Pontificia Universidad Católica de Chile

Master Thesis

Brauer group of abelian schemes

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1 Introduction

The Brauer group $\mathrm{Br}(k)$ of a field k was introduced by Richard Brauer in 1932 [1, p. 243] as an attempt to classify division algebras over k. It is central in the modern formulation of class field theory and it is also related to Galois cohomology as

$$\mathrm{Br}(k) \cong H^2(\mathrm{Gal}(k_s/k), k_s^\times) \tag{1}$$

by Theorem 1. This isomorphism helps us understand cohomological classes as concrete algebraic objects related to k and, on the other hand, helps to study these concrete objects via cohomological tools.

In 1951, Goro Azumaya in [2] defined and studied the Brauer group of a local ring, and Auslander&Goldman in [3] generalized this to any commutative ring. Finally, Grothendieck defined and studied the Brauer group of any scheme in a series of *Exposés* [9–11]. The Brauer group of a scheme has been proven quite useful. Indeed, its elements can be often be interpreted as obstructions. In arithmetic, it can be seen to obstruct the existence of rational points [40, Section 8.2]. In geometry, non-trivial Brauer classes obstruct stable rationality [48, Chapter 12]. In the theory of moduli, Brauer classes obstruct natural geometric constructions, see [60, Cor. 3.24, Prop. 3.26] and [38].

In fact, both sides of the isomorphism in Equation (1) generalize to schemes. More precisely, if X is a scheme we denote by $\mathrm{Br}(X)$ its Brauer group and by $\mathrm{Br}'(X) := H^2(X_{\text{ét}}, \mathbb{G}_m)_{\text{tors}}$ the *cohomological Brauer group* of X. An isomorphism between these two does not exist in general, in contrast with the case of a field, but there is always an injective group morphism, the *Brauer map*, $\mathrm{Br}(X) \to \mathrm{Br}'(X)$ and Grothendieck asked to determine when it is surjective. This question is known as the $\mathrm{Br} = \mathrm{Br}'$ problem and remains open. The first step towards an answer was given by Grothendieck himself in [10, Thm. 2.1], where he shows that cohomological Brauer classes on regular schemes are represented by Azumaya algebra away from codimension 3, thereby establishing the result for regular surfaces. In 1972 Berkovich [15, Prop. 1] and Hoobler [16, Prop. 3.3] simultaneosuly (and separately) proved that $\mathrm{Br} = \mathrm{Br}'$ for abelian varieties (up to a p-component if the characteristic of the base field is p). Later, in 1980, Hoobler proved $\mathrm{Br} = \mathrm{Br}'$ for smooth affine schemes over a field [20, Cor. 1]. A year after, Ofer Gabber proved $\mathrm{Br} = \mathrm{Br}'$ for affine schemes (and more generally, for a separated union of two affine schemes) [21, II, Thm 1]. In the 90's, all these results were subsumed by a powerful (unpublished) result of Gabber: $\mathrm{Br} = \mathrm{Br}'$ for any scheme with an ample line bundle (e.g. quasi-projective schemes). A different proof of this result was found by de Jong in the early 2000's using twisted sheaves [48, Chapter IV]. This perspective shifted the attention for Azumaya algebras towards vector bundles on stacks. This idea was used by Edidin, Hassett, Kresch and Vistoli in [28, Cor. 3.11] where they found the first counterexample to Grothendieck's question. This example is a non-separated normal surface over a field. Soon afterwards, Schröer in [29, Thm. 3.1] proves that every two-dimensional separated geometrically normal algebraic space that is of finite type over a field has $\mathrm{Br} = \mathrm{Br}'$. In fact, the geometrically normal hypothesis is superfluous as proven by Mathur in [49, Cor. 4].

The relationship with vector bundles on stacks opened up connections to other problems in algebraic geometry; the resolution property [31, 32, 42, 47, 49, 52, 57, 63], extension problems [51] and quotient stacks [45]. We also remark that many researchers have studied variants of the $\mathrm{Br} = \mathrm{Br}'$ problem. Perhaps most famous is the work of Toen who studies the existence of derived Azumaya Algebras [39]. Schröer [34] and Schröer-Huybrechts [30] studied the analog for complex analytic surfaces and Schröer-Heinloth [36] the representability by non-Azumaya algebras. More recently, de Jong, Lieblich and Shin have studied twisted sheaves of infinite rank [59]. As of today, the $\mathrm{Br} = \mathrm{Br}'$ problem is still studied, for stacks see Shin's work [46, 54, 61], but especially for separated schemes. For instance, the problem is still widely open for smooth algebraic threefolds!

The purpose of this thesis is to revisit Hoobler's work in [16] where he proves the following theorem, and erase all hypothesis on S based on a remark made by Hoobler loc. cit.

Theorem 1.1 ([16, Cor. 2.6, Prop. 3.3])**.** Let $A \to S$ be an abelian scheme with S reduced, connected and geometrically unibranch (eg. normal) with identity section e and $l \notin \mathrm{char}(S)$ a prime. Then for any cohomology class $\alpha \in \mathrm{Br}'(A)[l^\infty]$ and $e^*\alpha = 0 \in \mathrm{Br}'(S)$ is represented by an Azumaya algebra.

Raynaud proves in [6, Thm. XI.1.4] that an abelian scheme over an integral geometrically unibranch scheme is S-projective, so it seems that Hoobler's work is subsumed by Gabber's result. In this thesis, we argue that Hoobler's methods do not require any hypothesis on S, so in fact his methods genuinely construct new Azumaya algebras!

The theory needed to prove this result will be developed. Most of these material is covered in classical textbooks. The first chapter is on Brauer groups of fields, and the main references are [19], [40], [37], and [48]. The second chapter is about étale cohomology and the main references are [22], [40], [55] the Stacks Project [41] and EGA IV [5, 8].

2 Notation and conventions

A k-algebra for us will be always unital and associative, but not necessarily commutative. Given a field k, we will denote by $\overline{k}$ and k_s a fixed algebraic closure and separable closure respectively. An abelian or cylic extension L/k is always assumed finite and Galois.

Given a category $\mathcal{C}$ and objects $\{X_i\}_{i\in I}$ whose product exists, we will denote by

$$\mathrm{pr}_J \colon \prod_{i\in I} X_i \to \prod_{i\in J} X_i$$

the natural projections. If $J = \{i, j\}$ we denote pr_{ij} instead of pr_J and similarly for $J = \{i, j, k\}$. We also denote sometimes X_{ij} instead of the product $X_i \times X_j$ and similarly for three objects.

Chapter 1

The Brauer group of a field

1 Algebras over a field

Definition 1.1 (Algebras over a field)**.** Let k be a field.

(a) A *division algebra* D over k is a (non-zero) k-algebra where every non-zero element is invertible.

(b) A k-algebra A is *central* if the image of the structure morphism $k \to A$ is the center of A. We say A is *simple* if it is simple as a ring, i.e, the only two-sided ideals of A are the zero ideal and A itself.

(c) A k-algebra that is finite-dimensional over k, central and simple is a *central simple algebra* (or CSA).

(d) An *Azumaya algebra* over k is a k-algebra A such that $A \otimes_k \overline{k} \cong M_n(\overline{k})$ as $\overline{k}$-algebras for some $n \geq 1$. The number n here is the *degree* of A. If $A \otimes_k L \cong M_n(L)$ already for an algebraic extension L/k we say L *splits* A, or A is *split* by L.

Example 1.2. The set of $n \times n$ matrices over k, $M_n(k)$, is a central simple k-algebra.

Example 1.3 (Cyclic algebras)**.** Let L/k be a cyclic extension of degree n. Choose σ a generator of $G = \mathrm{Gal}(L/k)$ and define $\chi \colon G \to \mathbb{Z}/n\mathbb{Z}$, by $\sigma \mapsto 1$. Given $b \in k^\times$, we define the *cyclic algebra* $D_k(b, \chi) := L\{y \mid y^n = b,\ \lambda y = y\sigma(\lambda),\ \forall \lambda \in L\}$. It is a central simple k-algebra of degree n [37, Construction 2.5.1, Prop. 2.5.2].

Assume k contains a primitive nth-root of unity and $(n, \mathrm{char}\, k) = 1$. In this case $L = k(\sqrt[n]{a})$ for some $a \in k^\times \setminus k^{\times n}$ by Kummer theory and a generator of $\mathrm{Gal}(L/K)$ is determined by the choice of primitive nth-root of unity, say ω. The corresponding cyclic algebra is denoted by $D_k(a, b, \omega)$ and admits $k\{i, j \mid i^n = a, j^n = b, ij = \omega ji\}$ as a description. For $n = 2$ these are called *quaternion algebras*. For example $Q_{\mathbb{R}}(-1, -1)$ is the $\mathbb{R}$-algebra of Hamilton quaternions.

The following theorem characterizes Azumaya algebras over a field k and also relates the different kind of algebras we've defined.

Theorem 1.4 (Characterization of Azumaya algebras)**.** Let k be a field. The following conditions on a k-algebra A are equivalent:

(a) A is central simple algebra over k;

(b) A is an Azumaya algebra over k;

(c) There exists a field extension L/k that splits A;

(d) There exists a finite separable extension L/k that splits A (so an Azumaya algebra already splits over the separable closure);

(e) There is an integer $r \geq 1$ and a division algebra D over k such that $A \cong M_r(D)$. Moreover, r is uniquely determined and D is unique up to isomorphism.

Proof. [37, Chapter II, Thm. 2.1.3, Cor. 2.1.7, Thm. 2.2.1, Cor. 2.2.11, Cor. 2.2.12]. □

Definition 1.5. Given a k-algebra A we define A^{opp} to be the same k-vector space but with ring multiplication given by $a \cdot b := ba$. It is called the *opposite algebra* of A.

