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Towards a new proof of the Intermediate Disorder Regime for Directed Polymers in Random Environments

Tesis para optar al grado de Magíster en Matemática

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"For Yahweh gives wisdom;
From His mouth come knowledge and discernment."
Proverbs 2:6

Hacia una nueva demostración del Régimen de Desorden Intermedio para Polímeros Aleatorios en Medios Aleatorios

En dimensión $1+1$, si las trayectorias aleatorias del polímero se reescalan difusivamente y, simultáneamente, la temperatura inversa se reescala con la longitud del polímero n como $\beta_n := \frac{n^{-1/4}}{\sqrt{2}}\beta$, se observa un comportamiento particular (denominado *Régimen de Desorden Intermedio*). En 2014, Alberts, Khanin y Quastel [AKQ14b] demostraron que, bajo el escalado anterior, la función de partición punto a punto normalizada $Z_{n,\sqrt{n}x}(\omega, \beta_n)e^{-n\lambda(\beta_n)}$ del polímero aleatorio converge en distribución a la solución $\mathcal{Z}_\beta(1, x)$ de la *Ecuación de Calor Estocástica* con Ruido Blanco multiplicativo ξ, que se expresa clásicamente por su Expansión de Caos.

Damos una nueva demostración (parcial) de esta convergencia, basada en resultados recientes de Quastel, Ramírez y Virág [QRV22], donde expresan $\mathcal{Z}_\beta(t, x)$ como el límite de una martingala uniformemente integrable de una filtración (natural) de ξ. Demostramos que ciertas aproximaciones de $Z_{n,\sqrt{n}x}(\omega, \beta_n)e^{-n\lambda(\beta_n)}$ convergen en distribución a las aproximaciones correspondientes de $\mathcal{Z}_1(1, x)$ dadas por estos términos de martingala.

Towards a new proof of the Intermediate Disorder Regime for Directed Polymers in Random Environments

In dimension $1+1$, if the random polymer paths are re-scaled diffusively and, simultaneously, the inverse temperature is re-scaled with the polymer length n as $\beta_n := \frac{n^{-1/4}}{\sqrt{2}}\beta$, a particular behavior is observed (called the *Intermediate Disorder Regime*). In 2014, Alberts, Khanin and Quastel [AKQ14b] proved that, under the above scaling, the normalized point-to-point partition function $Z_{n,\sqrt{n}x}(\omega, \beta_n)e^{-n\lambda(\beta_n)}$ of the random polymer converges in distribution to the solution $\mathcal{Z}_\beta(1, x)$ of the *Stochastic Heat Equation* with multiplicative White Noise ξ, which is classically expressed by its Chaos Expansion.

We give a new (partial) proof of this convergence, based on recent results by Quastel, Ramírez and Virág [QRV22], where they express $\mathcal{Z}_\beta(t, x)$ as the limit of a uniformly integrable martingale of a (natural) filtration of ξ. We prove that certain approximations of $Z_{n,\sqrt{n}x}(\omega, \beta_n)e^{-n\lambda(\beta_n)}$ converge in distribution to the corresponding approximations of $\mathcal{Z}_1(1, x)$ given by these martingale terms.

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Chapter 1

Introduction

We start by introducing the *Directed Polymer in Random Environment* model on $\mathbb{Z}^d$. This model was initially introduced in the context of statistical mechanics (for $d = 1$) as a toy model to study the interfaces of the planar Ising model with random coupling constants, see [HH85]. It was later generalized to arbitrary dimension [Bol89, IS88]. For $d \geq 2$, the directed polymer in random environment should be better interpreted as a model for the behaviour of an (actual) polymer stretched in a medium with impurities. It has deep connections with the Stochastic Heat Equation (SHE) and the Kardar-Parisi-Zhang (KPZ) Equation, as we will discuss later.

Recent reviews on the subject can be found in [Com17, Zyg24]; particularly, see [Com17, Chapter 8] to read more about its connection with SHE and KPZ Equation. We, also, will follow the definitions of [Com17] and throughout all this thesis we will frequently refer to it.

After introducing the model, we will give a first statement of the main theorem of this thesis. We end up the introduction chapter describing the structure of the document.

1.1 Directed Polymers in Random Environments

The polymer measure

We start by considering a *simple random walk*, $S = (S_n)_{n\geq 0}$, on $\mathbb{Z}^d$ starting from 0. More precisely, we consider the random sequence S on the probability space $(\Lambda, \mathcal{A}, P)$, with $\Lambda = (\mathbb{Z}^d)^{\mathbb{N}}$ the path space, $\mathcal{A}$ the product $\sigma-$algebra and P a probability measure such that, under P, $P(S_0 = 0) = 1$, the increments $S_1 - S_0, \ldots, S_n - S_{n-1}$ are independent and

$$P(S_n - S_{n-1} = \pm e_j) = (2d)^{-1} \text{ for each } j \in \{1, \ldots, d\} \text{ and } n \in \mathbb{N},$$

where $(e_j)_{j=1,\ldots,d}$ is the canonical basis of $\mathbb{R}^d$. In the sequel, $P[X]$ will denote the P-expectation of a random variable X defined on $(\Lambda, \mathcal{A}, P)$.

Next, we consider the product space $\mathbb{R}^{\mathbb{N}\times\mathbb{Z}^d}$, whose elements we will call *environments* and denote by $\omega = \{\omega(n, x) : n \in \mathbb{N}, x \in \mathbb{Z}^d\}$. Given a fixed environment ω, we define the *polymer measure in the environment* ω, or *quenched polymer measure*, as the measure

$P_n^{\beta,\omega}$ on $(\Lambda, \mathcal{F})$ defined for each $n \in \mathbb{N}$ as

$$P_n^{\beta,\omega}(\mathrm{d}\mathbf{x}) = P(\mathrm{d}\mathbf{x}) \frac{1}{Z_n(\omega,\beta)} \exp\{\beta H_n^\omega(\mathbf{x})\}$$

where $\beta > 0$ is a given parameter (called the *inverse temperature*),

$$H_n^\omega(\mathbf{x}) := H_n(\mathbf{x}) := \sum_{i=1}^n \omega(i, x_i)$$

is the energy of the path $\mathbf{x} = (x_1, \ldots, x_n)$ (sometimes called *Hamiltonian potential*), and

$$Z_n(\omega,\beta) := P\Big[\exp\Big\{\beta \sum_{i=1}^n \omega(i, S_i)\Big\}\Big] = P[\exp\{\beta H_n^\omega(S)\}] \tag{1.1}$$

is the *partition function*, a normalizing constant making $P_n^{\beta,\omega}$ a probability measure.

We interpret that, under the polymer measure $P_n^{\beta,\omega}$, the walk will prefer paths which visit sites x with higher values of $\omega(i,x)$ and avoid those with lower values. Notice that, as the inverse temperature β increases, this "preference" becomes stronger.

We can even think of this model as a game where the player (the walk) has to collect coins distributed over a map ($\mathbb{N}\times\mathbb{Z}^d$), where the values of the coins depend on ω; the player will obviously try to follow the most rewarding paths. We could think of the parameter β as an impediment to distinguish between the values of the coins, that is, when β is close to zero (or zero) all values will become similar or equal, while if β is large the values of the coins will become more different between them.

The random environment

The final ingredient to define the model is to consider a *random environment* $\omega = \{\omega(n,x) : n \in \mathbb{N}, x \in \mathbb{Z}^d\}$, consisting of a family of real valued, non-constant and *i.i.d.* random variables defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and such that

$$\mathbb{P}[\exp\{\beta\omega(n,x)\}] < \infty \quad \forall \beta \in \mathbb{R}. \tag{1.2}$$

Here, and in the sequel, $\mathbb{P}[X]$ denotes the $\mathbb{P}$-expectation of a random variable X defined on $(\Omega, \mathcal{G}, \mathbb{P})$, and for $A \in \mathcal{G}$, $\mathbb{P}[X; A] = \mathbb{P}[X \mathbb{1}_A]$.

We then define the *directed polymer in random environment* model by specifying its distribution: the *annealed polymer measure* $\mathbf{P}_{\beta,n}$ is given by

$$\mathbf{P}_{\beta,n}(A \times B) := \int_A P_n^{\beta,\omega}(B) \mathrm{d}\mathbb{P}.$$

for $A \in \mathcal{G}$ and $B \in \mathcal{F}$, with B depending on at most the first n jumps of the walk.

Remark 1.1. We can not define this measure simultaneously for all n, since $(P_n^{\beta,\omega})_{n\in\mathbb{N}}$ (and therefore $(\mathbf{P}_{\beta,n})_{n\in\mathbb{N}}$) is not a consistent family of measures.

This model represents the behaviour of a polymer in a medium with impurities. If the impurities are negligible (in a sense), the polymer would move freely through the fluid. If the impurities affect more the energy of the polymer, trajectories such that the waste of energy is minimized will be preferred. This disorder of the media is captured by the environment ω and the parameter β. We want to measure how "far" from being pure is the medium; this will be discussed a forthcoming section.

Simple Random Walk vs Last Passage Percolation

As

$$P_n^{0,\omega}(\mathrm{d}\mathbf{x}) = P(\mathrm{d}\mathbf{x}),$$

we see that when $\beta = 0$ (which corresponds to an infinite temperature) we are just in the case of the simple random walk. On the other hand we note that for fixed n,

$$\begin{aligned}\lim_{\beta\to\infty} \beta^{-1} \log Z_n(\omega,\beta) &= \lim_{\beta\to\infty} \beta^{-1} \log(P[\exp\{\beta H_n(S)\}]) \\ &= \lim_{\beta\to\infty} \beta^{-1} \log\left((2d)^{-n} \sum_{\mathbf{x}} \exp\{\beta H_n(\mathbf{x})\}\right)\end{aligned}$$

where the index $\mathbf{x}$ ranges over all $(2d)^n$ possible paths of length n of the simple random walk. The factor $(2d)^{-n}$ will vanish in the limit since it will result in a constant term divided by β. Now, using properties of the logarithm and writing

$$\hat{v} = (\exp\{H_n(\mathbf{x})\})_{\mathbf{x}}$$

as a vector in $\mathbb{R}^{(2d)^n}$, we obtain

$$\begin{aligned}\lim_{\beta\to\infty} \beta^{-1} \log Z_n(\omega,\beta) &= \lim_{\beta\to\infty} \log\left(||\hat{v}||_\beta\right) \\ &= \log(||\hat{v}||_\infty) \\ &= \max_{\mathbf{x}} H_n(\mathbf{x}),\end{aligned}$$

where $||\cdot||_\beta$ and $||\cdot||_\infty$ denote the β-norm and the supremum norm in $\mathbb{R}^{(2d)^n}$, respectively.

This tells us that, as $\beta \to \infty$, $Z_n(\beta,\omega) \simeq \exp\{\beta \max_{\mathbf{x}} H_n(\mathbf{x})\}$. In other words, as $\beta \to \infty$, the contributions of all paths which do not maximize the energy become negligible, thus the quenched polymer measure concentrates purely on these maximizing paths. In summary, as $\beta \to \infty$,

$$\begin{cases} \beta^{-1} \log Z_n(\omega,\beta) & \longrightarrow & \max_{\mathbf{x}} H_n(\mathbf{x}) \\ P_n^{\omega,\beta} & \text{concentrates on} & \arg\max_{\mathbf{x}} H_n(\mathbf{x}). \end{cases} \tag{1.3}$$

The right-hand sides of (1.3) are respectively the passage time and the set of geodesics of a *last passage percolation* problem.

In general, the Last Passage Percolation model (LPP) can be defined as follows. Let $d \geq 2$. Consider a family $\{w_x\}_{x\in\mathbb{Z}^d}$ of i.i.d. random variables. Fix $N \in \mathbb{N}$. Consider the set of "up-right" paths of length N starting at 0 as

$$\Pi_N := \{\mathbf{x} = \{x_i\}_{i=0}^{N} : x_i = 0 \ \wedge \ \forall i \in \{0, ..., N-1\} \exists j \in \{1, ..., d\} \ x_{i+1} - x_i = e_j\}$$

where e_j denote the canonical vectors in $\mathbb{Z}^d$. The (point-to-line) *last passage time* in defined as

$$L_N := \max_{\mathbf{x} \in \Pi_N} \sum_{i=1}^{N} w_{x_i}$$

and the corresponding geodesics is defined as the arg max over Π_N. *Point-to-point last passage percolation* can be defined similarly for paths from 0 to $\hat{N} = (N_1, ..., N_d) \in \mathbb{Z}^d$ if we take

$$\Pi_{\hat{N}} := \{\mathbf{x} = \{x_i\}_{i=0}^{||\hat{N}||_1} : x_i = 0 \ \wedge x_{||\hat{N}||_1} = \hat{N} \wedge \ \forall i \in \{0, ..., ||\hat{N}||_1 - 1\} \ x_{i+1} - x_i = e_j\}$$

The model can be generalized in many directions. See [Rem23, Section 2] for a introduction of LPP and connections with other important models, particularly with the *totally asymmetric simple exclusion process* (TASEP).

Remark 1.2. Intuitively, the polymer measure interpolates as β ranges from 0 to ∞ between the law of a simple random walk and the distribution concentrated on the set of geodesics of last passage percolation (in the case of a unique geodesic, it is a Dirac distribution).

1.2 First formulation of main results

Assume $d = 1$. For simplicity, we will assume that environment ω has mean zero and variance one:

Assumption 1. The variables $\omega(i, x)$, $(i, x) \in \mathbb{N} \times \mathbb{Z}$, which are i.i.d. with exponential moments of all orders (1.2), also satisfy the following:

$$\mathbb{P}[\omega(i, x)] = 0 \text{ and } \mathbb{P}[\omega^2(i, x)] = 1.$$

We need to give some definitions before stating the theorem. Define the *logarithmic moment generating function* λ of $\omega(i, m)$ as

$$\lambda(\beta) := \log \mathbb{P}[\exp\{\beta\omega(i, m)\}]$$

for $\beta \geq 0$, which is finite by Assumption 1.2. For $x \in \mathbb{Z}$, we define the *point-to-point* partition function

$$Z_{n,x}(\omega, \beta) := P\Big[\exp\Big\{\beta \sum_{i=1}^{n} \omega(i, S_i)\Big\} \mathbb{1}_{S_n = x}\Big]. \tag{1.4}$$

Now fix $\beta > 0$. We define

$$\beta_n := \tfrac{n^{-1/4}}{\sqrt{2}} \beta.$$

The aim of this thesis is to provide a new proof of the following famous result by Alberts, Khanin and Quastel [AKQ14b], establishing the so-called intermediate disorder regime for the model of directed polymers in random environments (in dimension $d = 1$).

Theorem 1.1 ([AKQ14b]). *Under Assumption 1, the normalized partition function of the discrete random polymer at temperature β_n converges in law to the partition function of the continuum random polymer:*

$$Z_n(\omega, \beta_n) \exp\{-n\lambda(\beta_n)\} \xrightarrow{d} \mathcal{Z}_\beta(1, *) := \int \mathcal{Z}_\beta(1, x)\mathrm{d}x, \tag{1.5}$$

where $\mathcal{Z}_\beta(1, \cdot)$ denotes the solution of the Stochastic Heat Equation in dimension $1+1$ with multiplicative space-time white noise ξ and initial condition δ_0, i.e.,

$$\partial_t \mathcal{Z}_\beta = \tfrac{1}{2}\partial_x^2 \mathcal{Z}_\beta + \beta\xi\mathcal{Z}_\beta, \quad \mathcal{Z}_\beta(0, \cdot) = \delta_0(\cdot). \tag{1.6}$$

Moreover, for the point-to-point partition function,

$$\tfrac{\sqrt{n}}{2} Z_{n,\sqrt{n}x}(\omega, \beta_n) \exp\{-n\lambda(\beta_n)\} \xrightarrow{d} \mathcal{Z}_\beta(1, x) \tag{1.7}$$

with the topology of uniform convergence on compact sets.

The precise meaning of solution to the Stochastic Heat Equation in (1.6) and other concepts such as *white noise*, *intermediate disorder* and *continuum random polymer* will be discussed in the next chapter.

Remark 1.3. In [AKQ14b], they actually prove convergence of the whole process: First, they generalize the convergence in Theorem 1.1 to

$$\tfrac{\sqrt{n}}{2} P_n^{\beta_n,\omega}\Big(S_{nt} = \sqrt{n}y \Big| S_{ns} = \sqrt{n}x\Big) \xrightarrow{d} \tfrac{\mathcal{Z}_\beta(s,x;t,y)\mathcal{Z}_\beta(t,y;1,*)}{\mathcal{Z}_\beta(s,x;1,*)},$$

then using tightness of the space-time process, they prove that, after diffusive scaling, the discrete polymer in the Intermediate Disorder Regime $\beta_n = O(n^{-1/4})$ converges in distribution to the Continuum Random Polymer. We did not work in that generality, but we think that it is possible to adapt our techniques to obtain similar results.

In this thesis, we attempt to provide a new proof of Theorem 1.1 using a totally different approach, based on a characterization of the solution of the Stochastic Heat Equation established by Quastel, Ramírez and Virág [QRV22], which we will discuss in detail in Section 2.4. Even if we are still not able to provide a complete alternative proof of Theorem 1.1, in this thesis we will present some important advances in this direction. More precisely, we will show that a certain approximation of the partition function (the *quasi-partition function*), when suitably rescaled, converges in distribution to the corresponding approximations of the solution of the Stochastic Heat Equation given by [QRV22], giving us a partial result in the point-to-point (1.7) and the point-to-line (1.5) cases.

Chapter 2

Preliminaries

In this chapter we collect some preliminary notions and concepts that we will need to state the main results of this thesis and for their proofs. We begin by discussing the disorder regimes of the model of directed polymers.

2.1 Disorder regimes

We come back to arbitrary dimension $d \geq 1$. An important topic for the directed polymer model is the so-called localization transition. We can think of "disorder" as a description of how the environment affects the polymer path, how different it gets compared to the simple random walk. Two main disorder regimes are observed, namely: the weak disorder regime and the strong disorder regime. Being in one or the other depends on the lattice dimension d and the (inverse) temperature β.

These regimes can be described in the words of [JL24]. At low temperature, the polymer trajectories are localized in a narrow corridor where the environment is most favorable [Bat18, Bat21, BC20, CH02, CH06, CSY03]. At high temperature, the effect of the disorder disappears on large scales, and the rescaled polymer trajectories have the same scaling limit as the simple random walk, that is to say, standard Brownian Motion [Bol89, CY06, IS88, SZ96].

2.1.1 The annealed bound

To properly introduce and explain the different disorder regimes, we first need to present the so-called annealed bound.

As before, we define the *logarithmic moment generating function* λ of $\omega(i,x)$

$$\lambda(\beta) := \log \mathbb{P}[\exp\{\beta\omega(i,x)\}] \tag{2.1}$$

for $\beta \geq 0$, which is finite by Assumption (1.2).

We also define the *free energy* as the following limit:

$$p(\beta) := \lim_{n\to\infty} p_n(\omega,\beta),$$

where p_n is defined by

$$p_n(\omega, \beta) := \tfrac{1}{n} \log Z_n(\omega; \beta).$$

Remark 2.1. The value of $p(\beta)$ does not depend on ω but only on its law. This property, where the random sequence $p_n(\omega, \beta)$ becomes nonrandom in the limit is called *self-averaging*; it is due to the fact that the fluctuations become negligible as $n \to \infty$. We can think of it as some kind of Law of Large Numbers.

The convergence is $\mathbb{P}$-*a.s.* and in L^p, $p \in [0, \infty)$ and in fact

$$p(\beta) = \sup_n \tfrac{1}{n} \mathbb{P}[\log Z_n(\omega; \beta)] \tag{2.2}$$

as a consequence of the Subadditive Lemma (A.1 in the Appendix). A proof of these results can be found in [Com17, Theorem 2.1].

The expression (2.2) implies that $p(\beta) \leq \lambda(\beta)$. Indeed, for all paths S, since the $\omega(j, x)$ are i.i.d., by a simple computation we have

$$\begin{aligned} \mathbb{P}[\exp\{\beta H_n(S)\}] &= \prod_{i=1}^{n} \mathbb{P}\left[\exp\left\{\beta\omega(1,0)\right\}\right] \\ &= \exp\left\{\log\left(\mathbb{P}\left[\exp\left\{\beta\omega(1,0)\right\}\right]^n\right)\right\} \\ &= \exp\{n\lambda(\beta)\}, \end{aligned}$$

and by Fubini-Tonelli's theorem,

$$\mathbb{P}[Z_n] = \mathbb{P}[P \exp\{\beta H_n(S)\}] = P[\mathbb{P} \exp\{\beta H_n(S)\}] = \exp\{n\lambda(\beta)\}.$$

Then, by Jensen inequality,

$$\mathbb{P}[p_n(\omega, \beta)] = \tfrac{1}{n} \mathbb{P}[\log Z_n] \leq \tfrac{1}{n} \log \mathbb{P}[Z_n] = \lambda(\beta),$$

and hence

$$p(\beta) \leq \lambda(\beta). \tag{2.3}$$

This inequality is known as the *annealed bound*.

