and processes can be modeled by arrows.

Therefore getting to the concept of category.

There is gluing of concept via tensor product, and a nice union of concept via composition. Of course, we can start wondering, what would happen if we take 2-categories, n-categories?

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[7] One can always search "categorification" in arxiv and find many other examples. The examples cited are the ones the author finds more pleasant to read.

A wishful hypothesis would be that **learning** permits the brain to emulate the category of a process that was not previously integrated. It is a process of adaptation.

To what level is the brain adaptable? How many categories can the brain emulate? The author would conjecture that at least any category with finite number of objects.

Of course, the human brain does not need to restrict to categories, but this would be another conjecture:

Are all the brains functions modeled by categories?

In helping to the wishful hypothesis, note that for mechanical cars, there is not a function of the brain related to the clutch pedal, so the use of this pedal is a little harder (but not super hard) to learn.

Once learned an activity of certain kind, activities closely related (in which a morphism with many ability identifications exists) can be learned more easily.

That is why a musician who can play many instruments would not have a hard time learning a new instrument, or why a dancer would easily learn a new kind of dance. That is also why if you take a statistics book or a scientific calculus book you would learn it pretty quickly.

Similar thing happens on video games, martial arts, visual arts, sports, etc. Pretty much any human discipline.

The study of these "categories on the brain" can help us understand better how the human brain and the process of learning work. It can also help doing a precise program on how to learn a discipline (although the author is against the idea that an universal way of learning can exist. (I am just a mathematician trying to expand his theory), and closer to the idea that this depends on each human being.)

And so this document comes to an end. Thanks for reading.

References

[Alu09] Paolo Aluffi. *Algebra: Chapter 0 (Graduate Studies in Mathematics).* American Mathematical Society, 2009. ISBN: 0821847813.

[And+13] Moreno Andreatta et al. "Towards a Categorical Theory of Creativity for Music, Discourse, and Cognition". In: *Mathematics and Computation in Music.* Springer Berlin Heidelberg, 2013, pp. 19–37. DOI: `10.1007/978-3-642-39357-0_2`.

[Art04] R. D. Arthan. "The Eudoxus Real Numbers". In: (May 24, 2004). arXiv: `math/0405454v1 [math.HO]`.

[BD00] John C. Baez and James Dolan. "From Finite Sets to Feynman Diagrams". In: *In Mathematics Unlimited - 2001 and Beyond, vol. 1, eds. Björn Engquist and Wilfried Schmid, Springer, Berlin, 2001, pp. 29-50.* (Apr. 20, 2000). arXiv: `math/0004133v1 [math.QA]`.

[BD98] John C. Baez and James Dolan. "Categorification". In: *In Higher Category Theory, eds. Ezra Getzler and Mikhail Kapranov, Contemp. Math. 230, American Mathematical Society, Providence, Rhode Island, 1998, pp. 1-36.* (Feb. 5, 1998). arXiv: `math/9802029v1 [math.QA]`.

[Ben91] Ierce Benjamin. *Basic Category Theory for Computer Scientists.* MIT University Press Group Ltd, Aug. 11, 1991. ISBN: 0262660717.

[BS81] S. Burris and H. P. Sankappanavar. *A Course in Universal Algebra (Graduate Texts in Mathematics).* Springer, 1981. ISBN: 0387905782.

[Dia14] Rafael Diaz. "On the Categorification of the Mbius Function". In: (Feb. 10, 2014). arXiv: `1402.2131v1 [math.CO]`.

[EW13] Ben Elias and Geordie Williamson. "Soergel Calculus". In: *Represent. Theory 20 (2016), 295-374* (Sept. 3, 2013). arXiv: `1309.0865v1 [math.QA]`.

[Fon16] Brendan Fong. "The Algebra of Open and Interconnected Systems". In: (Sept. 17, 2016). arXiv: `1609.05382v1 [math.CT]`.

[FR11] Philipp Fahr and Claus Michael Ringel. "Categorification of the Fibonacci numbers using representations of quivers". In: (July 10, 2011). arXiv: `1107.1858v2 [math.RT]`.

[Hat] Allen Hatcher. *Algebraic Topology.* Cambridge University Pr. 556 pp. ISBN: 0521795400.

[Kho14] Mikhail Khovanov. "Heisenberg algebra and a graphical calculus". eng. In: *Fundamenta Mathematicae* 225.0 (2014), pp. 169–210. URL: `http://eudml.org/doc/282925`.

[Kho16] Mikhail Khovanov. "Linearization and categorification". In: (Mar. 27, 2016). arXiv: `1603.08223v1 [math.RT]`.

[Kho99] Mikhail Khovanov. "A categorification of the Jones polynomial". In: *Duke Math. J. 101 (2000), no. 3, 359–426* (Aug. 30, 1999). arXiv: `math/9908171v2 [math.QA]`.

[Lan10] Saunders Mac Lane. *Categories for the Working Mathematician.* Springer, Nov. 19, 2010. 332 pp. ISBN: 1441931236.

[Lau11] Aaron D. Lauda. “An introduction to diagrammatic algebra and categorified quantum sl(2)”. In: *Bulletin of the Institute of Mathematics Academia Sinica, Vol 7 (2012), No. 2, pp.165–270* (June 10, 2011). arXiv: `1106.2128v2 [math.QA]`.

[Lib17] Nicolas Libedinsky. “Gentle introduction to Soergel bimodules I: The basics”. In: (Jan. 31, 2017). arXiv: `1702.00039v1 [math.RT]`.

[LM13] Weiyun Lu and Aaron K. McBride. “Algebraic Structures on Grothendieck Groups”. In: (2013).

[LS11] Anthony Licata and Alistair Savage. “A survey of Heisenberg categorification via graphical calculus”. In: *Bull. Inst. Math. Acad. Sin. (N.S.) 7 (2012), pp. 291–321* (Apr. 25, 2011). arXiv: `1104.4789v4 [math.RT]`.

[Maz12] Volodymyr Mazorchuk. *Lectures on Algebraic Categorification (QGM Master Class Series)*. European Mathematical Society, 2012. ISBN: 3037191082.

[MK15] Radmila Sazdanovic Mikhail Khovanov. “Categorifications of the polynomial ring”. eng. In: *Fundamenta Mathematicae* 230.3 (2015), pp. 251–280. URL: `http://eudml.org/doc/282905`.

[Sav14] Alistair Savage. “Introduction to categorification”. In: (Jan. 23, 2014). arXiv: `1401.6037v2 [math.RT]`.