Proposition 1.6. Let k be a field.

(a) If A, B are Azumaya algebras, then so are $A \otimes_k B$ and A^{opp}.

(b) The natural morphism

$$A \otimes_k A^{\text{opp}} \to \operatorname{End}_k(A) \cong M_{n^2}(k)$$
$$a \otimes b \mapsto (x \mapsto axb)$$

where $n = \dim_k A$, is an isomorphism.

Proof. [37, Chapter II, Lem. 2.2.5, Prop. 2.2.6]. □

Definition 1.7. Two Azumaya algebras A, B are *equivalent* ($A \sim B$) if there are $m, n \geq 1$ such that

$$A \otimes_k M_n(k) \cong B \otimes_k M_m(k).$$

Remark 1.8. By Theorem 1.4.(e), every Azumaya algebra is isomorphic to a unique (up to isomorphism) division algebra D. It follows that two Azumaya algebras over k of the same rank are equivalent iff they are k-isomorphic.

Proposition 1.9 (Brauer group of a field, [37, Prop. 2.4.7])**.** The set of isomorphism classes of Azumaya algebras over k with the tensor operation forms a group with identity element given by (the class of) $M_n(k)$ (for any n). It is called the *Brauer group* of k and we denote it by $\operatorname{Br}(k)$.

Definition 1.10. The set of equivalence classes of Azumaya k-algebras of degree n is denoted by $\mathrm{Az}_{n,k}$. The set of equivalence classes of Azumaya algebras split by a fixed field extension L/k is denoted by $\operatorname{Br}(L/k)$. If L/k is an extension of fields, we get a natural map $\operatorname{Br}(k) \to \operatorname{Br}(L)$, given by $[A] \mapsto [A \otimes_k L]$ with kernel $\operatorname{Br}(L/k)$.

2 Cohomological interpretation

The goal of this section is to prove that the Brauer group of a field k admits a cohomological description, namely there is a functorial isomorphism of groups

$$\delta_k \colon \operatorname{Br}(k) \xrightarrow{\sim} H^2(k, \mathbb{G}_m(k_s)).$$

For basic notions of Galois cohomology we refer to [43, Chapter I-IV]. For H^0 and H^1 in the nonabelian case we refer to [19, Pages 123-126].

Proposition 2.1 ([40, Prop. 1.5.9])**.** There is a bijection of pointed sets

$$\mathrm{Az}_{n,k} / \sim \xrightarrow{\sim} H^1(k, \mathrm{PGL}_n(k_s)).$$

Proof. Recall that the group of k-automorphisms of $M_n(k)$ is isomorphic to $\mathrm{PGL}_n(k)$, see [37, Lem. 2.4.1]. We write $G_k := \mathrm{Gal}(k_s/k)$ and notice G_k acts on $\mathrm{PGL}_n(k_s)$ by acting on the entries of a matrix.

Let $A \in \mathrm{Az}_{n,k}$ and choose an isomorphism $\phi\colon A \otimes_k k_s \to M_n(k_s)$. Define a function $c_\phi\colon G_k \to \mathrm{PGL}_n(k)$ by $\sigma \mapsto \phi^\sigma \phi^{-1} \in \mathrm{Aut}_k(M_n(k_s)) \cong \mathrm{PGL}_n(k_s)$. We claim c_ϕ is a 1-cocycle and its cohomology class does not depend on the isomorphism chosen. The first assertion is a simple computation:

$$c_\phi(\sigma\tau) = \phi^{\sigma\tau}\phi^{-1} = \left(\phi^\sigma\phi^{-1}\right)^\tau \left(\phi^\tau\phi^{-1}\right) = c_\phi(\sigma)^\tau c_\phi(\tau).$$

Now choose another isomorphism $\psi\colon A \otimes_k k_s \to M_n(k_s)$ and let $\alpha = \phi\psi^{-1} \in \mathrm{PGL}_n(k_s)$. Then

$$\alpha^\sigma c_\psi(\sigma) = \phi^\sigma(\psi^{-1})^\sigma\psi^\sigma\psi^{-1} = \phi^\sigma\psi^{-1} = \phi^\sigma\phi^{-1}\phi\psi^{-1} = c_\phi(\sigma)\alpha$$

shows c_ϕ and c_ψ are cohomologous. If $A \sim B$, i.e, there is a k-isomorphism $\psi'\colon A \to B$ and hence a k_s-isomorphism $\psi\colon A \otimes_k k_s \to B \otimes_k k_s$. If $\phi\colon B \otimes_k k_s \to M_n(k_s)$ is a k_s-isomorphism, then $c_\phi = c_{\phi\psi}$. Thus, we have a well-defined map

$$\begin{aligned} \mathrm{Az}_{n,k} / \sim\ &\to H^1(k, \mathrm{PGL}_n(k_s)) \\ A &\mapsto [c_\phi], \end{aligned}$$

where $\phi\colon A \otimes_k k_s \to M_n(k_s)$ is any k_s-isomorphism.

For the injectivity, suppose $A, B \in \mathrm{Az}_k$ are such that $c(A)$ and $c(B)$ are cohomologous, i.e, there is $\alpha \in \mathrm{PGL}_n(k_s)$ such that

$$\alpha^\sigma c(A)(\sigma) = c(B)(\sigma)\alpha.$$

Choose isomorphisms ϕ and ψ as above so that $c(A) = [c_\phi]$ and $c(B) = [c_\psi]$. Then being cohomologous translates to

$$\alpha^\sigma\psi^\sigma\psi^{-1} = \phi^\sigma\phi^{-1}\alpha, \qquad \forall\sigma \in G_k.$$

Changing ψ by $\alpha\psi$ we get $\psi^\sigma\psi^{-1} = \phi^\sigma\phi^{-1}$ for every $\sigma \in G_k$. But this means $\left(\psi^{-1}\phi\right)^\sigma = \psi^{-1}\phi$ for all $\sigma \in G_k$ so $\psi^{-1}\phi\colon A \otimes_k k_s \to B \otimes_k k_s$ is really defined over k, proving the injectivity. The surjectivity is a special case of Galois descent, see [44, Thm. 14.85]. □

Consider the following exact sequence of groups

$$0 \longrightarrow \mathbb{G}_m(k_s) \longrightarrow \mathrm{GL}_n(k_s) \longrightarrow \mathrm{PGL}_n(k_s) \longrightarrow 0$$

By non-abelian cohomology [19, Prop. 2, p. 125], we get an exact sequence of pointed sets

$$H^1(k, \mathrm{GL}_n(k_s)) \longrightarrow \mathrm{Az}_{n,k} / \sim \xrightarrow{\delta_{n,k}} H^2(k, \mathbb{G}_m(k_s)),$$

where we used last proposition's identification. Now the main theorem of this section:

Theorem 2.2 ([19, X, §5, Lemma 1])**.** The *Brauer map* $\delta_k\colon \mathrm{Br}(k) \to H^2(k, \mathbb{G}_m(k_s))$ is an isomorphism of groups, functorial in k.

Proof. Functoriality comes from the non-abelian cohomology sequence. The map δ_k has an explicit description in terms of cocycles and it is possible to prove it is a morphism of groups, tracing all definitions. Unfortunately, it is too long and messy, so we won't include it.

For the injectivitity. Suppose $\delta_k([A]) = \delta_k([B])$ for some A, B of degree m and n respectively, so $\delta_{mn,k}([A \otimes_k B^{\text{opp}}]) = 0$. As $H^1(k, \mathrm{GL}_n(k_s)) = 0$, we get that $A \otimes M_{n^2}(k) \cong M_{mn}(k) \otimes_k B$, i.e, $A \sim B$. Now we prove surjectivity. As

$$H^2(k, \mathbb{G}_m(k_s)) = \varinjlim_{\substack{L/k \\ \text{finite Galois}}} H^2(\mathrm{Gal}(L/k), \mathbb{G}_m(L)),$$

it suffices to prove that for a finite Galois extension L/k with Galois group G, any 2-cocycle $c\colon G^2 \to L^\times$ is a 2-coboundary, i.e, it can be written as

$$c(\sigma,\tau)b(\sigma\tau) = b(\sigma)\sigma(b(\tau)) \tag{1.1}$$

for some function $b\colon G \to \mathrm{PGL}_n(L)$. Let V be a L-vector space with a basis $\{e_\rho,\ \rho \in G\}$, with Galois action given by $\sigma e_\tau = e_{\sigma\tau}$. Given $\sigma \in G$, let $b(\sigma)\colon L \to L$ be the L-morphism that sends e_τ to $c(\sigma,\tau)\sigma(e_\tau)$. It permutes and rescales the basis so it is an automophism, i.e, $b(\sigma) \in \mathrm{Aut}(V) \cong \mathrm{GL}_n(L)$. Now for each $\rho \in G$

$$[b(\sigma)\sigma(b(\tau))](e_\rho) = c(\sigma,\tau\rho)\cdot\sigma(c(\tau,\rho))\cdot e_{\sigma\tau\rho}$$
$$[c(\sigma,\tau)b(\sigma\tau)](e_\rho) = c(\sigma,\tau)\cdot c(\sigma\tau,\rho)\cdot e_{\sigma\tau\rho},$$

hence (1.1) follows from the fact that c is a 2-cocycle. Composing with the projection $\mathrm{GL}_n(L) \to \mathrm{PGL}_n(L)$ we get the desired function. □

Last theorem allows us to identify $\mathrm{Br}(L/k) \cong H^2(\mathrm{Gal}(L/K), \mathbb{G}_m(L))$.