2.1.2 Weak and Strong Disorder Regimes

We consider the *normalized partition function*

$$W_n^\beta := Z_n(\omega; \beta) \exp\{-n\lambda(\beta)\},\ n \geq 1,$$

which has expectation 1 by the previously computed equality $\mathbb{P}[Z_n(\omega; \beta)] = \exp\{n\lambda(\beta)\}$. It is easy to check that W^β is a positive, mean 1, martingale with respect to the filtration $(\mathcal{G}_n)_n$, where

$$\mathcal{G}_n := \{\omega(j, x) \,:\, j \leq n,\, x \in \mathbb{Z}^d\}.$$

In fact, this is a consequence of

$$w_n(\mathbf{x}) := \exp\{\beta H_n(\mathbf{x}) - n\lambda(\beta)\}$$

being a martingale with respect to the same filtration, where $\mathbf{x}$ is any fixed path. By Doob's martingale convergence theorem [Mie06, Corollary 2.3.1], the limit W_∞^β exists $\mathbb{P}-$a.s, and it is non-negative.

It is well-known that, for $\beta \neq 0$, the martingale w_n^β vanishes for all infinite paths $\mathbf{x}$. This is due to $H_n(\mathbf{x})$ being of order $\sqrt{n}$ as $n \to \infty$ (consequence of the Central Limit Theorem). But it is possible that the mean over all paths converges to a positive limit.

The first step to study the limit W_∞^β is showing that the event $\{W_\infty^\beta = 0\}$ is measurable with respect to the tail $\sigma-$field

$$\mathcal{T} := \bigcap_{n\geq 1} \mathcal{T}_n, \quad \mathcal{T}_n := \sigma\{\omega(j,x); j \geq n, x \in \mathbb{Z}^d\},$$

so therefore, by Kolmogorov's zero-one law [Mie06, Theorem 3.1.1] the event in consideration is trivial. We will not show that $\{W_\infty^\beta = 0\} \in \mathcal{T}$, but it is a straightforward calculation using a certain Markov property of the model; see [Com17, Section 3.1]. However, we summarize all this in the following statement.

Proposition 2.1. *The limit*

$$W_\infty^\beta = \lim_{n\to\infty} W_n^\beta$$

exists $\mathbb{P}$*-a.s. Moreover, we have a dichotomy. Either the limit* W_∞^β *is a.s. positive, or it is a.s. zero:*

$$\mathbb{P}\{W_\infty^\beta > 0\} = 1,$$

or

$$\mathbb{P}\{W_\infty^\beta = 0\} = 1.$$

Based on this proposition, we can define the disorder phases: The polymer is said to be in the *weak disorder* phase when $\mathbb{P}\{W_\infty^\beta > 0\} = 1$ holds, and in the *strong disorder* phase when $\mathbb{P}\{W_\infty^\beta = 0\} = 1$ holds.

It is possible to show that $\beta \mapsto W_\infty^\beta$ is non-increasing, so we can define a phase transition.

Proposition 2.2 (Proposition 3.1 in [Com17])**.** *There exists* $\beta_c = \beta_c(\mathbb{P}, d) \in [0, \infty]$ *such that*

$$\begin{cases} W_\infty^\beta > 0 & a.s. \quad \text{if } \beta \in [0, \beta_c), \\ W_\infty^\beta = 0 & a.s. \quad \text{if } \beta > \beta_c. \end{cases}$$

It has been shown that $\beta_c \in (0, \infty]$ for $d \geq 3$ [Bol89] and that $\beta_c = 0$ in dimensions $d = 1, 2$ [CH02, CY06]. In particular, in dimension $d = 1$ the weak disorder regime only occurs for $\beta = 0$.

2.1.3 Very Strong Disorder

A third notion of disorder is defined according to the rate of convergence of the limit $W_n^\beta \to W_\infty^\beta$. Precisely, if the rate of convergence is exponential, that is, if

$$\lim_{n\to\infty} \frac{1}{n} \log W_n^\beta < 0,$$

then it is said that *very strong disorder* occurs. An important aspect of studying the very strong disorder is that, under this condition, localization occurs [Bat18, BC20].

Notice that if $W_\infty^\beta > 0$, then a.s.

$$p(\beta) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\omega;\beta) = \lambda(\beta) + \lim_{n\to\infty} \frac{1}{n} \log W_n^\beta = \lambda(\beta). \tag{2.4}$$

In other words, $W_\infty^\beta > 0$ implies $p(\beta) = \lambda(\beta)$. In the case $W_\infty^\beta = 0$, the annealed bound equality may or may not be achieved, depending on the rate of convergence of W_n^β to zero.

By computations (2.4), being in the very strong disorder regime is equivalent to not achieve equality in the annealed bound (2.3).

In fact, we can define

$$\mathfrak{f}(\beta) := \lim_{n\to\infty} \frac{1}{n}\mathbb{P}[\log W_n^\beta] = \lim_{n\to\infty} \frac{1}{n} \log W_n^\beta = \sup_{n\geq 1} \frac{1}{n}\mathbb{P}[\log W_n^\beta],$$

where the second convergence holds $\mathbb{P}$-a.s. (the existence of these limits is established in [CSY03, Proposition 2.5]). With this notation, we have the following:

Proposition 2.3 ([CY06], Theorem 3.2)**.** *The following holds:*

(a) There exists $\beta_c \in [0,\infty]$ such that weak disorder holds for $\beta < \beta_c$ and strong disorder holds for $\beta > \beta_c$.

(b) The function $\beta \mapsto \mathfrak{f}(\beta)$ is non-increasing. In particular, there exists $\overline{\beta}_c \in [0,\infty]$ such that very strong disorder holds if and only if $\beta > \overline{\beta}_c$.

As a consequence of (2.4), Proposition 2.2 and Proposition 2.3, we have that $\beta_c \leq \overline{\beta}_c$.

Originally, it was proved that very strong disorder holds for all $\beta > 0$ in dimensions $d = 1$ [CV06] and $d = 2$ [Lac10], implying that $\overline{\beta}_c = \beta_c$ in these two cases. Also, it has been proved that in dimension $d \geq 3$, $\overline{\beta}_c \in (0,\infty]$, all leading to the conjecture that $\beta_c = \overline{\beta}_c$ for any dimension [BC20, CH06, Com17, CY06, Lac10, Viv23, Zyg24]. Recently, this conjecture has been confirmed under reasonable hypotheses by Junk and Lacoin [JL24].

Theorem 2.4 ([JL24], Theorem 2.1)**.** *For any $d \geq 3$, if the environment is bounded from above, that is, there exists $A \in \mathbb{R}$ such that $\mathbb{P}[\omega(1,0) \leq A] = 1$, then strong disorder and very strong disorder are equivalent. In other words, there is equality of the critical points, that is, $\beta_c = \overline{\beta}_c$.*

Remark 2.2. This theorem closes the "problem" of defining weak and strong disorder according to the a.s. positivity of W_∞^β and, *a priori*, not according to the localization of the polymer, since *a posteriori* they are equivalent. In other words, the intuition of "strong disorder $\iff$ localized behavior" and "weak disorder $\iff$ Brownian behavior" turns out to be correct.

2.1.4 Intermediate Disorder Regime

Throughout this subsection, we assume that $d = 1$. For $x \in \mathbb{Z}$, we recall the *point-to-point* partition function

$$Z_{n,x}(\omega, \beta) := P\Big[\exp\Big\{\beta \sum_{i=1}^{n} \omega(i, S_i)\Big\} \mathbb{1}_{S_n = x}\Big]. \tag{2.5}$$

Our objective is to observe the macroscopic behavior of the polymer in two cases: the *point-to-line* (P2L) case, where the endpoint of the polymer is free, and the *point-to-point* (P2P) case, where it follows a particular direction (when re-scaled properly, it will mean to fix the endpoint).

For that, we will re-scale the path diffusively, that is, $(j, S_j) \mapsto \left(\frac{j}{n}, \frac{S_j}{\sqrt{n}}\right)$, where n is the length of the polymer, and then take the limit as $n \to \infty$. We expect to obtain a locally Brownian object, since this same re-scaling leads to a Brownian motion in the simple random walk case, according to Donsker's Invariance Principle [Mie06, Theorem 6.7.1].

If we keep $\beta > 0$ fixed, the re-scaled (*point-to-line*) polymer trajectory will fall into a narrow corridor, that is, it will show a localized behavior. So our interest is making $\beta \to 0$ as $n \to \infty$ in a way that we can "look closer" to what happens near $\beta_c = 0$. In other words, we want to see how the polymer transitions from the strong disorder towards the weak disorder phase. In order to do that, we will scale the temperature as a function of the polymer length:

$$\beta_n = \tfrac{n^{-1/4}}{\sqrt{2}} \beta, \tag{2.6}$$

for some fixed $\beta > 0$. This is called the *intermediate disorder regime* and $n^{-1/4}$ is called the *critical scaling*.

The limiting object, as $n \to \infty$, will be locally Brownian, but with a singular measure with respect to the Wiener measure [AKQ14a, Section 4.4] (or with respect to a Brownian bridge measure in the *point-to-point* case).

A priori, it is not obvious why $n^{-1/4}$ is the correct scaling to capture the intermediate regime. To give some intuition behind this choice of scaling, we expand the exponential inside the partition function as a Taylor series

$$\begin{aligned} Z_n(\omega, \beta n^{-1/4}) &= P\left[\exp\left\{\beta n^{-1/4} \sum_{i=1}^{n} \omega(i, S_i)\right\}\right] \\ &= P\left[\sum_{k=0}^{\infty} \frac{1}{k!} \left(\beta n^{-1/4} \sum_{i=1}^{n} \omega(i, S_i)\right)^k\right] \end{aligned}$$

and we only take the terms up to first order, obtaining

$$\begin{aligned} Z_n(\omega, \beta n^{-1/4}) &\approx P\left[1 + \beta n^{-1/4} \sum_{i=1}^{n} \omega(i, S_i)\right] \\ &= 1 + \beta n^{-1/4} \sum_{i=1}^{n} \sum_{x \in \mathbb{Z}} \omega(i, S_i) P(S_i = x). \end{aligned}$$

Now, assume for simplicity that the $\omega(i, x)$ are i.i.d. with mean zero and variance one (Assumption 1). We can show the following result.

Proposition 2.5. *The term of order $n^{-1/4}$ of the partition function $Z_n(\omega, \beta_n)$ converges in law to a normal distribution. That is,*

$$\beta n^{-1/4} \sum_{i=1}^{n} \sum_{x \in \mathbb{Z}} \omega(i, S_i) P(S_i = x) \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

where $\sigma^2 = 2\beta^2/\sqrt{\pi}$.

Proof. See the Appendix. ■

Hence, at least the first order of the partition function converges in law under this critical scaling, though it might converge under other scalings[1]. In fact, it will converge up to all orders, as Theorem 1.1 states. To show convergence, the path followed by [AKQ14b] is defining a slightly modified partition function

$$\mathfrak{Z}_n^{\omega}(\beta_n) := P\left[\prod_{i=1}^{n} (1 + \beta_n \omega(i, S_i))\right]$$

which, up to first order, have the same terms as $Z_n(\omega, \beta n^{-1/4})$, but has finite expansion so it is a lot easier to study than the original partition function. This product is expanded and new terms appear, though these are still very difficult to analyze. Then, the terms of each order are studied using advanced theories, namely: Wiener chaos expansion and U-statistics. We will not elaborate more on their proof, since our objective in this thesis is to show a similar theorem but based on recent results by [QRV22] that do not depend on this chaos expansion theory that [AKQ14b] uses in a strong manner.

We end up this section with some remarks.

Remark 2.3. In [AKQ14b] they conjecture that the Intermediate Disorder Regime would also exist when dropping the condition of finite exponential moments of the environment (1.2) and only assuming finity of six moments, though in that case the normalization should be different, since the exponential moments assumption is necessary for the logarithmic moment generating function $\lambda(\beta)$ (2.1) to be defined.

[1] If we change the exponent 1/4 by a value $\alpha \in (1/4, 1]$, similar computations will tell us that the partition function will converge to a constant. If $\alpha \in (0, 1/4)$ it is not clear what the limiting object will be (See [AKQ10]).

In [DZ16], this conjecture is proved using a truncated version of the logarithmic moment generating function. They also prove similar results assuming *heavy-tailed* environments, though in those cases the regimes and limits are different. The scaling limits of the directed polymers with heavy-tailed random environments are studied with more generality in [BT19, BL21, CSZ16]. We did not work in any of those cases, but at least in the case of six finite moments we believe that our proofs can be adapted to obtain analogous results.

Remark 2.4. Closely related to the Directed Polymer in Random Environments model in dimension $1+1$ is the *stationary O'Connell–Yor model of semi-discrete directed polymers.* In [JF20] it is proved that, under the Intermediate Disorder Regime, the logarithm of the partition function of the semi-discrete directed polymer converges to the solution u of the *Stochastic Burgers Equation*

$$\partial_t u = \tfrac{1}{2}\partial_{x^2} u + u\partial_x u + \beta\partial_x \xi.$$

This equation is related to the Stochastic Heat Equation (1.6) via the transformation

$$u(t,x) = \partial_x \log \mathcal{Z}_\beta(t,x).$$

2.2 White Noise

To properly define the Stochastic Heat Equation appearing in Theorem 1.1, we first need to introduce the white noise ξ, which we do in this section. General definitions can be found in [Jan97], but we will just give the same definition as in [QRV22, Section 2.1].

Let $(\Xi, \mathcal{F}, \nu)$ be a probability space containing an independent standard Gaussian sequence. Take the measure space $([0,1]\times\mathbb{R}, \mathcal{B}([0,1]\times\mathbb{R}), \lambda)$, where λ is the Lebesgue measure. The *(space-time) white noise* defined on the probability space $(\Xi, \mathcal{F}, \nu)$ is a linear isometry

$$\xi : f \mapsto \langle f, \xi\rangle$$

from $L^2([0,1]\times\mathbb{R})$ to $(\Xi, \mathcal{F}, \nu)$, such that $\langle f, \xi\rangle$, $f \in L^2([0,1]\times\mathbb{R})$, form a mean zero Gaussian family. The isometry property says that

$$\nu[\langle f, \xi\rangle\langle g, \xi\rangle] = \langle f, g\rangle_{L^2([0,1]\times\mathbb{R})}$$

for all $f, g \in L^2([0,1]\times\mathbb{R})$. The bracket notation $\langle f, \xi\rangle$ suggests but does not rigorously mean an $L^2([0,1]\times\mathbb{R})$ inner product, since ξ cannot be realized in $L^2([0,1]\times\mathbb{R})$. To see this, suppose that $\xi \in L^2$ and take $(e_j)_{j\geq 1}$ any orthonormal basis of L^2. Then, for any $j, k \geq 1$,

$$\nu[\langle e_j, \xi\rangle\langle e_k, \xi\rangle] = \langle e_j, e_k\rangle_{L^2([0,1]\times\mathbb{R})} = \delta_{j,k} \tag{2.7}$$

implying that, by Parseval's identity,

$$\nu\left[||\xi||^2_{L^2}\right] = \nu\left[\sum_{j=1}^{\infty}\langle e_j, \xi\rangle^2\right]$$

and by the previous observation (2.7),

$$\nu\left[||\xi||_{L^2}^2\right] = \sum_{j=1}^{\infty} 1 = \infty.$$

As a consequence of the second Borel-Cantelli lemma ([Dur19, Theorem 2.3.7.]),

$$\nu(\langle e_j, \xi\rangle^2 > 1 \text{ i.o.}) = 1,$$

so $||\xi||_{L^2} = \infty$ ν-a.s and therefore $\xi \notin L^2([0,1]\times\mathbb{R})$ ν-a.s..

Now, even if the white noise cannot be realized in L^2, its inner product against functions in L^2 still make sense. The white noise can be constructed in the following elementary way. Fix again an orthonormal basis $(e_j)_j$ of L^2. From the above definitions and computations $\langle e_j, \xi\rangle$ should be independent $\mathcal{N}(0,1)$ random variables, so we define

$$\langle e_j, \xi\rangle := \xi_j,$$

where $(\xi_j)_j$ is the sequence of independent standard Gaussian variables in Ξ. For $f \in L^2([0,1]\times\mathbb{R})$, with $f = \sum_{j=1}^{\infty}\langle f, e_j\rangle e_j$, we can realize

$$\langle f, \xi\rangle := \sum_{j=1}^{\infty}\langle f, e_j\rangle \xi_j$$

which converges in $L^2(\Xi, \mathcal{F}, \nu)$ by Parseval's identity and satisfies the given definition of white noise.

For completeness, we could take $\Xi := \mathbb{R}^{\mathbb{N}}$, $\mathcal{F}$ the Borel σ-algebra on the product space, and ν the independent standard Gaussian product measure.

To conclude this section, we mention that, as a matter of fact, the white noise can be viewed as a convergent series, not in L^2 but rather in a negative Sobolev space. To be more precise, $\xi \in H_{-1-\delta,\mathrm{loc}}([0,1]\times\mathbb{R})$ for any $\delta > 0$ [JM18, Proposition C.5.].

2.3 Stochastic Heat Equation

In this thesis, we are interested in studying scaling limits of the directed polymer model as $\beta \to 0$ with a particular rate, with a suitable re-scaling of time and space. The model in $d = 1$ is expected to converge to a corresponding "continuous polymer model". As it turns out, the partition function of such a continuous polymer is described by the Stochastic Heat Equation, which we introduce now.

We consider the *stochastic heat equation* (SHE) in dimension $1+1$ with multiplicative noise:

$$\partial_t \mathcal{Z}_\beta = \tfrac{1}{2}\partial_x^2 \mathcal{Z}_\beta + \beta\xi\mathcal{Z}_\beta, \quad \mathcal{Z}_\beta(0,\cdot) = \varsigma(\cdot), \tag{2.8}$$

where $\xi \in (\Xi, \mathcal{F}, \nu)$ is a space-time Gaussian white noise and ς is a given measure on $\mathbb{R}$. The solution is known to exist for $(t,x) \in (0,\infty)\times\mathbb{R}$ (under certain conditions) and it is usually expressed as a Wiener chaos expansion (see [Jan97, Chapter 3] and [CSZ16]).

At this point, it is pertinent to define what a solution of SHE is: we say that a measurable function $\mathcal{Z}_\beta : (0,T] \times \mathbb{R} \times \Xi \to [0,\infty)$ is a *mild solution* of the *one dimensional stochastic heat equation with space-time white noise* ξ and initial condition ς, if for almost all $(t,x) \in (0,T] \times \mathbb{R}$ the random function

$$(s,y) \mapsto g(t-s, x-y)\mathcal{Z}_\beta(s,y)$$

is a square integrable, progressively measurable process, where $g(t,x) = (2\pi t)^{-1/2}\exp\{-x^2/2t\}$ is the heat kernel, and the equation

$$\mathcal{Z}_\beta(t,x) = \int_{\mathbb{R}} g(t,x-y)\mathrm{d}\varsigma(y) + \beta\int_0^t\int_{\mathbb{R}} g(t-s,x-y)\mathcal{Z}_\beta(s,y)\xi(s,y)\mathrm{d}y\mathrm{d}s, \tag{2.9}$$

is satisfied.

Classical solution

As mentioned above, the classical solution is given by a chaos representation, namely

$$\begin{aligned}\mathcal{Z}_\beta(t,x) = \sum_{k=0}^{\infty}\beta^k\int_{\Delta_{k,t}}\int_{\mathbb{R}^{k+1}} & g(t-s_k,x-y_k)g(s_k-s_{k-1},y_k-y_{k-1})\cdots g(s_1,y_1-y_0)\varsigma(y_0)\\ & \xi^{\otimes k}(s_1,\dots,s_k,y_1,\dots,y_k)d(y_0,y_1,\dots,y_k)d(s_1,\dots,s_k),\end{aligned}$$

where

$$\Delta_{k,t} = \{0 < s_1 < s_2 \ldots < s_k < t\}.$$

Note that when $k = 0$, the general term in the series just becomes $g(t,\cdot) * \varsigma$. This is obtained by expressing SHE in Duhamel form, that is, writing $\mathcal{Z}_\beta$ as in (2.9), and then iterating. This gives us a series expansion which is only defined as a limit in $L^2(\nu)$. We note that the randomness of this solution comes from the terms $\xi^{\otimes k}$ and that the convergence depends also on the initial condition ς. It is known that the solution exists and is unique for all initial conditions satisfying

$$\nu\left[\sup_{A\subset[-n,n]}\varsigma(A)\right] \leq Ce^{C'n}.$$

Properties of $\mathcal{Z}_\beta$

We consider important to enunciate some relevant properties of $\mathcal{Z}_\beta$, since they allow the *continuum random polymer* to be well defined a Markov process. Consider now $\mathcal{Z}_\beta(s,x;t,y)$ the solution of SHE starting from x at time s, that is, the unique solution of

$$\begin{cases} \frac{\partial \mathcal{Z}_\beta}{\partial t} = \frac{1}{2}\frac{\partial^2\mathcal{Z}_\beta}{\partial x^2} + \beta\mathcal{Z}_\beta\xi & (t > s, y \in \mathbb{R}) \\ \lim_{t\to s^+}\mathcal{Z}_\beta(s,x;t,y) = \delta_x(y) & \end{cases}$$

We will not prove the following statement, but a complete proof can be found in [AKQ14a, Theorem 3.1] (except for the continuity, which is based on standard arguments for SPDE's [Wal86], and the positivity, which can be found in [Mue91, MF14]).