Example 2.3. By last theorem,

$$\mathrm{Br}(\mathbb{R}) \cong H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times) \cong \widehat{H}^0(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^\times) = \mathbb{R}^\times/N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times) = \mathbb{R}^\times/\mathbb{R}^{>0} \cong \mathbb{Z}/2\mathbb{Z},$$

where we used Tate modified cohomology [43, Defi. 2.3] and the cohomology of a cyclic group [43, Thm. 2.16].

3 Severi-Brauer varieties

There is a third interpretation of the Brauer group using schemes. By Galois descent 3.12 and Theorem 7.1, $H^1(k, \mathrm{PGL}_n(k_s))$ classifies k-schemes X such that $X \times_k k_s \cong \mathbb{P}^{n-1}_{k_s}$, as

$$\mathrm{Aut}(\mathbb{P}^{n-1}_{k_s}) = \mathrm{PGL}_{n,k_s}.$$

Definition 3.1. A *Severi-Brauer k-variety* of dimension n is a k-scheme X such that $X \times_k k_s \cong \mathbb{P}^n_{k_s}$.

By faithfully flat descent a Severi-Brauer variety is smooth, proper and geometrically integral as $\mathbb{P}^n_k$ has these properties, see [5, 2.7.1.(vii), 4.6.5.(i)] and [8, 17.7.3.(ii)].

Proposition 3.2. Let SB^{n-1}_k be the set of k-isomorphism classes of Severi-Brauer k-varieties of dimension $n-1$. There is a bijection of pointed sets

$$SB^{n-1}_k \xrightarrow{\sim} H^1(k, \mathrm{PGL}_n(k_s)).$$

Proof. This follows from Galois descent 3.12. □

Example 3.3 (Severi-Brauer varieties of dimension 1 are conics)**.** Let X be a Severi-Brauer variety of dimension 1. As $X \to k$ is smooth, the tangent bundle $\mathcal{T}_{X/k}$ is locally free. Let $f\colon \mathbb{P}^1_{k_s} \to X$ be the projection. Then $f^*\mathcal{T}_{X/k} \cong \mathcal{T}_{\mathbb{P}^1_{k_s}/k_s} \cong \mathcal{O}_{\mathbb{P}^1_{k_s}}(2)$. By flat base change [44, Cor. 12.8],

$$H^0(\mathbb{P}^1_{k_s}, \mathcal{O}_{\mathbb{P}^1_{k_s}}(2)) \cong H^0(X, \mathcal{T}_{X/k}),$$

hence we have three sections in $H^0(X, \mathcal{T}_{X/k})$ that generate $\mathcal{T}_{X/k}$ globally, corresponding to $x_0, x_1, x_2 \in H^0(\mathbb{P}^1_{k_s}, \mathcal{O}_{\mathbb{P}^1_{k_s}}(2))$. These define an immersion into $\mathbb{P}(\mathcal{T}_{X/k}) \cong \mathbb{P}^2_k$ which is closed since X is proper. After base change, this is the 2-uple embedding, hence X is realized as plane conic as well.

By construction, a Severi-Brauer k-variety is trivial if and only if it is k-isomorphic to $\mathbb{P}^n_k$ for some $n \geq 1$. Moreover

Theorem 3.4 (Châtelet, [40, Prop 4.5.10])**.** Let X be a Severi-Brauer k-variety of dimension $n-1$. The following are equivalent:

(a) X is k-isomorphic to $\mathbb{P}^{n-1}_k$.

(b) X is k-birational to $\mathbb{P}^{n-1}_k$.

(c) X has a k-rational point.

Proof. $(a) \implies (b)$ is trivial. Now we prove $(b) \implies (c)$. If k is a finite field, then every Severi-Brauer k-variety is isomorphic to $\mathbb{P}^{n-1}_k$ by Example 4.3.(a), so it obviously has a k-rational point. If k is infinite, then a dense open subset of $\mathbb{P}^{n-1}_k$ contains a k-rational point, hence so does X.

For $(c) \implies (a)$, take $x \in X(k)$. Now take an isomorphism $X_{k_s} \cong \mathbb{P}^{n-1}_{k_s}$ and compose it with an automorphism so that x maps to the point $P := (1 : 0 : \cdots : 0) \in \mathbb{P}^{n-1}_{k_s}$. Then (X, x) can be seen as a twist of $(\mathbb{P}^{n-1}_k, P)$ (I hope the reader is already familiar with the notion of twist). Thus, for our purposes it suffices to show that $H^1(k, \mathrm{Aut}(\mathbb{P}^{n-1}_{k_s}, P))$ is trivial. The group $\mathrm{Aut}(\mathbb{P}^{n-1}_{k_s}, P)$ is a subgroup of $\mathrm{PGL}_n(k_s)$ isomorphic to

$$\begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix} \pmod{k)_s^\times} \quad \cong \quad \begin{pmatrix} 1 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix}$$

The map from the group on the right to $\mathrm{GL}_{n-1}(k_s)$ defined by forgetting the first row and column is surjective and fits in an exact sequence

$$0 \longrightarrow (k_s)^{n-1} \longrightarrow \mathrm{Aut}(\mathbb{P}^{n-1}_{k_s}, P) \longrightarrow \mathrm{GL}_{n-1}(k_s) \longrightarrow 0.$$

By Hilbert's 90 10.9 and the fact k_s^{n-1} is cohomologically trivial [43, Cor. 6.3], the group in the middle has trivial H^1 as desired. □

4 Examples of Brauer groups

Definition 4.1. A field k is C_r, $r \geq 0$, if any homogeneous polynomial of degree d in $n > d^r$ variables has a nontrivial zero in k^n.

For example, a field is C_0 if and only if it is algebraically closed.

Proposition 4.2. If k is C_0 or C_1, then $\mathrm{Br}(k)$ is trivial.

Proof. If k is C_0, it is algebraically closed. Take D a finite-dimensional central division k-algebra and $x \in D$. Take f the minimal polynomial of x, so that $k(x) \cong k[t]/(f)$. Since k is algebraically closed, $k(x) \cong k$ and hence $x \in k$. This settles the first case.

For the second case, take k a C_1 field and D a central division k-algebra of degree n and $\mathrm{nr}\colon D \to k$ the reduced norm (see [40, 1.5.3]). Take $v_1, \dots, v_{n^2} \in D$ a k-basis of D. Then the polynomial

$$f(x_1, \dots, x_{n^2}) = \mathrm{nr}(x_1 v_1 + \cdots + x_{n^2} v_{n^2})$$

is a homogeneous polynomial of degree n in n^2-variables. If $n > 1$, then the C_1-property of k implies nr has a non-trivial zero, which is impossible as any non-zero element of D is invertible, and hence has non-zero reduced norm. Thus, $n = 1$, and the claim follows. □

For C_r-fields there is the following important transition theorem.

Example 4.3. (a) Finite fields are C_1, hence have trivial Brauer group (see [17, I, §2, Thm. 3]).

(b) Function fields of curves are C_1 by [40, Thm. 1.2.7], hence have trivial Brauer group. This is known as Tsen's theorem.

(c) (Local class field theory) For any (non-archimedean) local field, there is an isomorphism $\mathrm{inv}_k\colon \mathrm{Br}(k) \to \mathbb{Q}/\mathbb{Z}$ (see [43, Thm. 8.9]).

(d) (Global class field theory) For any global field k, there is an exact sequence

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \bigoplus_{\nu \in \Omega_k} \mathrm{Br}(k_\nu) \xrightarrow{\mathrm{inv}_k} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

where $\mathrm{inv}_k = \sum_{\nu \in \Omega_k} \mathrm{inv}_{k_\nu}$ (see [43, Thm. 14.11]).

Chapter 2

Étale and flat cohomology

In this chapter we will develop some of the étale (and flat) cohomology theory needed in order to define and study the cohomological Brauer group of a scheme.

1 Grothendieck topologies and sites

Definition 1.1. Let $\mathcal{C}$ be a category where fibered products exist. A *Grothendieck topology* on $\mathcal{C}$ consists of a set $\mathrm{Cov}(X)$ of collections of morphisms $\{X_i \to X\}_{i\in I}$ for each $X \in \mathrm{Ob}(\mathcal{C})$ that satisfy:

(a) Any isomorphism $V \xrightarrow{\sim} X$ is in $\mathrm{Cov}(X)$.

(b) If $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$ and $Y \to X$ is an arrow in $\mathcal{C}$, then

$$\{X_i \times_X Y \to Y\}_{i\in I} \in \mathrm{Cov}(Y).$$

(c) If $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$ and $\{X_{ij} \to X_i\}_{j\in J_i} \in \mathrm{Cov}(X_i)$ for each $i \in I$, then

$$\{X_{ij} \to X\}_{i\in I,\, j\in J_i} \in \mathrm{Cov}(X).$$

The collections $\{X_i \to X\} \in \mathrm{Cov}(X)$ are called *coverings* of X. A category equipped with a Grothendieck topology is called a *site*.

Example 1.2 (Classic topology)**.** If X is a topological space, we may consider the category $\mathrm{Op}(X)$ whose objects are open subsets of X, and

$$\mathrm{Hom}(U,V) = \begin{cases} \{U \to V\} & \text{if } U \subseteq V \\ \emptyset & \text{otherwise.} \end{cases}$$

For each open U, we define $\mathrm{Cov}(U)$ as the collections $\{U_i \to U\}_{i\in I}$ such that $U = \bigcup_{i\in I} U_i$.

Example 1.3 (The E-topology, [22, p. 47])**.** The sites we will study come from scheme theory. Given a scheme S, consider the category (Sch/S) of S-schemes with S-morphisms. Next we consider a class of morphisms of schemes E such that:

(i) every isomorphism is in E;

(ii) the composition of two morphisms in E is in E;

(iii) any base change of a morphism in E is in E.

The full subcategory of (Sch/S) whose structure morphism is in E will be denoted by E/S and we say a morphism in E is an E-morphism.

As examples of these classes we have:

- (zar): all open immersions;
- (ét): all étale morphisms;
- (sm): all smooth morphisms;
- (fppf): all flat and locally of finite presentation morphisms;
- (fpqc): all fpqc morphisms: A morphism of schemes $f\colon X \to Y$ is *fpqc* if it is flat and every quasi-compact open subset of Y is the image of quasi-compact open subset of X.

An *E-covering* of an object $Y \in (\mathrm{Sch}/S)$ is a family of E-morphisms (which are also S-morphisms) $\{Y_i \xrightarrow{g_i} Y\}_{i\in I}$ such that

$$Y = \bigcup_{i\in I} g_i(Y_i).$$

Define the *big E-site* as the category (Sch/S) equipped with the *E-topology*: for each S-scheme X, we define $\mathrm{Cov}(X \to S) := \{E\text{-coverings of } X\}$. The properties of the class E ensures that this is certainly a site. The *small E-site* is defined as the category E/S equipped with the E-topology defined above.

Definition 1.4. Using the notation of the example above, the *big and small étale sites* are the big and small (ét)-sites respectively and are denoted by $X_{\text{Ét}}$ and $X_{\text{ét}}$ respectively.

The *flat site* will be the big (fppf)-site and is denoted by X_{Fppf}. The small (fppf)-site is denoted by X_{fppf}.

Using other classes of E-morphisms we define and denote the sites similarly. For example, X_{zar} is the small (zar)-site.

2 Sheaves on a site

Definition 2.1. Let $\mathcal{C}$ be a site. A *presheaf* of sets on $\mathcal{C}$ is a functor $F\colon \mathcal{C} \to \mathrm{Sets}$. A presheaf of abelian groups, groups, rings, etc... is defined similarly changing the codomain category accordingly.

Definition 2.2. A presheaf F is a *sheaf* if for every object X in $\mathcal{C}$ and covering $\{X_i \to X\}_{i\in I}$ of X the following sequence is exact:

$$0 \longrightarrow F(X) \longrightarrow \prod_{i\in I} F(X_i) \overset{\mathrm{pr}_i^*}{\underset{\mathrm{pr}_j^*}{\rightrightarrows}} \prod_{i,j\in I} F(X_i \times_X X_j)$$

Definition 2.3. The categories of abelian presheaves (resp. sheaves) on a site $\mathcal{C}$ will be denoted by $\hat{\mathcal{C}}$ (resp. $\tilde{\mathcal{C}}$). If the values are taken in Ab, we will say they are abelian. If the values are taken in Grp, we will talk about group (pre)sheaves.

Remark 2.4. Let $\mathcal{C}$ be S_E where $E =$ ét, fppf, fpqc. Let $X \in S_E$ and F a presheaf on S_E that transforms coproducts into products. If we have a cover $\{X_i \to X\}$, then $\{\coprod_i X_i \to X\}$ is also a cover of X, but now it has one member! This is not always possible, for instance, in the small Zariski site. Define $X' = \coprod_i X_i$ and $X'' = X' \times_X X'$. Then the sheaf condition of F for the cover $\{X_i \to X\}$ can be rewritten as the exactness of

$$0 \longrightarrow F(X) \longrightarrow F(X') \rightrightarrows F(X'') \ .$$

Theorem 2.5 (Sheafification)**.** Let $\mathcal{C}$ be a site.

(a) The forgetful functor $i\colon \tilde{\mathcal{C}} \to \hat{\mathcal{C}}$ has a left adjoint $-^{\#}\colon \hat{\mathcal{C}} \to \tilde{\mathcal{C}}$, i.e, ther are (bi)-functorial isomorphisms

$$\operatorname{Hom}_{\hat{\mathcal{C}}}(F, i(\mathcal{G})) \xrightarrow{\sim} \operatorname{Hom}_{\tilde{\mathcal{C}}}(F^{\#}, \mathcal{G}).$$

(b) There is a natural map $\theta\colon F \to F^{\#}$ for any presheaf which is an isomorphism if F is already a sheaf.

(c) A section $s \in F^{\#}(U)$ is the same as the data of a covering $\{U_i \to U\}$, and sections $s_i \in F(U_i)$ such that: $\theta(s_i) = s|_{U_i}$ and for each i, j there is a covering $\{U_{ijk} \to U_i \times_U U_j\}$ such that $s_i|_{U_{ijk}} = s_j|_{U_{ijk}}$ for each k.

Proof. [41, Tag 00WK] □

Proposition 2.6 (Properties of $\tilde{\mathcal{C}}$)**.** Let $\mathcal{C}$ be a site and let $\tilde{\mathcal{C}}$ be the category of abelian sheaves on $\mathcal{C}$. Then $\tilde{\mathcal{C}}$

(a) is an abelian category [22, II.2.15];

(b) has enough injectives [41, Tag 01DL].

3 Fpqc and Galois descent

The main goal of fpqc descent of quasi-coherent sheaves is, given an fpqc morphism of schemes $p\colon S' \to S$, describe the essential image of the pullback functor $p^*\colon \operatorname{QCoh}(S) \to \operatorname{QCoh}(S')$.

Example 3.1 (Prototype example)**.** We will start studying the particular case where $p\colon S' \to S$ is a "Zariski covering morphism", i.e, $S' = \coprod_{i\in I} U_i$ and $\bigcup_{i\in I} U_i = S$ is a Zariski open cover of S. The data of a sheaf $\mathcal{G}$ on S' is equivalent to the data of sheaves $\mathcal{G}_i$ on U_i for each i, and the existence of $\mathcal{F}$ on S with $p^*\mathcal{F} \cong \mathcal{G}$ is equivalent to the existence of $\mathcal{F}$ on S such that $\mathcal{F}|_{U_i} \cong \mathcal{G}_i$ for each i. [18, II, Exercise 1.22], tells us that such an $\mathcal{F}$ exists if there are isomorphisms

$$\varphi_{ij}\colon \mathcal{G}_i|_{U_i\cap U_j} \to \mathcal{G}_j|_{U_i\cap U_j}$$

that satisfy $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on triple intersections $U_i \cap U_j \cap U_k$ (this last condition is called the *cocycle condition*). If

$$S'' := S' \times_S S' = \left(\coprod_{i\in I} U_i\right) \times_S \left(\coprod_{j\in I} U_j\right) = \coprod_{i,j\in I} (U_i \times_S U_j)$$

and pr_1, pr_2 the two projections, then

$$(\mathrm{pr}_1^* \mathcal{G})|_{U_i\times_S U_j} = \mathcal{G}_i|_{U_i\cap U_j} \quad \text{and} \quad (\mathrm{pr}_2^* \mathcal{G})|_{U_i\times_S U_j} = \mathcal{G}_|\,|_{U_i\cap U_j},$$

hence the condition on 2-intersections can be understood as the the existence of an isomorphism

$$\varphi\colon\ \mathrm{pr}_1^* \mathcal{G} \xrightarrow{\sim} \mathrm{pr}_2^* \mathcal{G}.$$

The cocycle condition can be rephrased as follows: Set $S''' := S'' \times_S S' \times_S S'$ and take the obvious projections pr_{12}, pr_{13} and pr_{23}. Then we are asking for

$$\mathrm{pr}_{13}^* \varphi = \mathrm{pr}_{23}^* \varphi \circ \mathrm{pr}_{12}^* \varphi.$$

If $\mathcal{G} \cong p^*\mathcal{F}$ for some sheaf $\mathcal{F}$ on S, then $\mathcal{G}$ satisfies the conditions above, so $\mathcal{G}$ is in the essential image of p^* if and only if it satisfies the condition on 2-intersections and the cocycle condition. Since being quasi-coherent is Zariski local, if we assume $\mathcal{G}$ is quasi-coherent then so is any $\mathcal{F}$ whose pullback is $\mathcal{G}$.

Fpqc descent for quasi-coherent sheaves basically says that this is also true if $S' \to S$ is no longer a Zariski covering morphism but any fpqc morphism. We will state in a slightly different way after defining some terminology.

Definition 3.2 ([44, Def. 14.63])**.** Let $p\colon S' \to S$ be an fpqc morphism and $\mathcal{F}' \in \mathrm{QCoh}(S')$. Define S'', S''' and the multiple projections as above. A *descent datum* for $\mathcal{F}'$ is an S''-isomorphism $\varphi\colon \mathrm{pr}_1^* \mathcal{F}' \xrightarrow{\sim} \mathrm{pr}_2^* \mathcal{F}'$ satisfying the cocycle condition $\mathrm{pr}_{13}^* \varphi = \mathrm{pr}_{23}^* \varphi \circ \mathrm{pr}_{12}^* \varphi$.

A *morphism of quasi-coherent S'-modules with descent data* $(\mathcal{F}', \varphi) \to (\mathcal{G}', \psi)$ is a morphism $f\colon \mathcal{F}' \to \mathcal{G}'$ such that

$$\begin{array}{ccc} \mathrm{pr}_1^* \mathcal{F}' & \xrightarrow{\varphi} & \mathrm{pr}_2^* \mathcal{F}' \\ {\scriptstyle \mathrm{pr}_1^* f}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_2^* f} \\ \mathrm{pr}_1^* \mathcal{G}' & \xrightarrow[\psi]{} & \mathrm{pr}_2^* \mathcal{G}' \end{array}$$

commutes.

Thus, there is a category of quasi-coherent S'-modules with descent data which we denote by $\mathrm{QCoh}(S' \to S)$.