Theorem 2.6 ([AKQ14a])**.** *There exists a version of the field $\mathcal{Z}_\beta(s,x;t,y)$ which is jointly continuous in all four variables and has the following properties:*

1. $\nu\left[\mathcal{Z}_\beta(s,x;t,y)\right] = g(t-s,y-x)$;
2. ***stationarity:*** $\mathcal{Z}_\beta(s,x;t,y) \overset{d}{=} \mathcal{Z}_\beta(s+u_0,x+z_0;t+u_0,y+z_0)$;
3. ***scaling:*** $\mathcal{Z}_\beta(sr^2,rx;r^2t,ry) \overset{d}{=} r^{-1}\mathcal{Z}_{\beta\sqrt{r}}(s,x;t,y)$;
4. ***positivity:*** *[Mue91, MF14] with ν probability one, $\mathcal{Z}_\beta(s,x;t,y)$ is strictly positive for all tuples $(s,x;t,y)$ with $0 \le s < t$,*
5. *the law of $\mathcal{Z}_\beta(s,x;t,y)/g(t-s,y-x)$ does not depend on x or y,*
6. ***independent increments:*** *for any finite disjoint $\{(s_i,t_i]\}_{i=1}^n$ and any $x_i, y_i \in \mathbb{R}$, the random variables $\{\mathcal{Z}_\beta(s_i,x_i;t_i,y_i)\}_{i=1}^n$ are mutually independent,*
7. ***Chapman-Kolmogorov equations:*** *with ν-probability one, for all $0 \le s < r < t$ and $x, y \in \mathbb{R}$,*
$$\mathcal{Z}_\beta(s,x;t,y) = \int \mathcal{Z}_\beta(s,x;r,z)\mathcal{Z}_\beta(r,z;t,y)\mathrm{d}z.$$

We introduce the notation

$$\mathcal{Z}_\beta(s,x;t,*) := \int_{\mathbb{R}} \mathcal{Z}_\beta(s,x;t,y)\mathrm{d}y,$$

which will be used next. In the polymer literature, $\mathcal{Z}_\beta(s,x;t,y)$ are called (continuum) point-to-point partition functions and $\mathcal{Z}_\beta(s,x;t,*)$ are called (continuum) point-to-line partition functions.

The Continuum Random Polymer

Conditional on the white noise ξ, let $\mathbb{P}_\beta^{\xi,t}$ be the measure on $C([0,t])$ whose finite dimensional distributions are given by

$$\mathbb{P}_\beta^{\xi,t}(X_{s_1} \in \mathrm{d}x_1, ..., X_{s_k} \in \mathrm{d}x_k) = \frac{\prod_{i=1}^{k+1} \mathcal{Z}_\beta(s_{i-1},x_{i-1};s_i,x_i)}{\mathcal{Z}_\beta(0,0;t,*)}\mathrm{d}x_1...\mathrm{d}x_k, \tag{2.10}$$

with $(s_0,x_0) = (0,0)$ and $(s_{k+1},x_{k+1}) = (t,*)$. By Chapman-Kolmogorov (Theorem 2.6), for almost every realization of ξ, these finite dimensional distributions are consistent. The *point-to-line KPZ continuum random polymer* $\mathscr{X}_t$ with time horizon t is the Markov process (for a fixed environment ξ) on $C([0,1])$ with finite dimensional marginal distributions given by (2.10).

Analogously, we define the finite dimensional marginal distributions of the *point-to-point KPZ continuum random polymer* $\mathscr{X}_{t,x}$ with endpoint (t,x) by

$$\mathbb{P}_\beta^{\xi,t,x}(X_{s_1} \in \mathrm{d}x_1, ..., X_{s_k} \in \mathrm{d}x_k) = \frac{\prod_{i=1}^{k+1} \mathcal{Z}_\beta(s_{i-1}, x_{i-1}; s_i, x_i)}{\mathcal{Z}_\beta(0,0;t,x)} \mathrm{d}x_1 ... \mathrm{d}x_k, \tag{2.11}$$

with $(s_0, x_0) = (0,0)$ and $(s_{k+1}, x_{k+1}) = (t,x)$.

2.4 Martingale representation of SHE

In this thesis we make use of Quastel, Ramírez and Virág's work [QRV22], where they start with the intuition that the solution (for $\beta = 1$ and $\varsigma = \delta_0$) could be given by Feynman-Kac formula

$$\mathcal{Z}_1(t,x) = g(t,x) \lim_{K\to\infty} Q^{(t,x)} \left[\exp\left\{\int_0^t \xi_{[K]}(s, B_s^{(t,x)})ds - \mathsf{norm}(K,\xi)\right\}\right]$$

where $\xi_{[K]}$ is some mollification of the noise, norm is some normalization to make the expression a martingale, $B^{(t,x)}$ is a Brownian bridge from 0 in time 0 to x in time t independent of ξ and $Q^{(t,x)}$ is its law. This will result true, but first, we need to provide some important definitions.

2.4.1 Brownian Bridges

Recall that a *Brownian Motion* in $\mathbb{R}$ is a continuous stochastic process $(W_t)_{t\geq 0}$ that satisfies the following three conditions:

(i) $W_0 = 0$,

(ii) for every $0 \leq t_0 \leq t_1 \leq ... \leq t_k$, the increments $W_{t_1} - W_{t_0}, ..., W_{t_k} - W_{t_{k-1}}$ are independent, and

(iii) for every $0 \leq s < t$, $W_t - W_s$ has normal distribution $\mathcal{N}(0, t-s)$.

Remark 2.5. The Brownian Motion can be properly defined in the probability space $(C([0,\infty),\mathbb{R}), \mathcal{W}, \mathbf{W})$, that is, the *Wiener space* of continuous functions endowed with the product σ-algebra $\mathcal{W}$ and the probability measure $\mathbf{W}$ determined by the finite-dimensional marginal distribution of W.

The "standard" *Brownian bridge* is a stochastic process $B_t,\ 0 \leq t \leq 1$, obtained from a Brownian motion W by

$$B_t := W_t - tW_1,\ 0 \leq t \leq 1$$

and can be thought as W conditioned to the endpoint $W_0 = W_1 = 0$. It has a singular measure with respect to $\mathbf{W}$, which we will call Q. More generally, given $T > 0$ and $z \in \mathbb{R}$, a Brownian bridge $B_t^{(T,z)}$, $0 \le t \le T$, conditioned to the endpoint $B_T^{(T,z)} = z$ is defined by

$$B_t^{(T,z)} := \sqrt{T} B_{t/T} + \tfrac{t}{T} z,\ 0 \le t \le T, \tag{2.12}$$

and we denote by $Q^{(t,x)}$ its law.

2.4.2 Quastel, Ramírez and Virág's theorem

Quastel, Ramírez and Virág [QRV22] prove the following.

Theorem 2.7 (Theorem 4 in [QRV22])**.** *The stochastic heat equation with space-time noise* (2.8)*, for* $\beta = 1$ *and initial condition* δ_0*, has the following unique solution:*

$$\mathcal{Z}_1(t,x) = g(t,x) \lim_{K\to\infty} Q^{(t,x)}\left[\exp\left\{\sum_{j=1}^{K} m_j \xi_j - \tfrac{1}{2}\sum_{j=1}^{K} m_j^2\right\}\right], \tag{2.13}$$

$$m_j = \int_0^t e_j\big(s, B_s^{(t,x)}\big)\mathrm{d}s, \tag{2.14}$$

where $(e_j)_{j\ge 1}$ *are bounded functions forming an orthonormal basis of* $L^2(\mathbb{R}^2)$ *and* $\xi_j := \langle e_j, \xi\rangle$ *(it can be any orthonormal basis, provided that the* e_j*'s satisfy the boundedness condition). Here* $B^{(t,x)}$ *is a Brownian bridge from* 0 *in time* 0 *to* x *in time* t*. The limit exists in* $L^2(\nu)$ *and coincides with the chaos expansion solution of* (2.8)*. The corresponding polymer measure* M_ξ *coincides with the continuum directed polymer constructed in [AKQ14a].*

Remark 2.6. We recall that, by Brownian scaling (Theorem 2.6, property 3), the solution for any $\beta > 0$ can be recovered by scaling space and time:

$$\mathcal{Z}_\beta(t,x) \overset{d}{=} \beta^2 \mathcal{Z}_1(\beta^4 t, \beta^2 x). \tag{2.15}$$

We state the following corollary (absent in [QRV22]) which will be useful later.

Corollary 2.7.1. *Fix* $\beta > 0$*. The stochastic heat equation with space-time noise* (2.8) *and initial condition* δ_0*, has the following unique solution:*

$$\mathcal{Z}_\beta(\tau,x) = g(\tau,x) \lim_{K\to\infty} Q^{(\tau,x)}\left[\exp\left\{\beta\sum_{j=1}^{K} m_j \xi_j - \tfrac{\beta^2}{2}\sum_{j=1}^{K} m_j^2\right\}\right], \tag{2.16}$$

$$m_j = \int_0^\tau e_j(s, B_s^{(\tau,x)})\mathrm{d}s, \tag{2.17}$$

where $(e_j)_{j\ge 1}$ *are bounded functions forming an orthonormal basis of* $L^2(\mathbb{R}^2)$ *and* $\xi_j := \langle e_j, \xi\rangle$*. Here* $B^{(\tau,x)}$ *is a Brownian bridge from* 0 *in time* 0 *to* x *in time* τ*.*

The proof of this can be found in Section 4.1

Remark 2.7. In both the Theorem 2.7 and the Corollary 2.7.1, the orthonormal basis of $L^2(\mathbb{R}^2)$ can be replaced by an orthonormal basis of $L^2([0,\tau]\times\mathbb{R})$.

It is announced in the title of the section that there is a martingale involved. In fact, most of the difficulty in the proof given by [QRV22] lies in proving the fact that $\mathcal{Z}_1(\tau, x)$ is the limit of an $L^2(\nu)$-integrable martingale in K. To give an idea, we also will prove the following proposition in Section 4.1.

Proposition 2.8. *Define the filtration $\nu_K = \sigma(\xi_1, ..., \xi_K)$. The sequence of random variables $z = (z_K)_K$ given by*

$$z_K := Q^{(\tau,x)}\left[\exp\left\{\sum_{j=1}^{K} m_j \xi_j - \tfrac{1}{2}\sum_{j=1}^{K} m_j^2\right\}\right]$$

is a martingale with respect to ν_K.

As mentioned before, the random variable $\mathcal{Z}_\beta$ is often interpreted as the partition function of a continuum random polymer, and the solution given by [QRV22], in fact, resembles the point-to-point partition function for the (discrete) random polymer (1.4). However, this interpretation comes from a decade ago; Alberts, Khanin and Quastel [AKQ14a] constructed explicitly the *Continuum Random Polymer*, which partition function is exactly $\mathcal{Z}_\beta$, and for the construction they rely on the fact that the kernels $\mathcal{Z}_\beta(s, x; t, y)$ satisfy Chapman-Kolmogorov equations (Theorem 2.6, property 7).

Remark 2.8. The classical solution of SHE in [AKQ14b] requires to integrate with respect the white noise ξ. In [QRV22] they also use the Skorokhod integral that allows them to integrate multidimensional functions with respect to ξ, but for the one theorem that we extracted from their paper, the stochastic integrals involved can be thought as Itô or Skorokhod integrals indistinctly, as they coincide in the one-dimensional case. We still needed some Gaussian Hilbert Spaces theory (see e.g. [Jan97]) to define the white noise ξ, though we will not work with it directly but only with simpler objects.

Chapter 3

Main Results

We start this chapter defining the objects that will be present in the statement of the main theorem of this thesis. Then, we will explain their relation with the original result by Alberts, Khanin and Quastel (Theorem 1.1). Finally, we give the precise statement that we will prove and explain its differences with the statement in Theorem 1.1.

3.1 Some definitions

We want to approximate the partition function of the (discrete) directed polymer by an object similar to that of (2.7), in a way that it naturally captures the intermediate disorder regime.

Extension of discrete paths

Consider any path $\mathbf{x} : \mathbb{N} \to \mathbb{Z}$. We define the natural continuous extension of $\mathbf{x}$ as the curve that interpolates between the discrete values, namely, for $t \in [0, \infty)$

$$\mathbf{x}_t := (1 - \{t\})\mathbf{x}_{\lfloor t \rfloor} + \{t\}\mathbf{x}_{\lfloor t \rfloor + 1},$$

where $\{t\} = t - \lfloor t \rfloor$. In the following, we will use the same notation for the discrete path and its continuous extension.

Re-scaled environment

We start with the environment $\omega = \{\omega(i, m) : i \in \mathbb{N}, m \in \mathbb{Z} \text{ such that } m \equiv i \pmod 2\}$ consisting of i.i.d. random variables with exponential moments of every order, mean zero, variance one (Assumption 1).

We choose the environment with this particular index set for two reasons: first, the set of points with positive probability of being visited is exactly

$$L_n = \{z \in \mathbb{Z} \ : \ P(S_n = z) > 0\} = \{z \in \mathbb{Z} : |z| \leq n \text{ and } z \equiv n \pmod 2\}.$$

The second reason is that we will need to extend the environment to $[0, \infty) \times \mathbb{R}$ in a way that we can define the Hamiltonian of every path as an integral over the interpolated

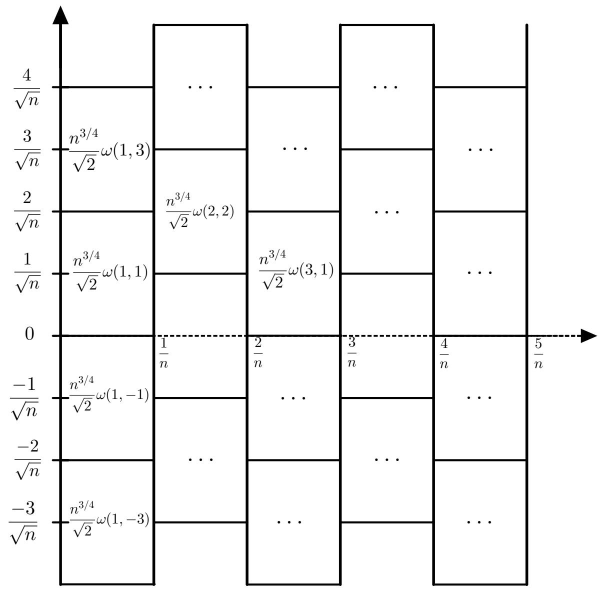

Figure 3.1: The environment $\xi^{(n)}$.

continuous path, and this choice of environment is ideal for that.

We now define the analog of the white noise, but in our discrete setting. Let $\xi^{(n)}$ be the random real-valued function on $(0, 1] \times \mathbb{R}$ given for each $(s, u) \in (0, 1] \times \mathbb{R}$ by

$$\xi^{(n)}(s, u) := 2^{-1/2} n^{3/4} \omega(sn, u\sqrt{n}),$$

where

$$\omega(s, u) := \omega(i, m)$$

if $s \in (i - 1, i]$ for some $i \in \{1, ...n\}$ and $u \in (m - 1, m + 1]$ for some $m \in \mathbb{Z}$ such that $m \equiv i \pmod 2$ (see Figure 3.1).

Remark 3.1. For all our computations, we will consider all paths shrunk into $[0, 1] \times \mathbb{R}$ diffusively. For example, we calculate the Hamiltonian potential of a path $\mathbf{x}$. Notice that when $t \in (n - 1, n]$, $\omega(t, \mathbf{x}_t) = \omega(n, \mathbf{x}_n)$ (see Figure 3.2), so

$$\begin{aligned}\int_0^1 \xi^{(n)}\left(s, \frac{\mathbf{x}_{ns}}{\sqrt{n}}\right) \mathrm{d}s &= \int_0^1 \frac{n^{3/4}}{\sqrt{2}} \omega(sn, \mathbf{x}_{sn}) \mathrm{d}s \\ &= \sum_{i=1}^n \int_{i-1}^i \frac{n^{-1/4}}{\sqrt{2}} \omega(i, \mathbf{x}_i) \mathrm{d}s = \frac{n^{-1/4}}{\sqrt{2}} \sum_{i=1}^n \omega(i, \mathbf{x}_i).\end{aligned}$$

In the sequel, we will be considering random variables $\langle \xi^{(n)}, f \rangle_{L^2}$, for $f \in L^2([0, 1] \times \mathbb{R})$. It is not clear what are these random variables, but they can be defined as follows: For

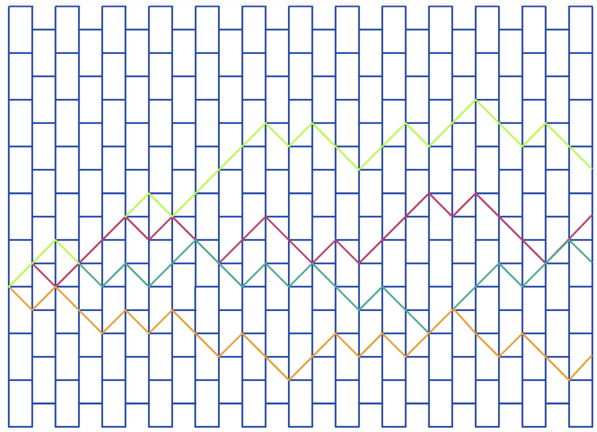

Figure 3.2: Simple random walk paths and the environment $\xi^{(n)}$.

fixed $f \in L^2$ and $M \in \mathbb{N}$, we consider $S_M := f\mathbb{1}_{[0,1]\times[-M,M]}$, where now we can write

$$\langle \xi^{(n)}, S_M \rangle = 2^{-1/2} n^{3/4} \sum_{i=1}^{n} \sum_{m \in i+2\mathbb{Z}} \omega(i,m) \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{m-1}{\sqrt{n}}}^{\frac{m+1}{\sqrt{n}}} S_M(s,u) \mathrm{d}u \mathrm{d}s,$$

which is a finite sum for every M. Then, by Assumption 1, for $M, N \in \mathbb{N}$,

$$\mathbb{P}\Big[\Big(\langle \xi^{(n)}, S_{M+N} \rangle - \langle \xi^{(n)}, S_M \rangle\Big)^2\Big]$$

is equal to

$$\frac{n^{3/2}}{2} \sum_{i=1}^{n} \sum_{m \in i+2\mathbb{Z}} \left(\int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{m-1}{\sqrt{n}}}^{\frac{m+1}{\sqrt{n}}} S_{M+N}(s,u) - S_M(s,u) \mathrm{d}u \mathrm{d}s \right)^2,$$

which by Jensen's inequality ([Dur19, Theorem 1.5.1]) with $\varphi(x) = x^2$ (note that the size of each integration region is exactly $\frac{2}{n^{3/2}}$) is less or equal to

$$\sum_{i=1}^{n} \sum_{m \in i+2\mathbb{Z}} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{m-1}{\sqrt{n}}}^{\frac{m+1}{\sqrt{n}}} \Big(S_{M+N}(s,u) - S_M(s,u) \Big)^2 \mathrm{d}u \mathrm{d}s = ||S_{M+N} - S_M||_{L^2}^2,$$

that vanishes as $M \to \infty$, since $f \in L^2$. Therefore, $\langle \xi^{(n)}, S_M \rangle$ is a Cauchy sequence in $L^2(\mathbb{P})$ and we can define

$$\langle \xi^{(n)}, f \rangle = \lim_{M \to \infty} \langle \xi^{(n)}, S_M \rangle \tag{3.1}$$

$$= 2^{-1/2} n^{3/4} \sum_{i=1}^{n} \sum_{m \in i+2\mathbb{Z}} \omega(i,m) \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{m-1}{\sqrt{n}}}^{\frac{m+1}{\sqrt{n}}} f(s,u) \mathrm{d}u \mathrm{d}s, \tag{3.2}$$

where the first equality is in $L^2(\mathbb{P})$ and the second is a convenient series representation. To check that the definition is consistent, take any other sequence $\tilde{S}_M$ of functions with compact support such that $\lim_{M\to\infty} ||\tilde{S}_M - f||_{L^2} = 0$, then

$$\mathbb{P}\Big[\Big(\langle\xi^{(n)}, S_M\rangle - \langle\xi^{(n)}, \tilde{S}_M\rangle\Big)^2\Big] \leq ||S_M - \tilde{S}_M||^2,$$

where the inequality comes from the same argument with Jensen's inequality used some lines above, and using the triangle inequality (in $L^2([0,1]\times\mathbb{R})$)

$$\mathbb{P}\Big[\Big(\langle\xi^{(n)}, S_M\rangle - \langle\xi^{(n)}, \tilde{S}_M\rangle\Big)^2\Big] \leq \big(||S_M - f|| + ||\tilde{S}_M - f||\big)^2 \overset{M\to\infty}{\longrightarrow} 0,$$

so the definition of $\langle\xi^{(n)}, f\rangle$ as a limit in $L^2(\mathbb{P})$ is unique. Using the argument with Jensen's inequality again, we obtain the useful bound

$$\mathbb{P}\Big[\langle\xi^{(n)}, f\rangle^2\Big] \leq ||f||_{L^2}^2, \text{ for all } f \in L^2([0,1]\times\mathbb{R}). \tag{3.3}$$

In particular, if we fix $\{e_j\}_{j\in\mathbb{N}}$ any orthonormal basis of $L^2([0,1]\times\mathbb{R})$ consisting of bounded continuous functions, we will set $\xi_j^{(n)} := \langle\xi^{(n)}, e_j\rangle$.

The coefficients $m_j^{(n)}$

Take $\{e_j\}_{j\in\mathbb{N}}$ any basis of $L^2([0,1]\times\mathbb{R})$ consisting of bounded continuous functions. For a given discrete path $\mathbf{x} : \mathbb{N} \to \mathbb{Z}$, we define the quantities

$$m_j^{(n)} = m_j^{(n)}(\mathbf{x}) := \int_0^1 e_j\left(\frac{sn}{n}, \frac{\mathbf{x}_{sn}}{\sqrt{n}}\right) \mathrm{d}s.$$

Quasi-partition functions

Having introduced the re-scaled environment $\xi^{(n)}$ and the coefficients $m_j^{(n)}$, we are now ready to define the approximations of the polymer partition function that we will work with.

For $|z| \leq n$, we define $[z]_n$ as the unique point in $L_n \cap (z-1, z+1]$.

For each fixed $K \in \mathbb{N}$, we consider the *point-to-point quasi-partition function*

$$\zeta_{K,x}^{(m,n)}(\beta) := P\Big[\exp\Big\{\beta\sum_{j=1}^{K} m_j^{(n)}\xi_j^{(m)} - \Lambda_{K,x}^{(m,n)}(\beta)\Big\}\mathbb{1}_{S_n=[\sqrt{n}x]_n}\Big], \tag{3.4}$$

where

$$\Lambda_{K,x}^{(m,n)}(\beta) := \log\mathbb{P}\Big[\exp\Big\{\beta\sum_{j=1}^{K} m_j^{(n)}\xi_j^{(m)}\Big\}\Big]$$

is a normalization and the $m_j^{(n)}$'s are evaluated in the random walk S. Similarly, we consider the *point-to-line quasi-partition function*

$$\zeta_K^{(m,n)}(\beta) := P\Big[\exp\Big\{\beta\sum_{j=1}^{K} m_j^{(n)}\xi_j^{(m)} - \Lambda_K^{(m,n)}(\beta)\Big\}\Big], \tag{3.5}$$

where now, the normalization is so that $\mathbb{P}\big[\zeta_K^{(m,n)}(\beta)\big] = 1$.

Remark 3.2. These quasi-partition functions do not satisfy a martingale property in any of the indexes n, m, K.

3.2 The limit of the quasi-partition function is the partition function (Heuristics)

We defined the point-to-point quasi-partition function (3.4) the way we did for two main reasons. The first one is that $\beta\sum_{j=1}^{K} m_j^{(n)}\xi_j^{(m)}$ will converge to $\beta\sum_{j=1}^{K} m_j\xi_j$ in the double limit as $n, m \to \infty$. The second one, is that the normalization makes the $\mathbb{P}$-norm equal to 1 in the point-to-line case.

Now, if we take $K \to \infty$, fix m and set $n = m$, we expect that the limit of the quasi-partition function is the partition function. Precisely, under Assumption 1 we should have that for any fixed $\beta > 0$, $x \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $|x| \le \sqrt{n}$, the point-to-point quasi-partition function $\zeta_{K,x}^{(n,n)}(\beta)$ converges $\mathbb{P}$-almost surely to the corresponding usual point-to-point partition function as $K \to \infty$. In other words, we should have the following $\mathbb{P}$-almost sure limit

$$\lim_{K\to\infty} \zeta_{K,x}^{(n,n)}(\beta) = Z_{n,[x\sqrt{n}]_n}(\omega, \beta_n) e^{-n\lambda(\beta_n)},$$

where $\beta_n = \frac{n^{-1/4}}{\sqrt{2}}\beta$. And if this turns out to be correct, then we would have the following limit as an immediate corollary, for any $\beta > 0$ and $n \in \mathbb{N}$:

$$\lim_{K\to\infty} \zeta_K^{(n,n)}(\beta) = Z_n(\omega, \beta_n) e^{-n\lambda(\beta_n)}.$$

We provide some heuristics on the possible proof of the first limit.

Heuristics. We begin with the convergence of $\sum m_j^{(n)}\xi_j^{(n)}$. Recall the definitions

$$m_j^{(n)} = \int_0^1 e_j\left(s, \tfrac{S_{ns}}{\sqrt{n}}\right) \mathrm{d}s \text{ and } \xi_j^{(n)} = \langle e_j, \xi^{(n)}\rangle.$$

Fix $\mathbf{x}$ any path of the simple random walk, then

$$\begin{aligned}\sum_{j=1}^{K}\left(\int_0^1 e_j(s, \mathbf{x}_s)\mathrm{d}s\right)\left\langle e_j, \xi^{(n)}\right\rangle &= \sum_{j=1}^{K}\int_0^1 \langle e_j, \xi^{(n)}\rangle e_j(s, \mathbf{x}_s)\mathrm{d}s \\ &= \int_0^1 \sum_{j=1}^{K}\langle e_j, \xi^{(n)}\rangle e_j(s, \mathbf{x}_s)\mathrm{d}s\end{aligned}$$

and we would like to take the limit as $K \to \infty$

$$\sum_{j=1}^{\infty} \left(\int_0^1 e_j(s, \mathbf{x}_s) \mathrm{d}s \right) \left\langle e_j, \xi^{(n)} \right\rangle = \int_0^1 \sum_{j=1}^{\infty} \langle e_j, \xi^{(n)} \rangle e_j(s, \mathbf{x}_s) \mathrm{d}s$$
$$= \int_0^1 \xi^{(n)}(s, \mathbf{x}_s) \mathrm{d}s,$$

and by Remark 3.1, this is equal to

$$\frac{n^{-1/4}}{\sqrt{2}} \sum_{i=1}^{n} \omega(i, \mathbf{x}_i).$$

The problem with this computations is that the convergence

$$\lim_{K \to \infty} \sum_{j=1}^{K} \langle e_j, f \rangle e_j = f$$

in $L^2([0,1] \times \mathbb{R})$ does not imply point-wise convergence, nor convergence of the integral over paths (which have zero measure in $[0,1] \times \mathbb{R}$), so we are not really able to write the previous equalities in any sense.

Denote by $f_K(\mathbf{x})$ the sum

$$\beta \sum_{j=1}^{K} \left(\int_0^1 e_j(s, \mathbf{x}_s) \mathrm{d}s \right) \xi_j^{(n)}$$

and $f(\mathbf{x}) := \beta \int_0^1 \xi^{(n)}(s, \mathbf{x}_s) \mathrm{d}s$. As a direct consequence of the previous calculations, for any fixed path $\mathbf{x}$,

$$e^{f_K(\mathbf{x})} \left(\mathbb{P}\left[e^{f_K(\mathbf{x})} \right] \right)^{-1} \mathbb{1}_{\mathbf{x}_n = [\sqrt{n}x]_n} \xrightarrow{K \to \infty} e^{f(\mathbf{x})} \left(\mathbb{P}\left[e^{f(\mathbf{x})} \right] \right)^{-1} \mathbb{1}_{\mathbf{x}_n = [\sqrt{n}x]_n}.$$

Now, we take the average over the 2^n possible paths of the simple random walk and get

$$P\left[e^{f_K(S)} \left(\mathbb{P}\left[e^{f_K(S)} \right] \right)^{-1} \mathbb{1}_{S_n = [\sqrt{n}x]_n} \right] \xrightarrow{K \to \infty} P\left[e^{f(S)} \left(\mathbb{P}\left[e^{f(S)} \right] \right)^{-1} \mathbb{1}_{S_n = [\sqrt{n}x]_n} \right].$$

Note that, as the $\omega(i, m)$ are i.i.d.,

$$\mathbb{P}\Big[\exp\Big\{ \beta_n \sum_{i=1}^{n} \omega(i, S_i) \Big\} \Big] = \prod_{i=1}^{n} \mathbb{P}\Big[\exp\Big\{ \beta_n \omega(i, S_i) \Big\} \Big]$$
$$= \exp\{ n \lambda(\beta_n) \},$$

by definition of $\lambda(\beta)$ (2.1), which is independent of P. Therefore,

$$P\left[e^{f(S)}\left(\mathbb{P}\left[e^{f(S)}\right]\right)^{-1}\mathbb{1}_{S_n=[\sqrt{n}x]_n}\right] = P\left[\exp\left\{\beta_n\sum_{i=1}^{n}\omega(i,S_i)\right\}\mathbb{1}_{S_n=[\sqrt{n}x]_n}\right]e^{-n\lambda(\beta_n)}$$
$$= Z_{n,[x\sqrt{n}]_n}(\omega,\beta_n)e^{-n\lambda(\beta_n)},$$

and we obtain the desired result. ∎

3.3 Statement of main results

Even if we are not able to prove the previous heuristics (yet), we will prove them in the case of a particular basis of $L^2([0,1]\times\mathbb{R})$, which will be defined in Section 4.2. The following proposition will be proved there.

Proposition 3.1. *Set $\beta_n := \frac{n^{-1/4}}{\sqrt{2}}\beta$ for fixed $\beta > 0$. There exists $(f_j)_{j\geq 1}$ an orthonormal basis of $L^2([0,1]\times\mathbb{R})$ consisting of bounded functions, such that the following is true: Under Assumption 1, for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $|x| \leq \sqrt{n}$, the point-to-point quasi-partition function $\zeta_{K,x}^{(n,n)}(\beta)$ converges $\mathbb{P}$-almost surely to the corresponding usual point-to-point partition function as $K \to \infty$. In other words, we have the following $\mathbb{P}$-almost sure limit*

$$\lim_{K\to\infty}\zeta_{K,x}^{(n,n)}(\beta) = Z_{n,[x\sqrt{n}]_n}(\omega,\beta_n)e^{-n\lambda(\beta_n)}.$$

And as an immediate corollary, we have the following.

Corollary 3.1.1. *Set $\beta_n := \frac{n^{-1/4}}{\sqrt{2}}\beta$ for fixed $\beta > 0$. There exists $(f_j)_{j\geq 1}$ an orthonormal basis of $L^2([0,1]\times\mathbb{R})$ consisting of bounded functions, such that the following is true: Under Assumption 1, we have that for any $n \in \mathbb{N}$, the point-to-line quasi-partition function $\zeta_K^{(n,n)}(\beta)$ converges $\mathbb{P}$-almost surely to the corresponding usual partition function as $K \to \infty$. In other words, we have the following $\mathbb{P}$-almost sure limit*

$$\lim_{K\to\infty}\zeta_K^{(n,n)}(\beta) = Z_n(\omega,\beta_n)e^{-n\lambda(\beta_n)}.$$

This allows us to re-state the result of Alberts, Khanin and Quastel (Theorem 1.1) in terms of the quasi-partition functions $\zeta_{K,x}^{(n,n)}$ and $\zeta_K^{(n,n)}$. Indeed, Theorem 1.1 states that, under Assumption 1, for any $\beta > 0$,

$$\lim_{n\to\infty}\lim_{K\to\infty}\zeta_K^{(n,n)}(\beta) \overset{d}{=} \int_{\mathbb{R}}\mathcal{Z}_\beta(1,x)\mathrm{d}x$$

and for all $x \in \mathbb{R}$

$$\lim_{n\to\infty}\lim_{K\to\infty}\tfrac{\sqrt{n}}{2}\zeta_{K,x}^{(n,n)}(\beta) \overset{d}{=} \mathcal{Z}_\beta(1,x).$$

Our main objective is to prove a similar limit, but letting $n \to \infty$ first and then $K \to \infty$. The idea is then to show that one can interchange the limits to truly recover Theorem 1.1, using some form of uniform convergence in one of the variables, but this will be the subject of future work.

However, we were still not able to prove the limits

$$\lim_{K\to\infty}\lim_{n\to\infty}\zeta_K^{(n,n)}(\beta)\overset{d}{=}\int_{\mathbb{R}}\mathcal{Z}_\beta(1,x)\mathrm{d}x \quad\text{and}\quad \lim_{K\to\infty}\lim_{n\to\infty}\tfrac{\sqrt{n}}{2}\zeta_{K,x}^{(n,n)}(\beta)\overset{d}{=}\mathcal{Z}_\beta(1,x)$$

directly either, which is why we used two indexes, n and m (instead of just n), when we defined the quasi-partition functions. When we decouple the $m_j^{(n)}$ and $\xi_j^{(m)}$ using different indexes, we can establish the desired convergence by first letting $m\to\infty$, then $n\to\infty$ and finally $K\to\infty$. Our results are condensed in the following statement, valid for any choice of orthonormal basis $(e_j)_j$ consisting of **continuous** and bounded functions.

Theorem 3.2. *Under Assumption 1, we have the convergence in distribution of the point-to-point quasi-partition function to the corresponding approximation of the solution of SHE, that is, for any fixed $x\in\mathbb{R}$,*

$$\lim_{n\to\infty}\lim_{m\to\infty}\tfrac{\sqrt{n}}{2}\zeta_{K,x}^{(m,n)}(\beta)\overset{d}{=}g(1,x)Q^{(1,x)}\left[\exp\left\{\beta\sum_{j=1}^K m_j\xi_j-\tfrac{\beta^2}{2}\sum_{j=1}^K m_j^2\right\}\right],$$

where the objects in the RHS are defined in Theorem 2.7. This iterated limit in distribution means that for every bounded and continuous function h,

$$\lim_{n\to\infty}\lim_{m\to\infty}\mathbb{P}\left[h\left(\tfrac{\sqrt{n}}{2}\zeta_{K,x}^{(m,n)}\right)\right]=\nu\left[h\left(g(1,x)Q^{(1,x)}\left[\exp\left\{\beta\sum_{j=1}^K m_j\xi_j-\tfrac{\beta^2}{2}\sum_{j=1}^K m_j^2\right\}\right]\right)\right]$$

In particular, if we let also $K\to\infty$, we recover the solution of the Stochastic Heat Equation, i.e.,

$$\lim_{K\to\infty}\lim_{n\to\infty}\lim_{m\to\infty}\tfrac{\sqrt{n}}{2}\zeta_{K,x}^{(m,n)}\overset{d}{=}\mathcal{Z}_\beta(1,x).$$

Furthermore, the same is true for the point-to-line case, that is,

$$\lim_{n\to\infty}\lim_{m\to\infty}\zeta_K^{(m,n)}(\beta)\overset{d}{=}\int_{\mathbb{R}}g(1,x)Q^{(1,x)}\left[\exp\left\{\beta\sum_{j=1}^K m_j\xi_j-\tfrac{\beta^2}{2}\sum_{j=1}^K m_j^2\right\}\right]\mathrm{d}x.$$

and, as a consequence,

$$\lim_{K\to\infty}\lim_{n\to\infty}\lim_{m\to\infty}\zeta_K^{(m,n)}\overset{d}{=}\int_{\mathbb{R}}\mathcal{Z}_\beta(1,x)\mathrm{d}x.$$

Theorem 3.2 will be a consequence of the following four results. The first two results, which will be proved in Section 4.3, deal with the limit as $m\to\infty$ for n and K fixed: the first of them being for the point-to-point case and the second one for the point-to-line case.

Proposition 3.3. *Let $\beta>0$, let $K,n\in\mathbb{N}$ and $x\in\mathbb{R}$ such that $|x|\leq n$. Then, under*

Assumption 1, we have the convergence in distribution:

$$\lim_{m\to\infty} P^{(n,x)}\left[e^{\beta\sum_{j=1}^K m_j^{(n)}\xi_j^{(m)}-\Lambda_{K,x}^{(n,m)}(\beta)}\right] \stackrel{d}{=} P^{(n,x)}\left[e^{\beta\sum_{j=1}^K m_j^{(n)}\xi_j-\tilde{\Lambda}_{K,x}^{(n)}(\beta)}\right],$$

where

$$\tilde{\Lambda}_{K,x}^{(n)}(\beta) := \log\nu\left[e^{\beta\sum_{j=1}^K m_j^{(n)}\xi_j}\right]$$

and $P^{(n,x)}$ is the law P of the random walk S conditioned on the endpoint $S_n = [\sqrt{n}x]_n$.

Proposition 3.4. *Let $\beta > 0$, let $K, n \in \mathbb{N}$. Then, under Assumption 1, we have convergence in distribution:*

$$\lim_{m\to\infty} P\left[e^{\beta\sum_{j=1}^K m_j^{(n)}\xi_j^{(m)}-\Lambda_{K}^{(n,m)}(\beta)}\right] \stackrel{d}{=} P\left[e^{\beta\sum_{j=1}^K m_j^{(n)}\xi_j-\tilde{\Lambda}_{K}^{(n)}(\beta)}\right]$$

where

$$\tilde{\Lambda}_{K}^{(n)}(\beta) := \log\nu\left[e^{\beta\sum_{j=1}^K m_j^{(n)}\xi_j}\right].$$

Remark 3.3. As the random variables ξ_j are independent and distributed as $\mathcal{N}(0,1)$, we actually have

$$\tilde{\Lambda}_{K,x}^{(n)}(\beta) = \tfrac{\beta^2}{2}\sum_{j=1}^K [m_j^{(n)}]^2.$$

and also

$$\tilde{\Lambda}_{K}^{(n)}(\beta) = \tfrac{\beta^2}{2}\sum_{j=1}^K [m_j^{(n)}]^2.$$

Then, the next two results, which will be proved in Section 4.4, deal with the limit as $n \to \infty$ and K fixed for the limiting objects in Proposition 3.3 and 3.4, respectively.

Proposition 3.5. *For any fixed realization of ξ, for $K \in \mathbb{N}$, $\beta > 0$ and $x \in \mathbb{R}$,*

$$\lim_{n\to\infty} P^{(n,x)}\Big[\exp\Big\{\beta\sum_{j=1}^K m_j^{(n)}\xi_j - \tfrac{\beta^2}{2}\sum_{j=1}^K [m_j^{(n)}]^2\Big\}\Big] = Q^{(1,x)}\Big[\exp\Big\{\beta\sum_{j=1}^K m_j\xi_j - \tfrac{\beta^2}{2}\sum_{j=1}^K m_j^2\Big\}\Big],$$

where $P^{(n,x)}$ is the law P of the random walk S conditioned on the endpoint $S_n = [\sqrt{n}x]_n$ and the convergence is in distribution.

Proposition 3.6. *For any fixed realization of ξ, for $K \in \mathbb{N}$ and $\beta > 0$,*

$$\begin{aligned}\lim_{n\to\infty} P\Big[\exp\Big\{\beta\sum_{j=1}^K m_j^{(n)}\xi_j - \tfrac{\beta^2}{2}\sum_{j=1}^K [m_j^{(n)}]^2\Big\}\Big]& \\ = \int_{\mathbb{R}} g(1,x) Q^{(1,x)}\Big[\exp\Big\{\beta\sum_{j=1}^K m_j\xi_j - \tfrac{\beta^2}{2}\sum_{j=1}^K m_j^2\Big\}\Big]\mathrm{d}x,\end{aligned}$$

where the convergence is in distribution.

Finally, we obtain Theorem 3.2 by taking the third limit as $K \to \infty$, which is exactly the Corollary 2.7.1 of Theorem 2.7.

Chapter 4

Proof of the main theorem

4.1 Martingale representation of SHE for arbitrary β

In this section we give the proofs of Corollary 2.7.1 and of Proposition 2.8.