Given $\mathcal{F} \in \mathrm{QCoh}(S)$ then $p^*\mathcal{F}$ has a canonical descent datum, namely the isomorphism

$$\varphi_{\mathrm{can}}\colon \mathrm{pr}_1^* p^*\mathcal{F} \cong (p \circ \mathrm{pr}_1)^*\mathcal{F} = (p \circ \mathrm{pr}_2)^*\mathcal{F} \cong \mathrm{pr}_2^* p^*\mathcal{F}.$$

Theorem 3.3 (Grothendieck, fpqc-descent for quasi-coherent sheaves)**.** For any fpqc morphism $S' \to S$ the functor

$$\begin{aligned} \mathrm{QCoh}(S) &\to \mathrm{QCoh}(S' \to S) \\ \mathcal{F} &\mapsto (p^*\mathcal{F}, \varphi_{\mathrm{can}}) \end{aligned}$$

is an equivalence of categories.

Proof. [44, Theorem 14.68]. ☐

We would also like to descend schemes. Let $p\colon S' \to S$ be an fpqc morphism of schemes, let $X \to S$ be an S-scheme, and denote by p^*X the fiber product $X \times_S S'$. We would like to describe the essential image of $p^*\colon (\mathrm{Sch}/S) \to (\mathrm{Sch}/S')$.

Given an S'-scheme X', a *descent datum* for X' is an S''-isomorphism $\varphi\colon \mathrm{pr}_1^* X' \xrightarrow{\sim} \mathrm{pr}_2^* X'$ satisfying the cocycle condition as before. The category of S'-schemes with descent data for p (with the obvious definition of morphism) is denoted by $(\mathrm{Sch}/S' \to S)$. Again, for an S-scheme X there is a canonical descent datum $(p^*, \varphi_{\mathrm{can}})$ attached to X which defines a functor

$$\begin{aligned} p^*_{\mathrm{can}}\colon (\mathrm{Sch}/S) &\to (\mathrm{Sch}/S' \to S) \\ X &\mapsto (p^*X, \varphi_{\mathrm{can}}). \end{aligned}$$

Unfortunately, it is no longer true that every S'-scheme with descent datum (X', φ) is effective, see [24, Section 6.7] and [41, Tag 08KF] for counterexamples. The situation is still manageable as we will see.

Definition 3.4 ([44, Def. 14.71])**.** Let (X', φ) be an S'-scheme with descent datum. An open subscheme U' of X' is *stable under* φ if φ restricts to an isomorphism $\mathrm{pr}_1^* U' \xrightarrow{\sim} \mathrm{pr}_2^* U'$, in which case $(U', \varphi|_{\mathrm{pr}_1^* U'})$ is a descent datum on U'.

With this in mind, there is the following theorem.

Theorem 3.5 (Fpqc descent for schemes, [44, Thm. 14.72])**.** Let $p\colon S' \to S$ be an fpqc morphism of schemes.

(a) The functor p^*_{can} is fully faithful.

(b) An S'-scheme with descent datum (X', φ) is in the essential image of p^*_{can} if and only if it can be covered by open subschemes which are quasi-affine (or just affine) and stable under φ.

Sketch of proof. (a) Let $X \to Y$ be an S-morphism. The question is local on S and Y. By the usual Zariski-gluing of morphisms of schemes, we can also reduce to the case X is affine. In that case, S-morphisms $X \to Y$ are nothing but $\mathcal{O}_S$-morphisms of quasi-coherent $\mathcal{O}_S$-algebras by [44, Prop. 11.1], and the same is true for S'-morphisms $X' \to Y'$. Thus, this case follows from Theorem 3.3 since the functor in that theorem is fully faithful.

(b) Let $X' = \bigcup_i U'_i$ be a cover as in the statement. Affine S'-schemes correspond to quasi-coherent $\mathcal{O}_{S'}$-algebras so the descent datum $(U_i, \varphi|_{\mathrm{pr}_1^* U'_i})$ is effective for each i, i.e, the schemes U'_i descend to affine S-schemes U_i. By part (a), the gluing data for the covering U'_i also descends to gluing data for the schemes S-schemes U_i. By gluing the schemes U_i with respect to this gluing data, we obtain an S-scheme X with $X \times_S S' \cong X'$.

□

A consequence of fpqc descent for schemes is Galois descent.

Definition 3.6 ([44, Def. 14.74])**.** A finite faithfully flat morphism of schemes $S' \to S$ is a *Galois covering* if there is a finite group $\Gamma \leq \mathrm{Aut}_S(S')$ such that the morphism $\coprod_{\sigma \in \Gamma} S' =: \Gamma \times S' \to S' \times S'$, given on T-valued points by $(\gamma, x) \mapsto (\gamma x, x)$, is an isomorphism. We say that p has Galois group Γ or that p is a Γ-covering.

Example 3.7. If L/k is a finite Galois extension, then $\operatorname{Spec} L \to \operatorname{Spec} k$ is Galois covering with group $\mathrm{Gal}(L/k)$ as we have an isomorphism $L \otimes_k L \xrightarrow{\sim} \prod_{\sigma \in \mathrm{Gal}(L/k)} L$.

Proposition 3.8 (Galois descent data I)**.** Let $S' \to S$ be a Γ-covering. To give a descent data on an S'-scheme X' (with respect to $S' \to S$) is equivalent to giving a right Γ-action on X' compatible with that of Γ on S', i.e, such that the structure morphism $X' \to S'$ is Γ-equivariant. Pictorically, if

$$\begin{array}{ccc} X' & \xrightarrow{\gamma} & X' \\ \downarrow & & \downarrow \\ S' & \xrightarrow[\gamma]{} & S' \end{array}$$

commutes for each $\gamma \in \Gamma$.

A morphism of S'-schemes with descent data is just a Γ-equivariant map. Finally, an open subscheme of X' is stable under the descent datum if and only if $\sigma(U) = U$.

Proof. [24, 6.2]. □

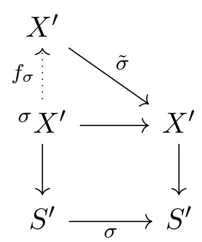
Proposition 3.9 (Galois descent data II, [40, Prop. 4.4.4])**.** Let $p\colon S' \to S$ be a Γ-covering and let X' be and S'-scheme. Given $\sigma \in \Gamma$, we denote by ${}^{\sigma}X'$ the base change of $X' \to S'$ by $S' \xrightarrow{\sigma} S'$. Then, giving a descent datum on X' with respect to $S' \to S$ is equivalent to giving a collection of isomorphisms $f_\sigma\colon {}^{\sigma}X' \to X'$ satisfying the cocycle condition $f_{\sigma\tau} = f_\sigma \cdot {}^{\sigma}(f_\tau)$ for $\sigma, \tau \in \Gamma$.

An isomorphism between S'-schemes with covering data $(X', f_\sigma) \to (Y', g_\sigma)$ is an S-isomorphism $b\colon X' \to Y'$ such that $bf_\sigma = g_\sigma({}^{\sigma}b)$ for any $\sigma \in \Gamma$.

An open subscheme $U \subseteq X'$ is stable under the descent datum if and only if $f_\sigma({}^{\sigma}U) = U$ for each $\sigma \in \Gamma$.

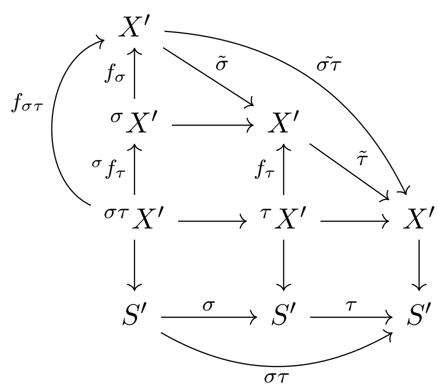
Proof. We use the previous proposition, and show that the new information is equivalent to the Γ-action on X' such that $X' \to S'$ is Γ-equivariant.

Since we have an isomorphism ${}^{\sigma}X' \to X'$ lying over σ, to give an isomorphism $\tilde{\sigma}\colon X' \to X'$ lying over σ is equivalent to give an S'-isomorphism $f_\sigma\colon {}^{\sigma}X' \to X'$ fitting in the following diagram

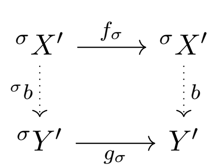


(2.1)

The next diagram shows $\tilde{\sigma\tau} = \tilde{\tau}\tilde{\sigma}$ is equivalent to $f_{\sigma\tau} = f_\sigma \cdot {}^{\sigma}(f_\tau)$



An isomorphism $b\colon X' \to Y'$ is an isomorphism of schemes with descent data, if and only if, for every $\sigma \in \Gamma$, the 3-dimensional diagram formed by putting together two copies of (2.1) and connecting them by $\mathrm{id}_{S'}\colon S' \to S'$, $b\colon X' \to Y'$ and ${}^{\sigma}b\colon {}^{\sigma}X' \to^{\sigma} X'$ where it corresponds, commutes. Equivalently, if and only if the following diagram commutes:



Finally, looking at (2.1) it is clear that $\tilde{\sigma}(U') = U'$ if and only if $f_\sigma({}^{\sigma}U') = U'$. □

Corollary 3.10 (Galois descent for schemes, [40, Cor. 4.4.6])**.** Let $S' \to S$ be a Γ-covering. Let X' be an S'-scheme together with isomorphisms $f_\sigma\colon {}^{\sigma}X' \to X'$ satisfying the cocycle conditions as in the previous proposition. If S is affine and X' is quasi-projective over S, then $X \times_S S' \cong X'$ for some S-scheme X.

Proof. The hypotheses are equivalent to give a right action of Γ on X' such that $X' \to S'$ is Γ-equivariant by Proposition 3.9. By Theorem 3.5, it suffices to cover X' by quasi-affine stable open subschemes. Suppose $S = \operatorname{Spec} A$ and fix an embedding $X' \to \mathbb{P}^n_A$. Given $x' \in X'$, we can choose a hypersurface in $\mathbb{P}^n_A$ that does not meet the Γ-orbit of x'. Certainly, let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the homegeneous prime ideals corresponding to the points in the Γ-orbit of x'. Since the irrelevant ideal A_+ in $k[x_0, \dots, x_n]$ is not contained in any of the $\mathfrak{p}_r$, then it is not contained in the union of them (by prime avoidance [41, Tag 00DS]). Hence, there is $f \in A_+ \setminus \cup_i \mathfrak{p}_i$ which defines a hypersurface $V(f)$ not meeting the Γ-orbit. Let $U' = X' \setminus V(f)$. Then $\cap_{\sigma \in \Gamma} \sigma(U')$ is quasi-affine stable open in X' and contains x'. □

Definition 3.11. Let $S' \to S$ be a Γ-covering. Fix an S-scheme X. An S-scheme Y is an *S'/S-twist* of X if $X \times_S S' \cong Y \times_S S'$.

Corollary 3.12 (Twists)**.** Let $S' \to S$ be a Γ-cover and X an S-scheme. Let $\operatorname{Twist}_{S'/S}(X)$ be the set of twists of X modulo S-isomorphism distinguished element X. Then we have a bijection of pointed sets

$$\operatorname{Twists}_{S'/S}(X) \xrightarrow{\sim} H^1(\Gamma, \operatorname{Aut}(X'_S))$$
$$Y \mapsto (\sigma \mapsto \phi^{-1}({}^{\sigma}\phi))$$

where ϕ is any isomorphism $\phi\colon Y_{S'} \xrightarrow{\sim} X_{S'}$.

Sketch of proof. Since X is already defined over S and any $\sigma \in \Gamma$ is an S-automorphism, we have $X_{S'} = {}^{\sigma}X_{S'}$ for any $\sigma \in \Gamma$. Then Proposition 3.10 proves that the map is surjective. □

4 Examples of sheaves

Let S be a scheme. Any sheaf on S is a sheaf on the small Zariski site by definition. Also, an S-scheme X (more precisely, its functor of points h_X) is itself a sheaf on the big Zariski site since morphisms of schemes glue.

The results of the preceding section lead us to examples of fpqc sheaves.

Proposition 4.1 (Quasi-coherent sheaves are fpqc sheaves, [40, Prop. 6.3.15])**.** Let S be a scheme and let $\mathcal{F} \in \operatorname{QCoh}(S)$. Define

$$\mathcal{F}_{\text{fpqc}}\colon S_{\text{fpqc}} \to \text{Sets}$$
$$(f\colon T \to S) \mapsto \Gamma(T, f^*\mathcal{F}).$$

Then $\mathcal{F}_{\text{fpqc}}$ is an S_{fpqc}-sheaf.

Proof. Notice that $\mathcal{F}_{\text{fpqc}}(f\colon T \to S) = \operatorname{Hom}_{\mathcal{O}_T}(f^*\mathcal{O}_S, f^*\mathcal{F})$. By Remark 2.4, it suffices to prove the sheaf conditions for a single fpqc S-morphism $p\colon X' \to X$. For this, we may replace $\mathcal{F}_{\text{fpqc}}$ by its restriction to X and call it $\mathcal{F}_{\text{fpqc}}$ again. We must then prove

$$0 \longrightarrow \mathcal{F}_{\text{fpqc}}(X) \longrightarrow \mathcal{F}_{\text{fpqc}}(X') \rightrightarrows \mathcal{F}_{\text{fpqc}}(X'')$$

is exact, which reduces to proving

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \longrightarrow \operatorname{Hom}_{\mathcal{O}_{X'}}(p^*\mathcal{O}_X, p^*\mathcal{F}) \rightrightarrows \operatorname{Hom}_{\mathcal{O}_{X''}}(q^*\mathcal{O}_X, q^*\mathcal{F})$$

where $q = p \circ \mathrm{pr}_1 = p \circ \mathrm{pr}_2\colon X'' \to X$, is exact. But this follows from the fully faithfullnes of the functor in Theorem 3.3. □

Proposition 4.2 (Representable sheaves are fpqc sheaves, [40, Prop. 6.3.16])**.** Let X be an S-scheme. Then h_X is a S_{fpqc}-sheaf.

Sketch of proof. The proof is similar to that of the previous proposition, now using that the functor in Theorem 3.5 is fully faithful. □

5 Functoriality of presheaves

For this section, the reader should have the following natural functors in mind:

(a) If $f\colon S' \to S$ is a morphism of schemes, then we have a functor $f\colon S_E \to S'_E$ by sending an object $T \to S$, to $T \times_S S' \to S'$, where E is any of the topologies defined in Example 1.3.

(b) For any scheme X, we have natural functors

$$X_{\text{zar}} \to X_{\text{ét}} \to X_{\text{Ét}} \to X_{\text{Fppf}} \to X_{\text{Fpqc}}.$$

(c) Any composition of (a) and (b).

Definition 5.1 ([41, Tag 00VC])**.** Let $u\colon \mathcal{C} \to \mathcal{D}$ be a functor between categories. Then there is an induced functor $u^p\colon \hat{\mathcal{D}} \to \hat{\mathcal{C}}$, given by $G \mapsto G \circ u$. This map admits a left adjoint u_p defined as follows: Take $V \in \mathcal{D}$ any object, and define a category $\mathcal{I}_V^u$ whose objects are arrows $\phi\colon V \to u(U)$ we denote by (U, ϕ) and a morphism $(U, \phi) \to (U', \phi')$ is a morphism $f\colon U \to U'$ such that $\phi' = u(f) \circ \phi$. For a presheaf F on $\mathcal{C}$ we define a presheaf $F_V\colon (\mathcal{I}_V^u)^{\text{op}} \to \text{Sets}$, $(U, \phi) \mapsto F(U)$. The categories $\mathcal{I}_V^u$ are not always filtered, but they are in the case of the examples of this section [41, Tag 00X3], so the following definition makes sense in these cases:

$$u_p(F)(V) = \operatorname{colim}_{\mathcal{I}_V^u} F_V = \left(\coprod_{(U,\phi) \in \mathcal{I}_V^u} F(U) \right) / \sim,$$

i.e, an element $s \in u_p(F)(V)$ is an equivalence class of sections $s_U \in F(U)$, with $\phi\colon V \to u(U)$ and where $s_U \in F(U)$, $s_{U'} \in F(U')$ are equivalent if there is an U'' and maps $f\colon U'' \to U$, $f'\colon U'' \to U'$ such that $F(f)(s_U) = F(f')(s_{U'})$.

Example 5.2. Let $f\colon X \to Y$ be a continuous map between topological spaces, and let $\mathcal{C} = \text{Op}(Y)$, $\mathcal{D} = \text{Op}(X)$ be the posetal categories of open subsets of X and Y respectively. Let $u\colon \mathcal{C} \to \mathcal{D}$ be defined by $U \mapsto f^{-1}(U)$. Then, if $F \in \hat{\mathcal{D}}$ we have $u^p(F)(V) = F(f^{-1}(V))$. Thus, $u^p = f_*$.

For u_p, let $G \in \hat{\mathcal{C}}$. By definition,

$$u_p(G)(U) = \operatorname*{colim}_{f(U) \subset V} G(V).$$

So, $u_p = f^{-1}$. If f is an open map, then $u_p(G)(U) = G(f(U))$.

Proposition 5.3 (Adjunction [41, Tag 00VE])**.** For any $F \in \hat{\mathcal{C}}$, $G \in \hat{\mathcal{D}}$ we have (bi)-functorial isomorphisms

$$\text{Hom}_{\hat{\mathcal{C}}}(F, u^p G) \xrightarrow{\sim} \text{Hom}_{\hat{\mathcal{D}}}(u_p F, G).$$

Definition 5.4 ([41, Tag 00WV])**.** If $\mathcal{C}$ and $\mathcal{D}$ are sites, then we say a functor $u\colon \mathcal{C} \to \mathcal{D}$ is *continuous* if

(i) If $\{U_i \to U\}$ covers U, then $\{u(U_i) \to u(U)\}$ covers $u(U)$.

(ii) If $T \to U$ is any morphism and $\{U_i \to U\}$ is a cover, then $u(T \times_U U_i) \to u(T) \times_{u(U)} u(U_i)$ is an isomorphis for each i.

The examples of this section are all continuous. For a continuous functor u, the functor u^p restricts to a functor $u^s\colon \tilde{\mathcal{D}} \to \tilde{\mathcal{C}}$.

Proposition 5.5 (Adjunction, [41, Tag 00WX])**.** Define $u_s\colon \tilde{\mathcal{C}} \to \tilde{\mathcal{D}}$ by $F \mapsto u_p(F)^{\#}$. For any $\mathcal{F} \in \tilde{\mathcal{C}}$, $\mathcal{G} \in \tilde{\mathcal{D}}$ we have (bi)-functorial isomorphisms

$$\mathrm{Hom}_{\tilde{\mathcal{C}}}(\mathcal{F}, u^s\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\tilde{\mathcal{D}}}(u_s\mathcal{F}, \mathcal{G}).$$

We conclude this section by defining a morphism of sites.