Proof of Corollary 2.7.1. Fix $(\tau, x) \in (0, \infty) \times \mathbb{R}$ and $(e_j)_{j \in \mathbb{N}}$ an orthonormal basis of $L^2(\mathbb{R}^2)$. By Remark 2.6, the Brownian scaling of $\mathcal{Z}_\beta$ implies that

$$\mathcal{Z}_\beta(\tau, x) \overset{d}{=} \beta^2 g(\beta^4\tau, \beta^2 x) \lim_{K \to \infty} Q^{(\beta^4\tau, \beta^2 x)}\left[\exp\left\{\sum_{j=1}^{K} \tilde{m}_j \tilde{\xi}_j - \frac{1}{2}\sum_{j=1}^{K} \tilde{m}_j^2\right\}\right],$$

with

$$\tilde{m}_j := \int_0^{\beta^4\tau} \tilde{e}_j(s, B_s^{(\beta^4\tau, \beta^2 x)}) \mathrm{d}s, \quad \tilde{\xi}_j := \langle \tilde{e}_j, \xi \rangle,$$

where $(\tilde{e}_j)_j$ can be chosen to be the following orthonormal basis of $L^2(\mathbb{R}^2)$: for every j, define $\tilde{e}_j : \mathbb{R}^2 \to \mathbb{R}$ by

$$\tilde{e}_j(s, t) := \beta^{-3} e_j(\beta^{-4}s, \beta^{-2}t).$$

It is easy to check that

$$\langle \tilde{e}_j, \tilde{e}_k \rangle = \delta_{jk}.$$

Now, for every j,

$$\begin{aligned}\tilde{m}_j &= \int_0^{\beta^4\tau} \tilde{e}_j(s, B_s^{(\beta^4\tau, \beta^2 x)}) \mathrm{d}s \\ &= \beta \int_0^{\tau} e_j(t, \beta^{-2} B_{\beta^4 t}^{(\beta^4\tau, \beta^2 x)}) \mathrm{d}t,\end{aligned}$$

where we used the definition of $\tilde{e}_j$ and the change of variables $s = \beta^4 t$. Using the definition of Brownian bridges that we give in (2.12), for $s \in [0,\tau]$,

$$\begin{aligned}\beta^{-2}B^{(\beta^4\tau,\beta^2 x)}_{\beta^4 s} &\stackrel{d}{=} \beta^{-2}\Big(\sqrt{\beta^4\tau}B_{\beta^4 s/\beta^4\tau} + \tfrac{\beta^4 s}{\beta^4\tau}\beta^2 x\Big)\\ &= \sqrt{\tau}B_{s/\tau} + \tfrac{s}{\tau}x \stackrel{d}{=} B^{(\tau,x)}_s,\end{aligned}$$

so, for every j,

$$\tilde{m}_j = \beta m_j.$$

Also, we have that $(\tilde{\xi}_j)_{j\in\mathbb{N}} \stackrel{d}{=} (\xi_j)_{j\in\mathbb{N}}$, because both are families of independent $\mathcal{N}(0,1)$ Gaussian variables, and as $\beta^2 g(\beta^4\tau, \beta^2 x) = g(\tau, x)$, we obtain the equality in distribution (2.16). ■

Proof of Proposition 2.8. It is clear that z is ν_K-adapted and that $\nu[|z_K|] < \infty$. Now fix $K \in \infty$, then

$$\begin{aligned}\nu[z_{K+1}|\nu_K] &= \nu\left[Q^{(t,x)}\left[\exp\left\{\sum_{j=1}^{K+1} m_j\xi_j - \tfrac{1}{2}\sum_{j=1}^{K+1} m_j^2\right\}\right]\middle|\nu_K\right]\\ &= Q^{(t,x)}\left[\nu\left[\exp\left\{\sum_{j=1}^{K+1} m_j\xi_j - \tfrac{1}{2}\sum_{j=1}^{K+1} m_j^2\right\}\middle|\nu_K\right]\right].\end{aligned}$$

Here, as ξ_j is ν_K-measurable for $j \le K$ and ξ_{K+1} is independent of ν_K, we obtain

$$\begin{aligned}\nu[z_{K+1}|\nu_K] &= Q^{(t,x)}\left[\exp\left\{\sum_{j=1}^{K} m_j\xi_j - \tfrac{1}{2}\sum_{j=1}^{K} m_j^2\right\}\nu\left[\exp\left\{m_{K+1}\xi_{K+1}\right\}\middle|\nu_K\right]\exp\left\{-\tfrac{m_{K+1}^2}{2}\right\}\right]\\ &= Q^{(t,x)}\left[\exp\left\{\sum_{j=1}^{K} m_j\xi_j - \tfrac{1}{2}\sum_{j=1}^{K} m_j^2\right\}\nu\left[\exp\left\{m_{K+1}\xi_{K+1}\right\}\right]\exp\left\{-\tfrac{m_{K+1}^2}{2}\right\}\right].\end{aligned}$$

As ξ_{K+1} is a standard Gaussian variable, $\nu\Big[\exp\{m_{K+1}\xi_{K+1}\}\Big] = \exp\Big\{\frac{m_{K+1}^2}{2}\Big\}$, and we obtain $\nu[z_{K+1}|\nu_K] = z_K$. ■

4.2 The limit of the quasi-partition function is the partition function

The objective of this section is proving Proposition 3.1. For this, we need to introduce the orthonormal basis $(f_j)_{j\ge 1}$ of $L^2([0,1]\times\mathbb{R})$ that we will make use of.

Haar wavelets

In [Haa10], it is introduced a particular class χ of functions over the interval $[0,1]$, that turns out to be an orthonormal basis of $L^2([0,1])$. These functions are usually called *Haar wavelets.* We denote the elements of χ by $\chi_{n,k}$, with $n, k \in \mathbb{N}\cup\{0\}$, and $0 \le k < 2^{n-1}$.

$\chi_{0,0}$ is the constant function equal to 1, while $\chi_{n,k}$, for $n > 0$ and $0 \le k < 2^{n-1}$ are defined by

$$\chi_{n,k}(t) := \begin{cases} \sqrt{2^{n-1}}, & \frac{2k}{2^n} \le t < \frac{2k+1}{2^n}, \\ -\sqrt{2^{n-1}}, & \frac{2k+1}{2^n} \le t < \frac{2k+2}{2^n}, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that

$$\int_0^1 \chi_{n,k} := \begin{cases} 1, & n = k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

and that

$$\int_0^1 \chi_{n_1,k_1}\chi_{n_2,k_2} = \delta_{n_1 n_2}\delta_{k_1 k_2},$$

thus forming an orthonormal system.

Also, if $f \in L^2([0,1])$ is perpendicular to every $\chi_{n,k} \in \chi$, then it is easy to extend the perpendicularity to indicator functions of intervals, then indicator functions of measurable sets. Using that simple functions (linear combinations of indicator functions of measurable sets) are dense in L^2 ([Ran02, Theorem 8.6.1]), we get that $f \equiv 0$. This is equivalent to χ being an orthonormal basis of $L^2([0,1])$. We re-enumerate the family χ to have $\mathbb{N}$ as index set.

The basis $(f_j)_{j\ge1}$

It is a simple functional analysis exercise to prove that the family $\chi^{\otimes 2}$ of functions $\chi_i \otimes \chi_j$, with $i, j \in \mathbb{N}$, defined by $\chi_i \otimes \chi_j(s,t) := \chi_i(s)\chi_j(t)$ for $(s,t) \in [0,1]^2$, provides an orthonormal basis for $L^2([0,1]^2)$. We re-enumerate the family $\chi^{\otimes 2}$ to have $\mathbb{N}$ as index set.

Fix $n \in \mathbb{N}$. The idea is to replicate the basis $\chi^{\otimes 2}$ in every rectangle where $\xi^{(n)}$ is constant and thus obtain a (useful) orthonormal basis for $L^2([0,1] \times \mathbb{R})$. The following is just the precise definition.

For every function $\chi_j \in \chi^{\otimes 2}$ we define the function $\tilde{\chi}_j : [-n^{-1}, 0] \times [-n^{-1/2}, n^{-1/2}] \to \mathbb{R}$ by

$$\tilde{\chi}_j(s,t) = \frac{n^{3/4}}{\sqrt{2}}\chi_j(ns+1, (n^{1/2}t+1)/2), \quad \text{for } (s,t) \in [-n^{-1}, 0] \times [-n^{-1/2}, n^{-1/2}].$$

The collection of functions $\tilde{\chi}_j$ is an orthonormal basis of $L^2([-n^{-1}, 0] \times [-n^{-1/2}, n^{-1/2}])$.

Finally, we consider the family of indexes $I_n := \{(i,m) \in \mathbb{N} \times \mathbb{Z} \ : \ i \le n, \ m \in i + 2\mathbb{Z}\}$, and for every $j \in \mathbb{N}$ and $(i,m) \in I_n$ we define $f_{j,(i,m)} : [0,1] \times \mathbb{R} \to \mathbb{R}$ by

$$f_{j,(i,m)}(s,t) := \begin{cases} \tilde{\chi}_j(s - \frac{i}{n}, t - \frac{m}{\sqrt{n}}) & \text{if } (s,t) \in [\frac{i-1}{n}, \frac{i}{n}] \times [\frac{m-1}{\sqrt{n}}, \frac{m+1}{\sqrt{n}}], \\ 0 & \text{otherwise.} \end{cases}$$

This way, the functions $f_{j,(i,m)}$, for $j \in \mathbb{N}$ and $(i,m) \in I_n$, form an orthonormal basis of $L^2([0,1] \times \mathbb{R}$. We finally re-enumerate the new class of functions to have $\mathbb{N}$ as index set. This is the orthonormal basis $(f_j)_{j\ge1}$ of $L^2([0,1] \times \mathbb{R})$ consisting of bounded functions that we will be considering.

Proof of Proposition 3.1

We now make use of the basis $(f_j)_j$ and the properties of Haar functions to prove the Proposition 3.1. It will be very similar to the heuristics in Section 3.2.

Proof of Proposition 3.1. As n is fixed, the random walk S has 2^n possible trajectories and can visit only $N := \sum_{i=1}^n i+1 = \frac{n^2+3n}{2}$ sites of the form $(i,m) \in \mathbb{N}\times\mathbb{Z}$. Take $K \in \mathbb{N}$ such that for all these N indexes (i,m) we have

$$g_{i,m} := \tfrac{n^{3/4}}{\sqrt{2}} \mathbb{1}_{\left[\frac{i-1}{n},\frac{i}{n}\right]\times\left[\frac{m-1}{\sqrt{n}},\frac{m+1}{\sqrt{n}}\right]} \in \big\{f_1,...,f_K\big\}. \tag{4.1}$$

Now fix a path $\mathbf{x}$ of the simple random walk. By the nature of the basis $(f_j)_j$, only the functions $g_{i,m}$, for $(i,m)=(i,\mathbf{x}_i)$, are such that (possibly) $m_j^{(n)}(\mathbf{x})\xi_j^{(n)} \neq 0$, so we obtain the equality

$$\sum_{j=1}^K m_j^{(n)}(\mathbf{x})\xi_j^{(n)} = \sum_{i=1}^n \Big(\int_0^1 g_{i,\mathbf{x}_i}(s,\mathbf{x}_s)\mathrm{d}s\Big)\big\langle g_{i,\mathbf{x}_i},\xi^{(n)}\big\rangle.$$

Using (4.1) and the series expansion (3.1), the expressions in braces become

$$\begin{aligned}\big\langle g_{i,\mathbf{x}_i},\xi^{(n)}\big\rangle &= \tfrac{n^{3/4}}{\sqrt{2}}\omega(i,\mathbf{x}_i)\int_{\frac{i-1}{n}}^{\frac{i}{n}}\int_{\frac{x_i-1}{\sqrt{n}}}^{\frac{x_i+1}{\sqrt{n}}} \tfrac{n^{3/4}}{\sqrt{2}}\mathrm{d}u\mathrm{d}s\\ &= \omega(i,\mathbf{x}_i).\end{aligned}$$

Using again the definition of $g_{i,\mathbf{x}_i}$, we get

$$\begin{aligned}\sum_{i=1}^n \Big(\int_0^1 g_{i,\mathbf{x}_i}(s,\mathbf{x}_s)\mathrm{d}s\Big)\big\langle g_{i,\mathbf{x}_i},\xi^{(n)}\big\rangle &= \sum_{i=1}^n \omega(i,\mathbf{x}_i)\int_0^1 \tfrac{n^{3/4}}{\sqrt{2}}\mathbb{1}_{\left[\frac{i-1}{n},\frac{i}{n}\right]\times\left[\frac{\mathbf{x}_i-1}{\sqrt{n}},\frac{\mathbf{x}_i+1}{\sqrt{n}}\right]}(s,\mathbf{x}_s)\mathrm{d}s\\ &= \sum_{i=1}^n \omega(i,\mathbf{x}_i)\int_{\frac{i-1}{n}}^{\frac{i}{n}} \tfrac{n^{3/4}}{\sqrt{2}}\mathrm{d}s = \tfrac{n^{-1/4}}{\sqrt{2}}\sum_{i=1}^n \omega(i,\mathbf{x}_i).\end{aligned}$$

Therefore,

$$\lim_{K\to\infty}\sum_{j=1}^K m_j^{(n)}(\mathbf{x})\xi_j^{(n)} = \tfrac{n^{-1/4}}{\sqrt{2}}\sum_{i=1}^n \omega(i,\mathbf{x}_i).$$

Now notice that, as the variables $\omega(i,m)$ are i.i.d.,

$$\begin{aligned}\mathbb{P}\Big[\exp\Big\{\beta_n\sum_{i=1}^n\omega(i,\mathbf{x}_i)\Big\}\Big] &= \prod_{i=1}^n \mathbb{P}\Big[\exp\Big\{\beta_n\omega(i,\mathbf{x}_i)\Big\}\Big]\\ &= \exp\{n\lambda(\beta_n)\},\end{aligned}$$

by definition of $\lambda(\beta)$ (2.1), which is independent of P.

For each $K \in \mathbb{N}$ and path $\mathbf{x}$, denote by $\phi_K(\mathbf{x})$ the sum

$$\beta \sum_{j=1}^{K} m_j^{(n)}(\mathbf{x}) \xi_j^{(n)}.$$

As a direct consequence of the previous calculations, for each fixed path $\mathbf{x}$,

$$e^{\phi_K(\mathbf{x})} \Big(\mathbb{P}\big[e^{\phi_K(\mathbf{x})} \big] \Big)^{-1} \mathbb{1}_{\mathbf{x}_n = [\sqrt{n}x]_n} \overset{K\to\infty}{\longrightarrow} e^{\beta_n \sum_{i=1}^n \omega(i,\mathbf{x}_i) - n\lambda(\beta_n)} \mathbb{1}_{\mathbf{x}_n = [\sqrt{n}x]_n},$$

and taking average over the 2^n possible paths of the simple random walk S we get

$$P\Big[e^{\phi_K(S)} \Big(\mathbb{P}\big[e^{\phi_K(S)} \big] \Big)^{-1} \mathbb{1}_{S_n = [\sqrt{n}x]_n} \Big] \overset{K\to\infty}{\longrightarrow} P\Big[e^{\beta_n \sum_{i=1}^n \omega(i,S_i) - n\lambda(\beta_n)} \mathbb{1}_{S_n = [\sqrt{n}x]_n} \Big],$$

that can be re-written as

$$\lim_{K\to\infty} \zeta_{K,x}^{(n,n)}(\beta) = Z_{n,[x\sqrt{n}]_n}(\omega, \beta_n) e^{-n\lambda(\beta_n)}.$$

■

Remark 4.1. The basis we use in Proposition 3.1 consists of discontinuous functions, while Theorem 3.2 needs continuous functions, as we will see in the proof of Propositions 3.5 and 3.6.

4.3 Convergence of the environment

Our first objective is proving the convergence of the environment ω, under the diffusive scaling, to the white noise ξ. The main result of the section is the following.

Proposition 4.1. *For any finite family of functions $\mathcal{C} \subset L^2([0,1] \times \mathbb{R})$, we have the following convergence in distribution:*

$$\Big(\langle \xi^{(m)}, f \rangle : f \in \mathcal{C} \Big) \xrightarrow{d} \left(\langle \xi, f \rangle : f \in \mathcal{C} \right)$$

To prove this statement, we begin proving two lemmas. For the first one we will use the following Central Limit Theorem.

Lemma 4.2 (Theorem 3.4.10. in [Dur19]. The Lindeberg-Feller theorem)**.** *For each n, let $X_{n,m}$, $1 \le m \le n$, be independent random variables with $\mathbb{P}[X_{n,m}] = 0$. Suppose*

(a) $\sum_{m=1}^n \mathbb{P}[X_{n,m}^2] \to \sigma^2 > 0$.

(b) For all $\varepsilon > 0$, $\lim_{n\to\infty} \sum_{m=1}^n \mathbb{P}[|X_{n,m}|^2; |X_{n,m}| > \varepsilon] = 0$.

Then $S_n = X_{n,1} + ... + X_{n,n} \xrightarrow{d} \mathcal{N}(0, \sigma^2)$ *as* $n \to \infty$.

Lemma 4.3. *The collection of random variables*

$$\{\langle \xi^{(n)}, \mathbb{1}_{\mathcal{R}}\rangle \ : \ \mathcal{R} \subset [0,1]\times\mathbb{R} \text{ is a rectangle with rational coordinates}\}$$

converges in distribution to

$$\{\langle \xi, \mathbb{1}_{\mathcal{R}}\rangle \ : \ \mathcal{R} \subset [0,1]\times\mathbb{R} \text{ is a rectangle with rational coordinates}\},$$

meaning that every finite vector $\left(\langle \xi^{(n)}, \mathbb{1}_{\mathcal{R}_1}\rangle, ..., \langle \xi^{(n)}, \mathbb{1}_{\mathcal{R}_N}\rangle\right)$ *converges in distribution, as* $n\to\infty$*, to* $\left(\langle \xi, \mathbb{1}_{\mathcal{R}_1}\rangle, ..., \langle \xi, \mathbb{1}_{\mathcal{R}_N}\rangle\right)$.

Proof. Consider $s,t,x,y \in \mathbb{Q}$ such that $0 \le s < t \le 1;\ x < y$. Consider $f := \mathbb{1}_{[s,t]\times[x,y]}$. We have

$$\begin{aligned}\langle \xi^{(n)}, f\rangle &= \int_0^1 \int_{-\infty}^{\infty} 2^{-1/2} n^{3/4} \omega(rn, un) f(r,u) \mathrm{d}u \mathrm{d}r \\ &= 2^{-1/2} n^{3/4} \int_{sn}^{tn} \int_{x\sqrt{n}}^{y\sqrt{n}} n^{-3/2} \omega(r,u) \mathrm{d}u \mathrm{d}r\end{aligned}$$

Here we note that ω takes constant (random) values in rectangles of lengths 1×2. So this integral becomes the sum of approximately $n(t-s)\frac{\sqrt{n}}{2}(y-x)$ integrals of i.i.d. random variables over rectangles of area 2. More precisely, if we set $\Omega := [sn, tn]\times[x\sqrt{n}, y\sqrt{n}]$, $R_{i,m} := (i-1, i]\times(m-1, m+1]$, and

$$I_\Omega := \{(i,m)\in\mathbb{N}\times\mathbb{Z} \ : \ i \equiv m (\mathrm{mod}\ 2), R_{i,m}\cap\Omega \neq \varnothing\},$$

we get

$$\begin{aligned}\langle \xi^{(n)}, f\rangle &= 2^{-1/2} n^{-3/4} \sum_{(i,m)\in I_\Omega} \int_{R_{i,m}\cap\Omega} \omega(i,m) \\ &= 2^{-1/2} n^{-3/4} \left\{ \sum_{(i,m)\in I_\Omega : R_{i,m}\subset\Omega} 2\omega(i,m) + \sum_{(i,m)\in I_\Omega : R_{i,m}\not\subset\Omega} c_{i,m}\omega(i,m) \right\}.\end{aligned}$$

where $0 \le c_{i,m} \le 2$ (note that the rectangles $R_{i,m}$ have area 2). Here we can count the cardinality of the index sets: the first one is asymptotically $n(t-s)\frac{\sqrt{n}}{2}(y-x)$, while the second one is at least $2\left\lfloor \frac{n(t-s)}{2} \right\rfloor$ and less than $2(t-s)n + (y-x)\sqrt{n} + 4$ (see Figure 4.1); call those *exact* quantities E_n and F_n, respectively. Re-naming the i.i.d. variables $\omega(i,m)$ and the constants $c_{i,m}$, we obtain

$$\langle \xi^{(n)}, f\rangle = \sqrt{2} n^{-3/4} \sum_{k=1}^{E_n} \omega_{n,k} + 2^{-1/2} n^{-3/4} \sum_{l=1}^{F_n} c_{n,l}\omega_{n,l}$$

with $\omega_{n,k}$ and $\omega_{n,l}$ satisfying Assumption 1. We need to show that the second sum converges to zero in distribution, and for that we will use Lemma 4.2. For each n (sufficiently large),

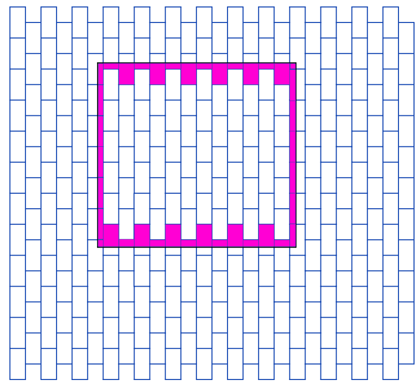

Figure 4.1: The boundary of the rectangle Ω intersects with some of the rectangles $R_{i,m}$. The colored area corresponds to the quantities $c_{n,l}$.

define the independent random variables $X_{n,l}$, $1 \leq l \leq F_n$, as

$$X_{n,l} := \frac{c_{n,l}\omega_{n,l}}{\sqrt{\sum_{l=1}^{F_n} c_{n,l}^2}},$$

which are well defined because, among the F_n *truncated* rectangles, there are at least $2\left\lfloor \frac{n(t-s)}{2} \right\rfloor$ with area $c_{n,l} \geq 1$ (see Figure 4.1 again). In particular,

$$2\left\lfloor \frac{n(t-s)}{2} \right\rfloor \leq \sum_{l=1}^{F_n} c_{n,l}^2 \leq 4F_n.$$

By definition of the r.v.'s $X_{n,l}$, for each n,

$$\sum_{l=1}^{F_n} \mathbb{P}[X_{n,l}^2] = 1,$$

so they satisfy condition *(a)* in Lindeberg-Feller theorem. Now, let $\varepsilon > 0$. Majorizing each $c_{n,l}$ by 2, we obtain

$$\sum_{l=1}^{F_n} \mathbb{P}[X_{n,l}^2; |X_{n,l}| > \varepsilon] \leq \sum_{l=1}^{F_n} \frac{4}{\sum_{l=1}^{F_n} c_{n,l}^2} \mathbb{P}\left[\omega_{n,l}^2; |\omega_{n,l}| > \frac{\varepsilon}{2}\sqrt{\sum_{l=1}^{F_n} c_{n,l}^2}\right]$$

and now minimizing the sum of the $c_{n,l}^2$'s by $2\left\lfloor\frac{n(t-s)}{2}\right\rfloor$,

$$\begin{aligned}\sum_{l=1}^{F_n}\mathbb{P}[X_{n,l}^2;|X_{n,l}|>\varepsilon] &\leq \sum_{l=1}^{F_n}\frac{4}{2\left\lfloor\frac{n(t-s)}{2}\right\rfloor}\mathbb{P}\left[\omega_{n,l}^2;|\omega_{n,l}|>\frac{\varepsilon}{2}\sqrt{2\left\lfloor\frac{n(t-s)}{2}\right\rfloor}\right]\\ &= \frac{4F_n}{2\left\lfloor\frac{n(t-s)}{2}\right\rfloor}\mathbb{P}\left[\omega_{n,1}^2;|\omega_{n,1}|>\frac{\varepsilon}{2}\sqrt{2\left\lfloor\frac{n(t-s)}{2}\right\rfloor}\right].\end{aligned}$$

As $F_n = O(n)$, the fraction is bounded by a constant, and as $\omega_{n,1}^2$ is integrable and $\frac{\varepsilon}{2}\sqrt{2\left\lfloor\frac{n(t-s)}{2}\right\rfloor}\to\infty$ as $n\to\infty$, we obtain condition *(b)* in Lindeberg-Feller theorem:

$$\lim_{n\to\infty}\sum_{l=1}^{F_n}\mathbb{P}[X_{n,l}^2;|X_{n,l}|>\varepsilon]=0,$$

allowing us to conclude that, by Lemma 4.2,

$$\lim_{n\to\infty}\sum_{l=1}^{F_n}\frac{c_{n,l}\omega_{n,l}}{\sqrt{\sum_{l=1}^{F_n}c_{n,l}^2}}\stackrel{d}{=}\mathcal{N}(0,1).$$

As

$$2^{-1/2}n^{-3/4}\sum_{l=1}^{F_n}c_{n,l}\omega_{n,l}=2^{-1/2}\frac{\sqrt{\sum_{l=1}^{F_n}c_{n,l}^2}}{n^{3/4}}\sum_{l=1}^{F_n}\frac{c_{n,l}\omega_{n,l}}{\sqrt{\sum_{l=1}^{F_n}c_{n,l}^2}}$$

and $F_n\leq 2(t-s)n+(y-x)\sqrt{n}+4$,

$$\frac{\sqrt{\sum_{l=1}^{F_n}c_{n,l}^2}}{n^{3/4}}\leq\frac{Cn^{1/2}}{n^{3/4}}\to 0.$$

Then, by Slutsky's Lemma [vdV98, Lemma 2.8], the product goes to zero in distribution.