Definition 5.6 ([41, Tag 00X1])**.** A *morphism of sites* $\sigma\colon \mathcal{D} \to \mathcal{C}$ is a continuous functor $u\colon \mathcal{C} \to \mathcal{D}$ such that u_s is exact. In this case, we write $f^* := u_s$ and $f_* := u^s$.

For the examples of this section, all functors are continuous and satisfy u_s is exact [41, Tag 00X6], so they define morphisms of sites:

Example 5.7. The following are morphisms of sites:

(a) For any morphism of schemes $f\colon S' \to S$, the natural morphism $f_E\colon S'_E \to S_E$ given by pullback is a morphism of sites.

(b) For any scheme S we have the following morphisms of sites

$$S_{\mathrm{fpqc}} \to S_{\mathrm{Fppf}} \to S_{\text{Ét}} \to S_{\text{ét}} \to S_{\mathrm{zar}}.$$

(c) Any composition of (a) and (b).

6 Derived functors and cohomology

Since the category of abelian sheaves on $\mathcal{C}$ has enough injectives 2.6.(b) we may define:

Definition 6.1 ([22, III. Def. 1.5])**.** Let $\mathcal{C}$ with a terminal object T.

(a) The global sections functor $H^0(T, -)\colon \tilde{\mathcal{C}} \to \mathrm{Ab}$ is left exact an hence has right derived functors $H^q(\mathcal{C}, -)\colon \mathcal{C} \to \mathrm{Ab}$. If $\mathcal{F}$ is an abelian sheaf on $\mathcal{C}$, then $H^q(\mathcal{C}, \mathcal{F})$ is the q-th cohomology group with respect to $\mathcal{C}$. If $\mathcal{C} = X_E$, then we say $H^q(X_E, \mathcal{F})$ is the q-th E cohomology group.

(b) The inclusion functor $i\colon \tilde{\mathcal{C}} \to \hat{\mathcal{C}}$ is also left exact. Its right derived functors are $\underline{H}^q$.

(c) Let $f\colon \mathcal{D} \to \mathcal{C}$ be a morphism of sites and let $f_*\colon \tilde{\mathcal{D}} \to \tilde{\mathcal{C}}$ be the direct image functor. Again, this is a left exact functor and we write $R^q f_*$ for its right derived functors. If $\mathcal{F}$ is an abelian sheaf on $\mathcal{D}$, then the sheaves $R^q f_* \mathcal{F}$ are the *higher direct images* of $\mathcal{F}$.

Definition 6.2. An abelian sheaf $\mathcal{F}$ is *flabby* if $\underline{H}^q(\mathcal{F}) = 0$ for every $q > 0$.

The following proposition helps us computing higher direct images:

Proposition 6.3. Let $f\colon \mathcal{D} \to \mathcal{C}$ be a morphism of sites induced by a continuous functor $u\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{F}$ an abelian sheaf on $\mathcal{D}$. Then $R^q f_* = (u^p \underline{H}^q(\mathcal{F}))^{\#}$, i.e, $R^q f_*$ is the sheaf associated to the presheaf $U \mapsto H^q(u(U), \mathcal{F})$.

Proof. [22, Chapter III, Prop. 1.13]. □

Remark 6.4. From this description it is clear that flabby sheaves are acyclic for f_*, so they may be used to compute higher direct images.

The following relation between cohomology and higher direct images is fundamental:

Theorem 6.5 (Leray spectral sequence)**.** Let $f\colon \mathcal{D} \to \mathcal{C}$ be a morphism of sites.

(a) For any abelian sheaf $\mathcal{F}$ on $\mathcal{D}$ we have an spectral sequence

$$E_2^{p,q} = H^p(\mathcal{C}, R^q f_* \mathcal{F}) \implies H^{p+q}(\mathcal{D}, \mathcal{F}).$$

(b) If $g\colon \mathcal{E} \to \mathcal{D}$ is another morphism of sites, then there is an spectral sequnece

$$E_2^{p,q} = R^p g_*(R^q f_* \mathcal{F}) \implies R^{p+q}(g \circ f)\mathcal{F}.$$

Proof. These are just Grothendieck spectral sequences for composition of functors [55, Prop. F.212]:

(a) $\Gamma(\mathcal{C}, -) \circ f_* = \Gamma(\mathcal{D}, -)$;

(b) $g_* \circ f_* = (g \circ f)_*$.

Also we use the fact that f_* sends flabby sheaves to flabby sheaves [22, Chapter III, Cor. 2.13] and that flabby sheaves are acyclic for both g_* and $\Gamma(\mathcal{C}, -)$. □

7 Galois vs. étale cohomology

Fix a field k and a separable closure k_s/k. If $\mathcal{F}$ is a sheaf on $(\operatorname{Spec} k)_{\text{ét}}$, define $\mathcal{F}(k_s) = \varinjlim \mathcal{F}(L)$ where the limit runs over all finite separable (or Galois) fields $k \subset L \subset k_s$. It is clear then that G_k acts on $\mathcal{F}(k_s)$ continuously. We write G_k-sets and G_k-Mod for the categories of continuous G_k-sets and continuous G_k-modules respectively.

Theorem 7.1 (Étale cohomology of a point, [40, Thm. 6.4.6])**.** Let $X = \operatorname{Spec} k$ with k a field.

(a) The functor $\operatorname{Sh} X_{\text{ét}} \to G_k$-sets, defined by $\mathcal{F} \mapsto \mathcal{F}(k_s)$ is an equivalence of categories. It follows that the global sections functor $\mathcal{F} \to \mathcal{F}(k)$ corresponds to the functor of G_k-invariants on G_k-sets, i.e, $S \mapsto S^{G_k}$.

(b) The same functor restricts to an equivalence between $\tilde{X}_{\text{ét}}$ and G_k-Mod.

(c) There are natural isomorphisms

$$H^q_{\text{ét}}(\operatorname{Spec} k, \mathcal{F}) \cong H^q(k, \mathcal{F}(k_s))$$

for each $q \geq 0$.

Proof.

(a) We construct an inverse functor. Let S be G_k-set. For a finite separable extension $k \subset L \subset k_s$ we define $\mathcal{F}(L) = S^{\mathrm{Gal}(k_s/L)}$. Since every étale k-scheme $U \to k$ is of the form

$$U = \coprod \operatorname{Spec} L$$

for finite separable field extensions L/k, we define

$$\mathcal{F}(U) = \prod \mathcal{F}(L).$$

If $k \subset L \subset M \subset k_s$ are two finite separable field extensions, the restrictions are nothing else than the inclusion of sets

$$\mathcal{F}(L) = S^{\mathrm{Gal}(k_s/L)} \hookrightarrow S^{\mathrm{Gal}(k_s/M)} = \mathcal{F}(M).$$

It is a sheaf, see [22, II. Lem. 1.8]. The assertion about the global sections functor is clear.

(b) Clear from (i).

(c) Take derived functors to the global sections functor on both sides of the equivalence, i.e, to $\Gamma(-, \mathcal{F})$ and $-^{G_k}$.

□

This shows étale cohomology is more interesting than Zariski cohomology, as a point has no Zariski cohomology for $q > 1$.

8 Points and stalks in the étale site

We will develop the theory of points and stalks just in étale site, since it admits a better description.

Definition 8.1. Let S be a scheme. A *geometric point* of S is a morphism $\overline{s}\colon \operatorname{Spec}(k) \to S$ from the spectrum of a separably closed field k to S. We write s for the image of $\overline{s}$, or we say that $\overline{s}$ *lies* over $s \in S$. We also write $k(\overline{s})$ for k.

Given an abelian presheaf F on $S_{\text{ét}}$, we define the *stalk* of F at a geometric point $\overline{s}$ as

$$F_{\overline{s}} := \overline{s}_p F(k(\overline{s}))$$

where here $\overline{s}$ is also considered as the functor $S_{\text{ét}} \to k(\overline{s})_{\text{ét}}$.

Remark 8.2. Unravelling the definition of $\overline{s}_p$ we see that $F_{\overline{s}} = \varinjlim F(U)$ where the limit runs over all arrows

$$k(\overline{s}) \to U_{k(\overline{s})}$$

where $U \in S_{\text{ét}}$, or equivalently, commuting triangles

$$\begin{array}{ccc} k(\overline{s}) & \xrightarrow{\overline{u}} & U \\ & {\scriptstyle \overline{s}} \searrow & \downarrow \\ & & S \end{array}$$

Such triangle is called an *étale neighbourhood* of $\overline{s}$ and will be denoted by $(U, \overline{u}) \to (S, \overline{s})$.

Remark 8.3. An important class of rings when studying the étale topology is that of *(strictly) henselian local rings*. They arise naturally as the stalks in the étale topology, i.e, if X is a scheme, then $(\mathcal{O}_X)_{\text{ét}}$ is an étale sheaf on X an its stalk at $\overline{x}$ is the strict henselization $\mathcal{O}_{X,\overline{x}} = \mathcal{O}_{X,x}^{\text{sh}}$ of $\mathcal{O}_{X,x}$. As the name suggets, this is a strictly henselian local ring. However, we won't develop the theory of henselian and strictly henselian local rings. We refer the reader to [22, I.4] and [41, Tag 04GE].

Definition 8.4. We say a scheme X is *strictly local* if it is the spectrum of a strictly henselian local ring.

Just as in the case of Zariski sheaves, there are many properties of sheaves and morphisms of sheaves that can be checked at stalks:

Proposition 8.5 (Checking at stalks)**.** Let S be a scheme.

(a) A sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F}'' \longrightarrow 0$ of abelian sheaves on $S_{\text{ét}}$ is exact if and only if the corresponding sequence $0 \longrightarrow \mathcal{F}_{\overline{s}} \longrightarrow \mathcal{F}'_{\overline{s}} \longrightarrow \mathcal{F}''_{\overline{s}} \longrightarrow 0$ is exact for every geometric point of S.