Now, ignoring the second sum (by the above argument and again by Slutsky's Lemma), the Central Limit Theorem gives us that, as $n\to\infty$, we have the convergence in distribution

$$\begin{aligned}\lim_{n\to\infty}\langle\xi^{(n)},f\rangle &\stackrel{d}{=}\lim_{n\to\infty}\frac{\sqrt{(t-s)(y-x)}}{\sqrt{\frac{1}{2}n^{3/2}(t-s)(y-x)}}\sum_{k=1}^{E_n}\omega_{n,k}\\ &\stackrel{d}{=}\sqrt{(t-s)(y-x)}\mathcal{N}(0,1)\stackrel{d}{=}\mathcal{N}(0,(t-s)(y-x))\end{aligned}$$

because of Assumption 1. Similarly,

$$\begin{aligned}\langle \xi, f\rangle &\overset{d}{=} \mathcal{N}(0, \|f\|_2)\\ &\overset{d}{=} \mathcal{N}(0, (t-s)(y-x)).\end{aligned}$$

Now, consider N rational rectangles $\mathcal{R}_1, ..., \mathcal{R}_N$. All their possible intersections are too rational rectangles, so we can partition and re-index in M disjoint sub-rectangles such their union is the same as the union of the other N rectangles. The new random variables $\langle \xi^{(n)}, \mathbb{1}_{\mathcal{R}_j}\rangle$ are possibly dependent; it can happen that a rectangle of the form $\left(\frac{i-1}{n}, \frac{i}{n}\right] \times \left(\frac{m-1}{\sqrt{n}}, \frac{m+1}{\sqrt{n}}\right]$ intersects with two disjoint rectangles $\mathcal{R}_j$, $\mathcal{R}_k$ simultaneously, so the random variable $\omega(i, m)$ would be involved in the definition of both $\langle \xi^{(n)}, \mathbb{1}_{\mathcal{R}_j}\rangle$ and $\langle \xi^{(n)}, \mathbb{1}_{\mathcal{R}_k}\rangle$. So now we consider independent subsets, for $n \in \mathbb{N}$ and $j = 1, ..., M$,

$$\mathcal{R}_j^{(n)} := \bigcup \left\{ r_{i,m}^{(n)} :\ i \in \{1, ..., n\},\ m \in i + 2\mathbb{Z} \text{ and } r_{i,m}^{(n)} \subseteq \mathcal{R}_j \right\}$$

where

$$r_{i,m}^{(n)} := \left(\tfrac{i-1}{n}, \tfrac{i}{n}\right] \times \left(\tfrac{m-1}{\sqrt{n}}, \tfrac{m+1}{\sqrt{n}}\right].$$

By a similar argument as before, the remaining parts $\mathcal{R}_j \setminus \mathcal{R}_j^{(n)}$ are finite sums of small rectangles, where each $\langle \xi^{(n)}, \mathbb{1}_{\mathcal{R}_j \setminus \mathcal{R}_j^{(n)}}\rangle$ tends to zero in distribution as $n \to \infty$ (using Lindeberg-Feller theorem and Slutsky's Lemma). Now, the vector

$$\begin{aligned}&\left(\langle \xi^{(n)}, \mathbb{1}_{\mathcal{R}_1^{(n)}}\rangle + \langle \xi^{(n)}, \mathbb{1}_{\mathcal{R}_j \setminus \mathcal{R}_j^{(n)}}\rangle, ..., \langle \xi^{(n)}, \mathbb{1}_{\mathcal{R}_M^{(n)}}\rangle + \langle \xi^{(n)}, \mathbb{1}_{\mathcal{R}_j \setminus \mathcal{R}_j^{(n)}}\rangle\right)\\ =& \left(\langle \xi^{(n)}, \mathbb{1}_{\mathcal{R}_1^{(n)}}\rangle, ..., \langle \xi^{(n)}, \mathbb{1}_{\mathcal{R}_M^{(n)}}\rangle\right) + \left(\langle \xi^{(n)}, \mathbb{1}_{\mathcal{R}_j \setminus \mathcal{R}_j^{(n)}}\rangle, ..., \langle \xi^{(n)}, \mathbb{1}_{\mathcal{R}_j \setminus \mathcal{R}_j^{(n)}}\rangle\right)\end{aligned}$$

converges in distribution as $n \to \infty$ to

$$\left(\langle \xi, \mathbb{1}_{\mathcal{R}_1}\rangle, ..., \langle \xi, \mathbb{1}_{\mathcal{R}_M}\rangle\right)$$

because the first part has independent coordinates converging to the corresponding (also independent) coordinates in the limit, the second part converges jointly to zero by Slutsky's Lemma ([vdV98, Lemma 2.8]) and the limit of the sum is the sum of the limits, again by Slutsky's Lemma. This convergence implies the convergence of the joint distribution with the N original rectangles as well. As the finite-dimensional distributions converge, we conclude. ■

Lemma 4.4. *$\langle \xi^{(n)}, f\rangle$ converges in distribution to $\langle \xi, f\rangle$ for all $f \in L^2([0,1] \times \mathbb{R})$.*

Proof. Fix $f \in L^2([0,1] \times \mathbb{R})$. We need to show that for every h bounded and uniformly continuous function,

$$\lim_{n\to\infty} \mathbb{P}\left[h\left(\left\langle \xi^{(n)}, f\right\rangle\right)\right] = \nu\left[h\left(\langle \xi, f\rangle\right)\right], \tag{4.2}$$

by the Portmanteau Theorem [Bil99, Theorem 2.1].

Take h bounded and uniformly continuous. Fix $\varepsilon_1, \varepsilon_2 > 0$. Take $g = \sum_i \lambda_i \mathbb{1}_{\mathcal{R}_i}$ a (finite) linear combination of indicator functions of rational rectangles, such that, $||f-g||_{L^2}^2 < \varepsilon_1$. Such g exists because simple functions (linear combinations of measurable sets indicators) are dense in L^p, $1 \le p < \infty$ [Ran02, Theorem 8.6.1], also every measurable set is almost the finite sum of rectangles (Littlewood's first principle; [Ran02, Theorem 4.2.2.]) and every rectangle is almost a rational rectangle.

We want to show that the difference

$$\left|\mathbb{P}\left[h\left(\left\langle \xi^{(n)}, f\right\rangle\right)\right] - \nu\left[h\left(\langle \xi, f\rangle\right)\right]\right|$$

is small. For that, we use the triangle inequality.

$$\begin{aligned}\left|\mathbb{P}\left[h\left(\left\langle \xi^{(n)}, f\right\rangle\right)\right] - \nu\left[h\left(\langle \xi, f\rangle\right)\right]\right| \le & \left|\mathbb{P}\left[h\left(\left\langle \xi^{(n)}, f\right\rangle\right) - h\left(\left\langle \xi^{(n)}, g\right\rangle\right)\right]\right| \\ & + \left|\mathbb{P}\left[h\left(\left\langle \xi^{(n)}, g\right\rangle\right)\right] - \nu\left[h\left(\langle \xi, g\rangle\right)\right]\right| \\ & + \Big|\nu\left[h\left(\langle \xi, g\rangle\right) - h\left(\langle \xi, f\rangle\right)\right]\Big|.\end{aligned}$$

First of all, by the previous lemma we have that $\langle \xi^{(n)}, g\rangle \xrightarrow{d} \langle \xi, g\rangle$, so the second term can be taken $< \varepsilon_2/3$ for n large. Secondly, by definition of ξ,

$$\nu[\langle \xi, g-f\rangle^2] = ||g-f||_2^2 < \varepsilon_1.$$

And similarly, note that by (3.3)

$$\mathbb{P}[\langle \xi^{(n)}, f-g\rangle^2] \le ||f-g||_{L^2}^2 < \varepsilon_1$$

by the choice of g. Now, as h is uniformly continuous, we could have chosen $\varepsilon_1 > 0$ so that the first and third terms in the triangle inequality are both $< \varepsilon_2/3$. This shows (4.2) and we conclude that $\langle \xi^{(n)}, f\rangle \xrightarrow{d} \langle \xi, f\rangle$. ∎

Proof. (Proof of Proposition 4.1) Analogous to the proof of Lemma 4.4. We take $f_1, ..., f_n \in L^2([0,1]\times\mathbb{R})$. Take $h : \mathbb{R}^n \to \mathbb{R}$ bounded and uniformly continuous. Fix $\varepsilon_1, \varepsilon_2 > 0$. Take $g_1, ..., g_n$ linear combinations of indicator functions of rational rectangles, such that, $||f_i - g_i||_{L^2}^2 < \varepsilon_1$ for each $i \in \{1, ..., n\}$. Then bound the difference

$$\left|\mathbb{P}\left[h\left(\left\langle \xi^{(m)}, f_1\right\rangle, ..., \left\langle \xi^{(m)}, f_n\right\rangle\right)\right] - \nu[h(\langle \xi, f_1\rangle, ..., \langle \xi, f_n\rangle)]\right|$$

by ε_2 using the triangle inequality and choosing ε_1 conveniently to bound some of the differences; to bound the terms with $\langle \xi^{(m)}, g_i\rangle$ and $\langle \xi, g_i\rangle$, simply let $m \to \infty$ and use Lemma 4.3, which can be used with fixed $g_1, ..., g_n$. ∎

Remark 4.2. The proof of Proposition 4.1 does not really use Lemma 4.4; we simply use the idea of its proof.

4.3.1 Proof of Propositions 3.3 and 3.4

Recall the definition of the *point-to-point quasi-partition function* (3.4):

$$\zeta_{K,x}^{(m,n)}(\beta) = P\Big[\exp\Big\{\beta\sum_{j=1}^{K} m_j^{(n)}\xi_j^{(m)} - \Lambda_{K,x}^{(m,n)}(\beta)\Big\}\mathbb{1}_{S_n=[\sqrt{n}x]_n}\Big]$$

Using conditional expectation we have

$$\zeta_{K,x}^{(m,n)}(\beta) = P\left(S_n = [\sqrt{n}x]_n\right) P^{(n,x)}\Big[\exp\Big\{\beta\sum_{j=1}^{K} m_j^{(n)}\xi_j^{(m)} - \Lambda_{K,x}^{(m,n)}(\beta)\Big\}\Big]$$

where $P^{(n,x)}$ denotes the law P of the polymer conditioned to end at the endpoint $(n, [\sqrt{n}x]_n)$. Now, by the Local Limit Theorem A.2.1 we directly get

$$\begin{aligned}\tfrac{\sqrt{n}}{2}\zeta_{K,x}^{(m,n)} &= \tfrac{\sqrt{n}}{2}\left(\tfrac{2}{\sqrt{n}}g(1,[\sqrt{n}x]_n n^{-\frac{1}{2}}) + O(n^{-\frac{3}{2}})\right) P^{(n,x)}\Big[\exp\Big\{\beta\sum_{j=1}^{K} m_j^{(n)}\xi_j^{(m)} - \Lambda_{K,x}^{(m,n)}(\beta)\Big\}\Big]\\ &= \Big(g(1, x + O(n^{-\frac{1}{2}})) + O(n^{-1})\Big) P^{(n,x)}\Big[\exp\Big\{\beta\sum_{j=1}^{K} m_j^{(n)}\xi_j^{(m)} - \Lambda_{K,x}^{(m,n)}(\beta)\Big\}\Big],\end{aligned}$$

where $g(t,x) = (2\pi t)^{-1/2}\exp\{-x^2/2t\}$ is the heat equation kernel.

Before letting $n \to \infty$, we will work with this last expression, letting $m \to \infty$ first. We will need the following lemmas.

Lemma 4.5. *Let $g \in L^1(X, \mathcal{M}, \mu)$, where $(X, \mathcal{M}, \mu)$ is any measure space. Given $\varepsilon > 0$, there exists $\delta > 0$ such that, for $A \in \mathcal{M}$,*

$$\mu(A) < \delta \implies \int_A |g|\mathrm{d}\mu < \varepsilon.$$

Proof. Without loss of generality, assume $g \geq 0$. Since

$$\int_X g\mathrm{d}\mu = \sum_{k=0}^{\infty}\int_{\{k<g\leq k+1\}} g\mathrm{d}\mu < \infty,$$

there exists $M \in \mathbb{N}$ such that

$$\int_{\{g>M\}} g\mathrm{d}\mu < \tfrac{\varepsilon}{2}.$$

Now take $\delta > 0$ such that $2\delta M < \varepsilon$. Then, if $\mu(A) < \delta$,

$$\begin{aligned}\int_A g \mathrm{d}\mu &= \int_{A\cap\{g>M\}} g\mathrm{d}\mu + \int_{A\cap\{g\leq M\}} g\mathrm{d}\mu \\ &\leq \int_{\{g>M\}} g\mathrm{d}\mu + \int_A M\mathrm{d}\mu \\ &< \tfrac{\varepsilon}{2} + M\mu(A) < \varepsilon.\end{aligned}$$

■

We use this lemma to prove the following.

Lemma 4.6. *For every $f \in L^2([0,1]\times\mathbb{R})$,*

$$\sup_{m\in\mathbb{N}} \mathbb{P}\left[\exp\left\{\langle \xi^{(m)}, f\rangle\right\}\right] < \infty.$$

Proof. Let $f \in L^2([0,1]\times\mathbb{R})$. For $n\in\mathbb{N}$, let $f_n := f\mathbb{1}_{[0,1]\times[-n,n]}$. For all $n\in\mathbb{N}$, as f_n has compact support, we can write the (finite) product

$$\mathbb{P}\Big[e^{\langle\xi^{(m)},f_n\rangle}\Big] = \prod_{i=1}^{m}\prod_{k\in i+2\mathbb{Z}} \mathbb{P}\Big[e^{\left(\frac{m^{3/4}}{\sqrt{2}}\iint_{R^{(m)}_{i,k}} f_n\right)\omega(i,k)}\Big]. \tag{4.3}$$

Take $\varepsilon > 0$ small such that: if $|\theta| < \varepsilon$, then

$$\mathbb{P}\Big[e^{\theta\omega}\Big] = e^{\log\{1+\frac{\theta^2}{2}+O(\theta^3)\}} \leq e^{\frac{\theta^2}{2}+C|\theta|^3},$$

for some $C > 0$. Now, using Lemma 4.5 with $f^2 \in L^1([0,1]\times\mathbb{R})$, take $M\in\mathbb{N}$ such that $|A| \leq 2M^{-3/2}$ implies $\iint_A f^2 < \frac{\varepsilon^2}{4}$. For all n and for all $m \geq M$,

$$\begin{aligned}\mathbb{P}\Big[e^{\langle\xi^{(m)},2f_n\rangle}\Big] &= \prod_{i=1}^{m}\prod_{k\in i+2\mathbb{Z}} \mathbb{P}\Big[e^{\left(\frac{m^{3/4}}{\sqrt{2}}\iint_{R^{(m)}_{i,k}} 2f_n\right)\omega(i,k)}\Big] \\ &\leq \prod_{i=1}^{m}\prod_{k\in i+2\mathbb{Z}} e^{\left(\frac{m^{3/4}}{\sqrt{2}}\iint_{R^{(m)}_{i,k}} 2f_n\right)^2\left(\frac{1}{2}+C\left|\frac{m^{3/4}}{\sqrt{2}}\iint_{R^{(m)}_{i,k}} 2f_n\right|\right)},\end{aligned}$$

since by Jensen's inequality [Dur19, Theorem 1.5.1] with $\varphi(x) = x^2$ and the choice of M,

$$\begin{aligned}\Big(\frac{m^{3/4}}{\sqrt{2}}\iint_{R^{(m)}_{i,k}} 2f_n\Big)^2 &\leq \iint_{R^{(m)}_{i,k}} (2f_n)^2 \\ &\leq 4\iint_{R^{(m)}_{i,k}} f^2 < \varepsilon^2.\end{aligned}$$

And using this bound again,

$$\mathbb{P}\Big[e^{\langle \xi^{(m)}, 2f_n\rangle}\Big] \leq \prod_{i=1}^{m} \prod_{k\in i+2\mathbb{Z}} e^{2\int_{R_{i,k}^{(m)}} f_n^2(1+C'\varepsilon)}$$
$$= e^{2||f_n||^2(1+C'\varepsilon)} \leq e^{2||f||^2(1+C'\varepsilon)}.$$

This is a uniform bound on n and $m \geq M$, so the sequence $e^{\langle \xi^{(m)}, f_n\rangle}$ (on n) is uniformly integrable, because its square has finite $\mathbb{P}$-norm ([vdV98, Theorem 2.20]). Now, as also $f_n \xrightarrow{L^2} f$ and exp is a continuous function, we can take the limit $n \to \infty$ in (4.3) and get

$$\mathbb{P}\Big[e^{\langle \xi^{(m)}, f\rangle}\Big] = \prod_{i=1}^{m} \prod_{k\in i+2\mathbb{Z}} \mathbb{P}\Big[e^{\left(\frac{m^{3/4}}{\sqrt{2}}\iint_{R_{i,k}^{(m)}} f\right)\omega(i,k)}\Big],$$

for $m \geq M$. Also, repeating the previous calculations, we obtain that

$$\sup_{m\geq M} \mathbb{P}\left[\exp\left\{\langle \xi^{(m)}, f\rangle\right\}\right] \leq e^{\frac{1}{2}||f||^2(1+C'\varepsilon)} < \infty.$$

Since this bound is uniform for m sufficiently large, we obtain that the supremum over m is, again, bounded. ■

Now we are able to prove Proposition 3.3. We begin proving that the normalizations converge.

Lemma 4.7. *Assume the same conditions as in Proposition 3.3. For each fixed path* $\mathbf{x}$ *such that* $\mathbf{x}_n = [\sqrt{n}x]_n$*, as* $m \to \infty$*, we have*

$$\Lambda_{K,x}^{(n,m)}(\beta) \to \tilde{\Lambda}_{K,x}^{(n)}(\beta)$$

Proof. For every path $\mathbf{x}$ such that $x_n = [\sqrt{n}x]_n$, we define the function $\varphi_{\mathbf{x}} \in L^2([0,1]\times\mathbb{R})$ as

$$\varphi_{\mathbf{x}}(s,y) := \beta \sum_{j=1}^{K} \left(\int_0^1 e_j\left(\lambda, \tfrac{\mathbf{x}_{n\lambda}}{\sqrt{n}}\right) \mathrm{d}\lambda\right) e_j(s,y),\ \ (s,y) \in [0,1]\times\mathbb{R}.$$

By Lemma 4.4, we have the convergence in distribution

$$\langle \xi^{(m)}, \varphi_{\mathbf{x}}\rangle \xrightarrow{d} \langle \xi, \varphi_{\mathbf{x}}\rangle.$$

The sequence $\left(e^{\langle \xi^{(m)}, \varphi_{\mathbf{x}}\rangle}\right)_m$ is bounded in L^2. In fact,

$$\sup_{m\in\mathbb{N}} \mathbb{P}\left[\left(e^{\langle \xi^{(m)}, \varphi_{\mathbf{x}}\rangle}\right)^2\right] = \sup_{m\in\mathbb{N}} \mathbb{P}\Big[e^{\langle \xi^{(m)}, 2\varphi_{\mathbf{x}}\rangle}\Big] < \infty$$

by Lemma 4.6. So, the sequence $\left(e^{\langle \xi^{(m)}, \varphi_{\mathbf{x}} \rangle}\right)_m$ is uniformly integrable and, therefore,

$$\lim_{m\to\infty} \mathbb{P}\left[e^{\langle \xi^{(m)}, \varphi_{\mathbf{x}} \rangle}\right] = \nu\left[e^{\langle \xi, \varphi_{\mathbf{x}} \rangle}\right] \tag{4.4}$$

by [vdV98, Theorem 2.20]. And taking logarithm on both sides (which is a continuous function), we obtain the result. ■

Proof of Proposition 3.3. Using the same notation as in Lemma 4.7, by Proposition 4.1 we have convergence in distribution (as $m \to \infty$) of the vector

$$\left(\langle \xi^{(m)}, \varphi_{\mathbf{x}} \rangle \ : \ \mathbf{x} \text{ s.t. } \mathbf{x}_n = [\sqrt{n}x]_n\right) \xrightarrow{d} \left(\langle \xi^{(m)}, \varphi_{\mathbf{x}} \rangle \ : \ \mathbf{x} \text{ s.t. } \mathbf{x}_n = [\sqrt{n}x]_n\right)$$

By the result of Lemma 4.7 (prefer the equation (4.4)), and classical results of convergence ([vdV98, Theorem 2.7]), also the matrix

$$\left(\left(\langle \xi^{(m)}, \varphi_{\mathbf{x}} \rangle, \mathbb{P}\left[e^{\langle \xi^{(m)}, \varphi_{\mathbf{x}} \rangle}\right]\right) \ : \ \mathbf{x} \text{ s.t. } \mathbf{x}_n = [\sqrt{n}x]_n\right)$$

converges in distribution to

$$\left(\left(\langle \xi, \varphi_{\mathbf{x}} \rangle, \nu\left[e^{\langle \xi, \varphi_{\mathbf{x}} \rangle}\right]\right) \ : \ \mathbf{x} \text{ s.t. } \mathbf{x}_n = [\sqrt{n}x]_n\right).$$

Now, as the measure $P^{(n,x)}$ is defined averaging over finitely many paths, we can find a continuous function h such that

$$P^{(n,x)}\left[e^{\beta \sum_{j=1}^K m_j^{(n)} \xi_j^{(m)} - \Lambda_{K,x}^{(n,m)}(\beta)}\right] = h\left(\left(\langle \xi^{(m)}, \varphi_{\mathbf{x}} \rangle, \mathbb{P}\left[e^{\langle \xi^{(m)}, \varphi_{\mathbf{x}} \rangle}\right]\right)_{\mathbf{x}:\mathbf{x}_n = [\sqrt{n}x]_n}\right)$$

and

$$P^{(n,x)}\left[e^{\beta \sum_{j=1}^K m_j^{(n)} \xi_j - \tilde{\Lambda}_{K,x}^{(n)}(\beta)}\right] = h\left(\left(\langle \xi, \varphi_{\mathbf{x}} \rangle, \nu\left[e^{\langle \xi, \varphi_{\mathbf{x}} \rangle}\right]\right)_{\mathbf{x}:\mathbf{x}_n = [\sqrt{n}x]_n}\right).$$

Finally, as composing with a continuous functions preserves the convergence in distribution ([vdV98, Theorem 2.3]), we obtain the desired result.

■

Proof of Proposition 3.4. The result is an immediate consequence of dropping the condition $\mathbf{x}_n = [\sqrt{n}x]_n$ in the proof of Proposition 3.3.

■

4.4 Bridging the bridges; proof of Propositions 3.5 and 3.6

Our objective in this section, as stated in the title, is to prove Propositions 3.5 and 3.6. For that, we need some preliminary results.

4.4.1 KMT Embeddings

We will need to bound the difference between a simple random walk conditioned to end (when properly re-scaled) at a point $x \in \mathbb{R}$ and a Brownian bridge ending at the same point x. Also, we will need the same kind of bound in the case of the simple random walk and a standard Brownian motion. Any of the mentioned objects are necessarily defined on the same probability space, so we will need to use some coupling results even for those "differences" to be well defined. The objective of this subsection is providing some tools to achieve those bounds.

Recall that, given any path $\mathbf{x} : \mathbb{N} \to \mathbb{Z}$, we define the natural continuous extension of $\mathbf{x}$ as the curve that interpolates between the discrete values, namely, for $t \in [0, \infty)$

$$\mathbf{x}_t := (1 - \{t\})\mathbf{x}_{\lfloor t \rfloor} + \{t\}\mathbf{x}_{\lfloor t \rfloor + 1}.$$

We will use the same notation for the discrete path and its continuous extension.

Brownian bridge case

Consider S a simple random walk on $\mathbb{Z}$. Let $L_n := \{z \in \mathbb{Z} \ : \ P(S_n = z) > 0\}$. For $z \in L_n$, let $S^{(n,z)}$ denote the random walk $(S_j)_{1 \le j \le n}$ conditioned on the event $S_n = z$.

Also consider a Brownian bridge $B = (B_t, 0 \le t \le 1)$ such that $B_0 = B_1 = 0$. Recall the process $B^{(n,z)}$ defined by

$$B_t^{(n,z)} := \sqrt{n} B_{t/n} + \tfrac{t}{n} z, \quad 0 \le t \le n,$$

to be a Brownian bridge on $[0, n]$ tied to the endpoints $B_0 = 0$ and $B_n = z$. Now define

$$\Delta(n, z) := \sup_{0 \le t \le n} \left| B_t^{(n,z)} - S_t^{(n,z)} \right|$$

which is a priori not well defined, since B and S are not necessarily defined on the same probability space, but we have the following theorems which will help us fix this problem and also bound the quantity $\Delta(n, z)$.

Theorem 4.8 (Theorem 6.3 in [LTF07])**.** *For every $b > 0$, there exist constants $0 < c, a, \alpha < \infty$ such that for every positive integer n, there exists a coupling of a Brownian bridge B and the family of processes $\{S^{(n,z)} : z \in L_n\}$ such that, for $\Delta(n, z)$ as above,*

$$\overline{P}\left[e^{a\Delta(n,z)}\right] \le c n^{\alpha} e^{b|z|^2/n},$$

where $\overline{P}$ is the measure of the coupling.

Using Chebyshev's inequality ([Dur19, Theorem 1.6.4]), one obtains the following corollary.

Corollary 4.8.1 (Theorem 6.4 in [LTF07])**.** *For every $b > 0$, there exist constants $0 < c, \alpha < \infty$ such that for every positive integer n, there exists a coupling of a Brownian bridge*

B and a family of processes $\{S^{(n,z)} : z \in L_n\}$ such that for all $r > 0$,

$$\overline{P}\{\Delta(n,z) > rc\log n\} \leq cn^{\alpha - r}e^{bz^2/n}, \tag{4.5}$$

where $\overline{P}$ is the measure of the coupling.

We want to measure the difference between the re-scaled random walk bridge $\frac{S_{tn}^{(n,[\sqrt{n}x]_n)}}{\sqrt{n}}$ and the Brownian bridge $B_t + tx$ conditioned to the endpoint x. For that, we use the coupling in Corollary 4.8.1 and define

$$\overline{\Delta}(n,x) := \sup_{0\leq t\leq 1}\left|B_t + tx - \frac{S_{tn}^{(n,[\sqrt{n}x]_n)}}{\sqrt{n}}\right|$$

and notice that

$$\begin{aligned}\overline{\Delta}(n,x) &= \tfrac{1}{\sqrt{n}}\sup_{0\leq t\leq n}\left|\sqrt{n}B_{t/n} + \tfrac{t}{n}\left([x\sqrt{n}]_n + \left(x\sqrt{n} - [x\sqrt{n}]_n\right)\right) - S_t^{(n,[x\sqrt{n}]_n)}\right| \\ &\leq \tfrac{1}{\sqrt{n}}\Delta(n,[x\sqrt{n}]_n) + n^{-3/2},\end{aligned}$$

since $|x\sqrt{n} - [x\sqrt{n}]_n| \leq 1$. So, using Corollary 4.8.1 with $b = 1$, we get that there exist constants $0 < c, \alpha < \infty$ such that for all $\delta > 0$

$$\begin{aligned}\overline{P}\left(\overline{\Delta}(n,x) > \delta\right) &\leq \overline{P}(\ \Delta(n,[x\sqrt{n}]_n) > \delta\sqrt{n} - 1/n) \\ &\leq cn^{\alpha - \frac{\delta\sqrt{n}-1/n}{c\log n}}e^{\frac{[x\sqrt{n}]_n^2}{n}} \\ &= ce^{x^2+O(1/n)}e^{\alpha\log n - \frac{\delta}{c}\sqrt{n} + \frac{1}{cn}}\end{aligned}$$

where $\lim_{n\to\infty}\alpha\log n - (\delta/c)\sqrt{n} + 1/(cn) = -\infty$, so for all $\delta > 0$,

$$\lim_{n\to\infty}\overline{P}\left(\overline{\Delta}(n,x) > \delta\right) = 0. \tag{4.6}$$

Brownian motion case

Consider now a Brownian motion $(W_t)_{t\geq 0}$ in $\mathbb{R}$. We want to compare W with the simple random walk S, so we use the notation

$$\Delta(n) := \sup_{0\leq t\leq n}|W_t - S_t|$$

which is, a priori (again), not well defined, since W and S are not necessarily on the same probability. The following theorem will help us couple both processes in a space where $\Delta(n)$ is well defined and can be bounded.

Theorem 4.9 (Theorem 1.5 in [Cha12])**.** *It is possible to construct a version of the sequence $(S_k)_{k\geq 0}$ and a standard Brownian motion $(W_t)_{t\geq 0}$ on the same probability space (let us call $\tilde{P}$ its measure) such that for all n and all $x \geq 0$,*

$$\tilde{P}\left(\Delta(n) \geq C\log n + x\right) \leq Ke^{-\lambda x},$$

where C, K, and λ do not depend on n.

Similarly to the previous case, we define

$$\tilde{\Delta}(n) := \sup_{0 \leq t \leq 1} \left| \tfrac{W_{tn}}{\sqrt{n}} - \tfrac{S_{nt}}{\sqrt{n}} \right|,$$

where $W_{tn}/\sqrt{n}$ is also a Brownian motion by a scaling invariance, and then clearly

$$\tilde{\Delta}(n) = \tfrac{\Delta(n)}{\sqrt{n}}.$$

Let $\delta > 0$. Using Theorem 4.9 with $x = (\sqrt{n}\delta - C \log n)$, for n sufficiently large (so that $x \geq 0$),

$$\tilde{P}(\tilde{\Delta}(n) > \delta) \leq K e^{-\lambda \delta \sqrt{n} + \lambda C \log n}.$$

So, similarly to the previous case, for any $\delta > 0$,

$$\lim_{n \to \infty} \tilde{P}(\tilde{\Delta}(n) > \delta) = 0. \tag{4.7}$$

4.4.2 Bounds on the maximum of the Brownian bridge and Brownian motion

We need bounds on the maximum of the Brownian bridge and Brownian motion to obtain certain results of convergence in probability. By [KS91, (3.40)], given $\eta \geq 0$ and $\gamma > \eta$, we have the following result for the maximum value of the Brownian bridge B:

$$\overline{P} \left(\max_{0 \leq s \leq 1} B_s + s\eta \geq \gamma \right) = e^{-\gamma(\gamma - \eta)}.$$

Using the reflection properties of the Brownian bridge, we have the following bound for every $x \in \mathbb{R}$ and $\gamma > |x|$:

$$\overline{P} \left(\max_{0 \leq s \leq 1} |B_s + sx| \geq \gamma \right) \leq 2e^{-\gamma(\gamma - |x|)}. \tag{4.8}$$

A similar result is true for the Brownian motion. Given $\gamma > 0$, we use a symmetry property of the Brownian motion and [MP10, Remark 2.22] to obtain

$$\tilde{P}\Big(\max_{0 \leq s \leq 1} |W_s| \geq \gamma \Big) \leq 2\tilde{P}\Big(\max_{0 \leq s \leq 1} W_s \geq \gamma \Big) \leq \tfrac{2\sqrt{2}}{\gamma\sqrt{\pi}} \exp\left\{ -\tfrac{\gamma^2}{2} \right\} \tag{4.9}$$

4.4.3 Proof of Proposition 3.5

We proceed to prove Proposition 3.5. For that, we will use the coupling of Corollary 4.8.1, together with the bounds stated in the previous subsection.

Proof of Proposition 3.5. Let $\varepsilon, \varepsilon' > 0$. Recall $\overline{P}$ the measure of the coupling in Corollary

4.8.1. By the bound (4.8) for the Brownian bridge paths, there exists $\gamma > 0$ such that

$$\overline{P}\left(\max_{0\le s\le 1} |B_s + sx| > \gamma\right) \le \varepsilon.$$

Note that for every $j \in \{1, ..., K\}$ the functions e_j are **continuous** on $[0,1]\times[-\gamma-1,\gamma+1]$, therefore they are uniformly continuous. In particular, there exists $1 > \delta > 0$ such that for all $j \in \{1, ..., K\}$, for all $y, z \in [0,1]\times[-\gamma-1,\gamma+1]$, if $|z-y| < \delta$, then $|e_j(z) - e_j(y)| < \varepsilon$.

Also, note that by (4.6) there exist $N \in \mathbb{N}$ such that for all $n \ge N$,

$$\overline{P}(\overline{\Delta}(n,x) \ge \delta) < \varepsilon.$$

Now, under the events $\Delta_\delta := \{\overline{\Delta}(n,x) < \delta\}$ and $M_\gamma := \{\max_{0\le s\le 1} |B_s + sx| \le \gamma\}$,

$$|m_j^{(n)} - m_j| \le \int_0^1 \left| e_j\left(s, \frac{S_{sn}^{(n,[\sqrt{n}x]_n)}}{\sqrt{n}}\right) - e_j(s, B_s + sx)\right| \mathrm{d}s < \varepsilon,$$

for $j \in \{1, ..., K\}$. Otherwise, we have still have bounds

$$|m_j^{(n)} - m_j| \le 2M_j,$$

where $M_j := \sup\{|e_j(s,y)|,\ (s,y) \in [0,1]\times\mathbb{R}\} < \infty$, for $j = 1, ..., K$. By Chebyshev's Inequality ([Dur19, Theorem 1.6.4]),

$$\begin{aligned}\varepsilon'\overline{P}\left(|m_j^{(n)} - m_j| \ge \varepsilon'\right) &\le \overline{P}\left[|m_j^{(n)} - m_j|\right] \\ &= \overline{P}\left[|m_j^{(n)} - m_j|\mathbb{1}_{M_\gamma\cap\Delta_\delta}\right] + \overline{P}\left[|m_j^{(n)} - m_j|\mathbb{1}_{(M_\gamma\cap\Delta_\delta)^c}\right],\end{aligned}$$

where we can use the bounds on $|m_j^{(n)} - m_j|$ to obtain

$$\varepsilon'\overline{P}\left(|m_j^{(n)} - m_j| \ge \varepsilon'\right) \le \varepsilon\overline{P}[\mathbb{1}_{M_\gamma\cap\Delta_\delta}] + 2M_j\overline{P}[\mathbb{1}_{M_\gamma\cap\Delta_\delta^c} + \mathbb{1}_{M_\gamma^c}].$$

Here we bound the first expectation by 1, while by the choice of γ, δ and n, the second expectation can be bounded by 2ε. We obtain

$$\overline{P}\left(|m_j^{(n)} - m_j| \ge \varepsilon'\right) \le \left(\tfrac{1+4M_j}{\varepsilon'}\right)\varepsilon.$$

And as M_j is independent of ε, the RHS can be made arbitrarily small. Thus,

$$\lim_{n\to\infty} \overline{P}(|m_j^{(n)} - m_j| \ge \varepsilon') = 0,$$

for $j = 1, ..., K$, which, by [vdV98, Theorem 2.7], implies the joint convergence in $\overline{P}$-probability (as $n \to \infty$)

$$(m_1^{(n)}, ..., m_K^{(n)}) \xrightarrow{\overline{P}} (m_1, ..., m_K).$$

Using this, we obtain

$$\exp\Big\{\beta\sum_{j=1}^K m_j^{(n)}\xi_j - \tfrac{\beta^2}{2}\sum_{j=1}^K [m_j^{(n)}]^2\Big\} \xrightarrow{\overline{P}} \exp\Big\{\beta\sum_{j=1}^K m_j\xi_j - \tfrac{\beta^2}{2}\sum_{j=1}^K m_j^2\Big\},$$

for any fixed realization of ξ, since convergence in probability is preserved by composing with continuous functions ([vdV98, Theorem 2.3]). It is easy to check that $S^{(n,[\sqrt{n}x]_n)}$ has the same distribution under $P^{(n,x)}$ as under $\overline{P}$, so for every n,

$$P^{(n,x)}\Big[\exp\Big\{\beta\sum_{j=1}^K m_j^{(n)}\xi_j - \tfrac{\beta^2}{2}\sum_{j=1}^K [m_j^{(n)}]^2\Big\}\Big] = \overline{P}\Big[\exp\Big\{\beta\sum_{j=1}^K m_j^{(n)}\xi_j - \tfrac{\beta^2}{2}\sum_{j=1}^K [m_j^{(n)}]^2\Big\}\Big].$$

Similarly, the Brownian bridge $B_t + tx$ has the same distribution under $Q^{(1,x)}$ as under $\overline{P}$, so

$$Q^{(1,x)}\Big[\exp\Big\{\beta\sum_{j=1}^K m_j\xi_j - \tfrac{\beta^2}{2}\sum_{j=1}^K m_j^2\Big\}\Big] = \overline{P}\Big[\exp\Big\{\beta\sum_{j=1}^K m_j\xi_j - \tfrac{\beta^2}{2}\sum_{j=1}^K m_j^2\Big\}\Big].$$

Using that the realization of ξ is fixed, together with the fact that we can bound $|m_j^{(n)}|$ by M_j, we can easily show uniform integrability on n for the sequence

$$\exp\Big\{\beta\sum_{j=1}^K m_j^{(n)}\xi_j - \tfrac{\beta^2}{2}\sum_{j=1}^K [m_j^{(n)}]^2\Big\},$$

and therefore we obtain

$$\lim_{n\to\infty} P^{(n,x)}\Big[\exp\Big\{\beta\sum_{j=1}^K m_j^{(n)}\xi_j - \tfrac{\beta^2}{2}\sum_{j=1}^K [m_j^{(n)}]^2\Big\}\Big] \overset{d}{=} Q^{(1,x)}\Big[\exp\Big\{\beta\sum_{j=1}^K m_j\xi_j - \tfrac{\beta^2}{2}\sum_{j=1}^K m_j^2\Big\}\Big].$$

■

Remark 4.3. The previous convergence result implies that, for any fixed realization of ξ, for $K \in \mathbb{N}$, $\beta > 0$ and $x \in \mathbb{R}$,

$$\begin{aligned}\lim_{n\to\infty} \tfrac{\sqrt{n}}{2} P\Big[\exp\Big\{\beta\sum_{j=1}^K m_j^{(n)}\xi_j - \tfrac{\beta^2}{2}\sum_{j=1}^K [m_j^{(n)}]^2\Big\}\mathbb{1}_{\{S_n=[x\sqrt{n}]_n\}}\Big] \\ = g(1,x)Q^{(1,x)}\Big[\exp\Big\{\beta\sum_{j=1}^K m_j\xi_j - \tfrac{\beta^2}{2}\sum_{j=1}^K m_j^2\Big\}\Big].\end{aligned}$$

Remark 4.4. In the proof we used that the functions e_j are continuous. The proof should be easy to modify to include the case of functions with null set of discontinuities.

4.4.4 Proof of Proposition 3.6

We can do the exact same proof that we did in the previous subsection, replacing $P^{(n,x)}$ by P, $\overline{P}$ by $\tilde{P}$ and $Q^{(1,x)}$ by $\mathbf{W}$ (which is the law of the Brownian motion, see Remark 2.5), to obtain that

$$\lim_{n\to\infty} P\Big[\exp\Big\{\beta\sum_{j=1}^{K} m_j^{(n)}\xi_j - \tfrac{\beta^2}{2}\sum_{j=1}^{K}[m_j^{(n)}]^2\Big\}\Big] \stackrel{d}{=} \mathbf{W}\Big[\exp\Big\{\beta\sum_{j=1}^{K} m_j\xi_j - \tfrac{\beta^2}{2}\sum_{j=1}^{K} m_j^2\Big\}\Big]. \quad (4.10)$$

And the proof is the same since we have the same kind of bounds for the maximum of $B^{(1,x)}$ as for W, and the same kind of bound for the probability that the quantities $\overline{\Delta}(n,x)$ and $\tilde{\Delta}(n)$ are greater that $\gamma > 0$, so the arguments are analogous.

Now, it only remains to prove that

$$\mathbf{W}\Big[\exp\Big\{\beta\sum_{j=1}^{K} m_j\xi_j - \tfrac{\beta^2}{2}\sum_{j=1}^{K} m_j^2\Big\}\Big] = \int_{\mathbb{R}} g(1,x)Q^{(1,x)}\Big[\exp\Big\{\beta\sum_{j=1}^{K} m_j\xi_j - \tfrac{\beta^2}{2}\sum_{j=1}^{K} m_j^2\Big\}\Big]\mathrm{d}x. \quad (4.11)$$

To prove equation (4.11) we will use [KS91, Problem 6.13], that in our case can be re-stated as follows.

Lemma 4.10. *For $0 = t_0 < t_1 < ... < t_n < T$, $x_0 = 0$, and $(x_1,\dots,x_n) \in \mathbb{R}^n$, the conditional finite-dimensional distributions $\mathbf{W}(W_{t_1} \in \mathrm{d}x_1,\dots,W_{t_n} \in \mathrm{d}x_n \mid W_T = x)$ are given by*

$$\prod_{i=1}^{n} g(t_i - t_{i-1}, x_i - x_{i-1}) \cdot \frac{g(T-t_n, x-x_n)}{g(T,x)}\mathrm{d}x_1\dots\mathrm{d}x_n, \quad (4.12)$$

where $g(\cdot,\cdot)$ is the heat kernel, for Lebesgue-almost every $x \in \mathbb{R}$.

Which directly implies the following.

Lemma 4.11. *For any measurable function h and Lebesgue-a.e. $x \in \mathbb{R}$,*

$$Q^{(1,x)}\big[h(B^{(1,x)})\big] = \mathbf{W}\big[h(W)\big|W_1 = x\big],$$

where $W = (W_t)_{t\in[0,1]}$ is a Brownian motion and $B^{(1,x)} = (B_t^{(1,x)})_{t\in[0,1]}$, $x \in \mathbb{R}$, are Brownian bridges with endpoint $B_1^{(1,x)} = x$.

Remark 4.5. These lemmas implies that the Brownian bridge $B^{(1,x)}$ can be defined in three equivalent ways: from a Brownian motion as

$$B_t^{(1,x)} = W_t + t(x - W_1), \quad 0 \le t \le 1,$$

by its finite dimensional distribution (4.11), or as a Brownian motion $(W_t)_{0\le t\le 1}$ conditioned to $W_1 = x$.

We will not prove these two lemmas, but we will use them to prove the following.

Lemma 4.12. *For any function measurable function h,*

$$\mathbf{W}[h(W)] = \int_{\mathbb{R}} g(1,x) Q^{(1,x)}[h(B^{(1,x)})]\mathrm{d}x.$$

Proof. Using conditional expectation, we can write

$$\mathbf{W}\Big[h\big(W\big)\Big] = \mathbf{W}\Big[\mathbf{W}\Big[h\big(W\big)\Big|W_1\Big]\Big].$$

By the previous lemma, this is equal to

$$\mathbf{W}\Big[Q^{(1,W_1)}\Big[h\big(B^{(1,W_1)}\big)\Big]\Big] = \int_{\mathbb{R}} Q^{(1,x)}\Big[h\big(B^{(1,x)}\big)\Big]\mathbf{W}(W_1 \in \mathrm{d}x),$$

and the last equality is because $Q^{W_1}\Big[h\big(B^{(1,W_1)}\big)\Big]$ is a function depending only on the endpoint W_1. W_1 distributes as $\mathcal{N}(0,1)$, so we obtain

$$\mathbf{W}\Big[h\big(W\big)\Big] = \int_{\mathbb{R}} g(1,x) Q^{(1,x)}\Big[h\big(B^{(1,x)}\big)\Big]\mathrm{d}x.$$

■

Using this, we can end the proof of Proposition 3.6.

Proof of Proposition 3.6. Define $h : C([0,1],\mathbb{R}) \to \mathbb{R}$ by

$$h(f) = \exp\Big\{\beta \sum_{j=1}^{K} m_j(f)\xi_j - \tfrac{\beta^2}{2}\sum_{j=1}^{K} m_j(f)^2\Big\}, \quad f \in C([0,1],\mathbb{R}).$$

Using the result (4.10) with this function h we obtain equation (4.11). ■

Appendix A

A.1 Subadditive lemma

A real sequence $\{a_n\}_{n\geq 1}$ is called *subadditive* if

$$a_{n+m} \leq a_n + a_m, \qquad n, m \geq 1.$$

A real sequence $\{a_n\}_{n\geq 1}$ is *superadditive* if $(-a_n; n \geq 1)$ is subadditive.

The following result is standard, *e.g.*, [MS13], Lemma 1.2.2.

Lemma A.1. *Let $\{a_n\}_{n\geq 1}$ be a sequence of numbers which is subadditive. Then the limit $\lim_{n\to\infty} n^{-1}a_n$ exists in $[-\infty, \infty)$ and is equal to*

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n} \tag{A.1}$$

Proof. It suffices to show that

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_k}{k} \tag{A.2}$$

for every k, since taking the $\liminf_{k\to\infty}$ in A.2 gives the existence of the limit, and then the result can be obtained by taking the $\inf_{k\geq 1}$, again in A.2.

Fix k and let

$$A_k = \max_{1\leq r\leq k} a_r.$$

Given a positive integer n we let j denote the largest integer which is strictly less than n/k. Then $n = jk + r$ for some integer r with $1 \leq r \leq k$. Using subadditivity, we have

$$a_n \leq ja_k + a_r \leq \frac{n}{k}a_k + A_k.$$

Dividing by n and taking the $\limsup_{n\to\infty}$ then gives A.2. ■

A.2 Local Limit Theorem

Recall some notation: $n \leftrightarrow x$, for $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$ if $P(S_n = x) > 0$; $q^{(n)}(x) = P(S_n = x)$, and $\overline{q}^{(n)}(x) = 2(d/2\pi n)^{d/2} \exp\{-d|x|^2/2n\}$. We state the standard local limit theorem for the simple random walk on $\mathbb{Z}^d$, we refer to [Law13] for a complete proof, although we state it as in [Com17].

Theorem A.2.

$$\sup_{n\leftrightarrow x} |q^{(n)}(x) - \overline{q}^{(n)}(x)| = O(n^{-(1+d/2)}).$$

In particular,

$$\sup_{n,x} q^{(n)}(x) = O(n^{-d/2}),$$

and, for all $A \in (0,\infty)$ there exists a $c > 0$ such that

$$\inf\{q^{(n)}(x); n \leftrightarrow x, |x| \leq An^{1/2}\} \geq Cn^{-d/2}.$$

Corollary A.2.1. *In dimension $d = 1$, $P(S_n = x)$ has the following formula*

$$P(S_n = x) = \tfrac{2}{\sqrt{n}} g(1, xn^{-1/2}) + O(n^{-3/2})$$

uniformly for x and n of the same parity, where $g(t,x) = (2\pi t)^{-1/2}\exp\{-x^2/2t\}$ is the heat equation kernel.

We now use this theorem to show the result (Proposition 2.5) mentioned in the introduction.

Proposition A.3. *The term of order $n^{-1/4}$ of the partition function $Z_n(\omega,\beta_n)$ converge in law to a normal distribution. That is,*

$$\beta n^{-1/4} \sum_{j=1}^{n} \sum_{x\in\mathbb{Z}} \omega(j,S_j) P(S_j = x) \xrightarrow{(d)} \mathcal{N}(0,\sigma^2),$$

where $\sigma^2 = 2\beta^2/\sqrt{\pi}$.

Proof. It suffices to show this for $\beta = 1$. Define

$$Y_n := n^{-1/4} \sum_{j=1}^{n} \sum_{x\in\mathbb{Z}} \omega(j,S_j) P(S_j = x).$$

We will prove that the characteristic functions converges, since it is equivalent to the statement of the lemma, namely

$$Y_n \xrightarrow{\text{(d)}} \mathcal{N}(0,\sigma^2) \iff \varphi_{Y_n} \longrightarrow \varphi_{\mathcal{N}(0,\sigma^2)}$$

Also define

$$X_j := \sum_{x\in\mathbb{Z}} \omega(j,S_j) P(S_j = x),$$

such that

$$\begin{aligned}\mathbb{P}[X_j] &= \sum_{x\in\mathbb{Z}} \mathbb{P}[\omega(j,x)P(S_j = x)] \\ &= \sum_{x\in\mathbb{Z}} \mathbb{P}[\omega(1,1)]P(S_j = x) = 0\end{aligned}$$

by Assumption 1, and similarly

$$\begin{aligned}\mathbb{P}[X_j^2] &= \sum_{x\in\mathbb{Z}} \mathbb{P}[\omega(j,x)^2]P(S_j = x)^2 + 2\sum_{x\neq y\in\mathbb{Z}} \mathbb{P}[\omega(j,x)\omega(j,y)]P(S_j = x)P(S_j = y) \\ &= \sum_{x\in\mathbb{Z}} P(S_j = x)^2 \ =: \sigma_j^2.\end{aligned}$$

The *Local Limit Theorem* (Corollary A.2.1) for the simple random walk tells us that

$$P(S_j = x) = \tfrac{2}{\sqrt{j}} g(1, xj^{-1/2}) + O(j^{-3/2}) \tag{A.3}$$

uniformly for x and j of the same parity (Note that trivially $|g| \leq (2\pi t)^{-1/2}$). So we can estimate, for $k \geq 2$:

$$\begin{aligned}\mathbb{P}[X_j^k] &= \mathbb{P}[\omega(1,1)^k]\sum_{x\in\mathbb{Z}} P(S_j = x)^k \\ &= \mathbb{P}[\omega(1,1)^k]\sum_{|x|\leq j, x\equiv_2 j} P^k(S_j = x)\end{aligned}$$

and using the Local Limit Theorem,

$$\begin{aligned}\mathbb{P}[X_j^k] &= \mathbb{P}[\omega(1,1)^k]\sum_{|x|\leq j, x\equiv_2 j} \left(\tfrac{2}{\sqrt{j}} g(1, xj^{-1/2}) + O(j^{-3/2})\right)^k \\ &= \mathbb{P}[\omega(1,1)^k]\sum_{|x|\leq j, x\equiv_2 j} \left\{\tfrac{2^k}{j^{k/2}} g^k(j, xj^{-1/2}) + O(j^{-(k+2)/2}\right\} \\ &= \mathbb{P}[\omega(1,1)^k]\left\{(j+1)2^k j^{-k/2} O(1) + O(j^{-k/2})\right\}.\end{aligned}$$

we know that moments of all order exist for $\omega(1,1)$ by Assumption 1.2, so for $k = 2$ this just tells us that $\mathbb{P}[X_j^2] = O(j^{-1})$, and for $k = 3$ we get $\mathbb{P}[X_j^3] = O(j^{-1/2})$.

Now, for $t \in \mathbb{R}$,

$$\begin{aligned}
\varphi_{Y_n}(t) &= \mathbb{P}\left[e^{itn^{-1/4}\sum_{j=1}^n X_j}\right] \\
&= \mathbb{P}\left[\prod_{j=1}^n e^{itn^{-1/4}X_j}\right] \\
&= \prod_{j=1}^n \mathbb{P}\left[e^{itn^{-1/4}X_j}\right]
\end{aligned}$$

by independence. As $tn^{-1/4} \to 0$ as $n \to \infty$ and the characteristic function is smooth,

$$\begin{aligned}
\varphi_{Y_n}(t) &= \prod_{j=1}^n \varphi_{X_j}(tn^{-1/4}) \\
&= \prod_{j=1}^n \left(1 - \tfrac{\sigma}{2}(tn^{-1/4})^2 - i\mathbb{P}\left[X_i^3 e^{i\xi_j X_j}\right] \tfrac{t^3 n^{-3/4}}{6}\right)
\end{aligned}$$

where we use the first and second moments of X_j previously calculated and $|\xi_j| < tn^{-1/4}$. As the third moment is $O(j^{-1/2})$ we have

$$\begin{aligned}
\varphi_{Y_n}(t) &= \prod_{j=1}^n \left(1 - \tfrac{\sigma}{2}t^2 n^{-\frac{1}{2}} + t^3 n^{-\frac{3}{4}} O(j^{-\frac{1}{2}})\right) \\
&= \exp\left\{\sum_{j=1}^n \log\left(1 - \tfrac{\sigma}{2}t^2 n^{-\frac{1}{2}} + t^3 n^{-\frac{3}{4}} O(j^{-\frac{1}{2}})\right)\right\}
\end{aligned}$$

and since we will let $n \to \infty$ we can use the Taylor expansion of $\log(1+x)$

$$\begin{aligned}
\varphi_{Y_n}(t) &= \exp\left\{\sum_{j=1}^n \left(\frac{\sigma}{2}t^2 n^{-\frac{1}{2}} + t^3 n^{-\frac{3}{4}} O(j^{-\frac{1}{2}}) + \left(t^2 n^{-\frac{1}{2}} O(\tfrac{1}{j}) + t^3 n^{-\frac{3}{4}} O(j^{-\frac{1}{2}})\right)^2\right)\right\} \\
&= \exp\left\{\sum_{j=1}^n \left(\tfrac{\sigma}{2}t^2 n^{-\frac{1}{2}} + t^3 n^{-\frac{3}{4}} O(j^{-\frac{1}{2}}) + \tfrac{t^4}{n} O(j^{-2}) + t^6 n^{-\frac{3}{2}} O(\tfrac{1}{j}) + t^5 n^{-\frac{5}{4}} O(j^{-\frac{3}{2}})\right)\right\} \\
&= \exp\left\{\left(-\frac{t^2}{2n^{\frac{1}{2}}}\sum_{j=1}^n \sigma_j^2\right) + t^3 O(n^{-\frac{1}{4}}) + t^4 O(n^{-2}) + t^6 O(\log(n) n^{-\frac{3}{2}}) + t^5 O(n^{-\frac{7}{4}})\right\}
\end{aligned}$$

Thus, if $n \to \infty$,

$$\lim_{n\to\infty} \varphi_{Y_n}(t) = \exp\left\{-\tfrac{t^2}{2} \lim_{n\to\infty} n^{-\frac{1}{2}} \sum_{j=1}^n \sigma_j^2\right\}$$

We know that $\varphi_{\mathcal{N}(0,\sigma^2)} = e^{-\frac{t^2\sigma^2}{2}}$, so we only need to show that the limit inside the exponential is $\frac{2}{\sqrt{\pi}}$. We know that a walk S in the step j can only be on integers x such that $|x| \leq j$ and $x \equiv j (\mathrm{mod} 2)$, so

$$\begin{aligned}\sigma_j^2 &= \sum_{x \in \mathbb{Z}} P(S_j = x)^2 \\ &= \sum_{x=0}^{j} P(S_j = 2x - j)^2\end{aligned}$$

and using again the Local Limit Theorem

$$\begin{aligned}\sigma_j^2 &= \sum_{x=0}^{j} \left(\tfrac{2}{\sqrt{j}} g(1, (2x-j)j^{-1/2}) + O(j^{-3/2}) \right)^2 \\ &= \sum_{x=0}^{j} \left\{ \tfrac{4}{j} \left((2\pi)^{-1/2} e^{-(2x-j)^2/2j} \right)^2 + O(j^{-2}) \right\} \\ &= \left\{ \tfrac{4}{j} \sum_{x=0}^{j} (2\pi)^{-1} e^{-(2x-j)^2/j} \right\} + O(\tfrac{1}{j})\end{aligned}$$

So, since

$$n^{-1/2} \sum_{j=1}^{n} O(\tfrac{1}{j}) = n^{-1/2} O(\log n) \longrightarrow 0,$$

we can ignore that term (and the constant $\frac{2}{\pi}$) and just show that

$$\lim_{n \to \infty} n^{-1/2} \sum_{j=1}^{n} \tfrac{1}{j} \sum_{x=0}^{j} e^{-\frac{(2x-j)^2}{j}} = \sqrt{\pi}.$$

Here our intuition tells us that we should compare the expression with the Gaussian integral $\int_0^\infty e^{t^2} \mathrm{d}t$. That is exactly what we will do.

For simplification, denote $f_j = \exp\{-t^2/j\}$. We can visually check, rearranging the sums $\sum \exp\{-(2x-j)^2/j\}$ to fall "near" the intervals $[0, j+1]$, that

$$\begin{array}{llll} -1 + \int_0^{j+1} f_j & \leq \sum_{x=0}^{j} \exp\{-(2x-j)^2/j\} & \leq 2 + \int_0^{j+1} f_j, & \text{for even } j \\ \int_0^{j+1} f_j & \leq \sum_{x=0}^{j} \exp\{-(2x-j)^2/j\} & \leq 3 + \int_0^{j+1} f_j, & \text{for odd } j \end{array}$$

So we write

$$\sum_{x=0}^{j} e^{-(2x-j)^2/j} = \int_0^{j+1} e^{-t^2/j} \mathrm{d}t + O(1).$$

Then,

$$\begin{aligned}\lim_{n\to\infty} n^{-\frac{1}{2}} \sum_{j=1}^{n} j^{-1} \sum_{x=0}^{j} e^{-(2x-j)^2/j} &= \lim_{n\to\infty} n^{-\frac{1}{2}} \sum_{j=0}^{n} \left(j^{-1} \int_0^{j+1} e^{-t^2/j} \mathrm{d}t + O(\tfrac{1}{j}) \right) \\ &= \lim_{n\to\infty} n^{-\frac{1}{2}} \left(\sum_{j=1}^{n} j^{-\frac{1}{2}} \int_0^{\frac{j+1}{\sqrt{j}}} e^{-t^2} \mathrm{d}t + O(\log n) \right)\end{aligned}$$

by summing the harmonic sequence and substituting $t \to t\sqrt{j}$. Now, ignoring the vanishing $O(\log n/\sqrt{n})$, we bound by above (we will squeeze our limit):

$$\begin{aligned}\lim_{n\to\infty} n^{-\frac{1}{2}} \sum_{j=1}^{n} j^{-1} \sum_{x=0}^{j} e^{-(2x-j)^2/j} &= \lim_{n\to\infty} n^{-\frac{1}{2}} \sum_{j=1}^{n} j^{-\frac{1}{2}} \int_0^{\frac{j+1}{\sqrt{j}}} e^{-t^2} \mathrm{d}t \\ &\le \lim_{n\to\infty} n^{-\frac{1}{2}} \sum_{j=1}^{n} j^{-\frac{1}{2}} \int_0^{\infty} e^{-t^2} \mathrm{d}t\end{aligned}$$

where we recognize the Gaussian integral $\int_0^\infty e^{-t^2}\mathrm{d}t = \frac{\sqrt{\pi}}{2}$, then

$$\begin{aligned}&\lim_{n\to\infty} n^{-\frac{1}{2}} \sum_{j=1}^{n} j^{-1} \sum_{x=0}^{j} e^{-(2x-j)^2/j} \le \tfrac{\sqrt{\pi}}{2} \lim_{n\to\infty} n^{-\frac{1}{2}} \sum_{j=1}^{n} j^{-\frac{1}{2}} \\ &= \tfrac{\sqrt{\pi}}{2} \lim_{n\to\infty} n^{-\frac{1}{2}} (2n^{\frac{1}{2}} + O(1)) = \sqrt{\pi}\end{aligned}$$

For the other bound, let $\varepsilon > 0$ and take $K \in \mathbb{N}$ such that $\int_0^{\frac{K+1}{\sqrt{K}}} e^{-t^2}\mathrm{d}t > \frac{\sqrt{\pi}}{2} - \frac{\varepsilon}{2}$. Then,

$$\begin{aligned}\lim_{n\to\infty} n^{-\frac{1}{2}} \sum_{j=1}^{n} j^{-\frac{1}{2}} \int_0^{\frac{j+1}{\sqrt{j}}} e^{-t^2} \mathrm{d}t &\ge \lim_{n\to\infty} n^{-\frac{1}{2}} \sum_{j=K}^{n} j^{-\frac{1}{2}} \int_0^{\frac{j+1}{\sqrt{j}}} e^{-t^2} \mathrm{d}t \\ &\ge \lim_{n\to\infty} n^{-\frac{1}{2}} \sum_{j=K}^{n} j^{-\frac{1}{2}} \int_0^{\frac{K+1}{\sqrt{K}}} e^{-t^2} \mathrm{d}t\end{aligned}$$

where we will use the property required for the fixed K and again use that the harmonic series is asymptotic to the logarithm

$$\begin{aligned}\lim_{n\to\infty} n^{-\frac{1}{2}} \sum_{j=1}^{n} j^{-\frac{1}{2}} \int_0^{\frac{j+1}{\sqrt{j}}} e^{-t^2} \mathrm{d}t &> \lim_{n\to\infty} n^{-\frac{1}{2}} \sum_{j=K}^{n} j^{-\frac{1}{2}} \left(\tfrac{\sqrt{\pi}}{2} - \tfrac{\varepsilon}{2} \right) \\ &= \left(\tfrac{\sqrt{\pi}}{2} - \tfrac{\varepsilon}{2} \right) \lim_{n\to\infty} n^{-\frac{1}{2}} (2n^{\frac{1}{2}} + O(1)) = \sqrt{\pi} - \varepsilon.\end{aligned}$$

And since $\varepsilon > 0$ was arbitrary, we get the desired limits and therefore the wanted convergence in law. ■

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