(b) For any abelian presheaf $\mathcal{F}$ on $S_{\text{ét}}$ we have $\mathcal{F}^{\#}_{\overline{s}} = \mathcal{F}_{\overline{s}}$ for any geometric point $\overline{s}$ of S.

Proof. (a) [22, II. Thm. 2.1.5.(b)].

(b) [22, II. Remark 2.14.(c)]

□

Next we study how stalks work with functoriality, i.e, taking pullback, pushforward and derived versions.

Proposition 8.6 (Stalks and morphisms)**.** Let $f\colon Y \to X$ be a morphism of schemes.

(a) Let $\mathcal{F}$ be a sheaf on $X_{\text{ét}}$. Let $\overline{y}$ be a geometric point of Y, and let $\overline{f(y)} := f \circ \overline{y}$ the corresponding geometric point of X. Then $(f^*\mathcal{F})_y \cong \mathcal{F}_{\overline{f(y)}}$.

(b) Now assume f is quasi-compact and quasi-separated (qcqs). Let $\mathcal{G}$ be a sheaf on $Y_{\text{ét}}$. Let $\overline{x}$ be a geometric point of X and $\tilde{X} := \operatorname{Spec} \mathcal{O}_{X,\overline{x}}$ the strictly local scheme at $\overline{x}$. Then, for all $p \geq 0$,

$$(R^p f_* \mathcal{G})_{\overline{x}} \cong H^p(\tilde{X} \times_X Y, \tilde{\mathcal{G}})$$

where $\tilde{\mathcal{G}}$ is the pullback of $\mathcal{G}$ to $\tilde{X} \times_X Y$.

Proof. [22, II, Thm. 3.2] and [22, III, Thm. 1.15]. □

9 Čech cohomology

Definition 9.1 (Čech complex)**.** Let $\mathcal{C}$ be a site, and $\mathscr{U} = \{U_i \to U\}_{i\in I}$ a covering of U in $\mathcal{C}$. For any $(p+1)$-tuples $(i_0, \ldots, i_p) \in I^{p+1}$, we write $U_{i_0\cdots i_p}$ for the product $U_{i_0} \times_U \cdots \times_U U_{i_p}$. For any $0 \leq j \leq p$, forgetting the j-th factor of the product gives a restriction morphism $U_{i_0\cdots i_p} \to U_{i_0\cdots\hat{i}_j\cdots i_p}$.

Now let F be an abelian presheaf on $\mathcal{C}$. The restriction morphisms above define restriction morphisms

$$\text{res}_j\colon F(U_{i_0\cdots\hat{i}_j\cdots i_p}) \to F(U_{i_0\cdots i_p}).$$

We define the *Čech complex* $(C^\bullet(\mathscr{U}, F), d^\bullet)$ associated to F and $\mathscr{U}$ as follows:

$$C^p(\mathscr{U}, F) := \prod_{i\in I^{p+1}} F(U_{i_0\cdots i_p}) \quad \text{and} \quad d^p\colon C^p(\mathscr{U}, F) \to C^{p+1}(\mathscr{U}, F)$$

is the homomorphism such that, if $s = (s_{i_0 \cdots i_p})_{i \in I^{p+1}}$, then

$$(d^p s)_{i_0 \cdots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \operatorname{res}_j (s_{i_0 \cdots \hat{i}_j \cdots i_{p+1}}).$$

It is called Čech complex because it is indeed a complex. The verification of this fact is standard.

Definition 9.2 (Čech cohomology)**.** The *Čech cohomology groups* of F with respect to $\mathscr{U}$ are defined as

$$\check{H}^p(\mathscr{U}, F) := H^p(C^\bullet(\mathscr{U}, F), d^p) = \frac{\ker d^p}{\operatorname{im} d^{p-1}}.$$

A second covering $\mathscr{V} := \{V_j \to U\}_{j \in J}$ is a *refinement* of $\mathscr{U}$ if there is a map $\alpha \colon J \to I$ such that each map $V_j \to U$ factors through $U_{\alpha(j)} \to U$. Such a refinement induces a map $C^\bullet(\mathscr{U}, F) \to C^\bullet(\mathscr{V}, F)$ which in turn defines a map

$$\check{H}^p(\mathscr{U}, F) \to \check{H}^p(\mathscr{V}, F).$$

This map does not depend on the factorizations nor α. For more details, see [22, Chapter III, §2].

Let F be an abelian presheaf on a site $\mathcal{C}$, and $U \in \mathcal{C}$. Then the *p-th Čech cohomology group* of U is defined as

$$\check{H}^p(U, F) := \varinjlim \check{H}^p(\mathscr{U}, F)$$

where the limit is taken over all covering of U ordered by the refinement relation. We also define the *Čech cohomology presheaf* on $\mathcal{C}$ as the presheaf

$$\begin{aligned} \underline{\check{H}}^p(F) \colon \mathcal{C}^{\mathrm{op}} &\to \mathrm{Ab} \\ U &\mapsto \check{H}^p(U, F). \end{aligned}$$

It is easy to see that this presheaf is separated.

Even though the Čech cohomology groups were defined without any mention of derived functors they do arise in this manner too:

Proposition 9.3. The functors $\check{H}^p(\mathscr{U}, -)$ and $\check{H}^p(U, -)$ are the right derived functors of

$$\check{H}^0(\mathscr{U}, -) \colon \hat{\mathcal{C}} \to \mathrm{Ab} \quad \text{and} \quad \check{H}^0(U, -) \colon \hat{\mathcal{C}} \to \mathrm{Ab}$$

respectively, for any $U \in \mathcal{C}$ and any covering $\mathscr{U}$ of U.

Proof. [22, Chapter III, Proposition 2.3]. □

Remark 9.4. Thus, associated to a short exact sequence of presheaves, there is an associated long exact sequence in Čech cohomology groups. This is not true if we start with a short exact sequence of sheaves. It is also not true that Čech cohomology and derived functor cohomology agree always.

Theorem 9.5 (Čech-to-derived spectral sequence, [41, Tag 03OU])**.** Let $\mathcal{C}$ be a site, $U \in \mathcal{C}$ and $\mathscr{U}$ a covering of U. For any abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ there are spectral sequences

$$\begin{aligned} \check{H}^p(\mathscr{U}, \underline{H}^q(\mathcal{F})) &\implies H^{p+q}(U, \mathcal{F}) \\ \check{H}^p(U, \underline{H}^q(\mathcal{F})) &\implies H^{p+q}(U, \mathcal{F}) \end{aligned}$$

By varying U, we obtain an spectral sequence

$$\underline{\check{H}}^p(\underline{H}^q(\mathcal{F})) \implies \underline{H}^{p+q}(\mathcal{F}).$$

Proof. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. The following diagram is commutative

$$\begin{array}{ccc} \mathrm{Sh}^{\mathrm{Ab}}(\mathcal{C}) & \xrightarrow{\underline{\check{H}}^0(-,\mathcal{F})} & \mathrm{PSh}^{\mathrm{Ab}}(\mathcal{C}) \\ & \searrow^{H^0(-,\mathcal{F})} & \downarrow{\scriptstyle \check{H}^0(U,-)} \\ & & \mathrm{Ab} \end{array}$$

In fact, the first functor is just the forgetful functor from sheaves to presheaves, and the $\check{H}^0(U,\mathcal{F}) = H^0(U,F)$ for sheaves. An injective sheaf, is injective as a presheaf, so the first functor takes injectives into injectives. Hence this is an instance of a Grothendieck spectral sequence for compositions of functors. The same applies for $\mathscr{U}$ instead of U and the last spectral sequence follows by varying U. □

Proposition 9.6 ([22, III. Prop. 2.9])**.** In the context of the theorem above, we have $\underline{\check{H}}^0(\underline{H}^q(\mathcal{F})) = 0$ for any $q > 0$. In particular, for any $U \in \mathcal{C}$ we have

$$\check{H}^0(U,\mathcal{F}) \cong H^0(U,\mathcal{F})$$
$$\check{H}^1(U,\mathcal{F}) \cong H^1(U,\mathcal{F})$$

and an exact sequence

$$0 \to \check{H}^2(U,\mathcal{F}) \to H^2(U,\mathcal{F}) \to \check{H}^1(U,\underline{H}^1(\mathcal{F})) \to \check{H}^3(U,\mathcal{F}) \to H^3(U,\mathcal{F}).$$

Proof. It suffices to prove $\underline{\check{H}}^0(\underline{H}^q(\mathcal{F})) = 0$ for $q > 0$ as the remaining assertions then follow from the spectral sequence above and the associated exact sequence in low degrees. Take an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. For the sake of concreteness, given a map of sheaves $\mathcal{F} \to \mathcal{G}$ I will denote by $\ker_p$ and im_p the presheaf kernel an image, and by $\ker_s$ and im_s the sheaf kernel and sheaf image. Since sheafification is exact, it commutes with cohomology (to see what this means, see the next remark). Thus

$$\begin{aligned} \underline{H}^q(\mathcal{F})^\# &= \left(\frac{\ker_p(i(I^q) \to i(I^{q+1}))}{\mathrm{im}_p(i(I^{q-1}) \to i(I^q))}\right)^\# \\ &= \frac{\ker_s(i(I^q)^\# \to i(I^{q+1})^\#)}{\mathrm{im}_s(i(I^{q-1})^\# \to i(I^q)^\#)} \\ &= \frac{\ker_s(I^q \to I^{q+1})}{\mathrm{im}_s(I^{q-1} \to I^q)} \\ &= 0. \end{aligned}$$

Since $\underline{\check{H}}^0(\underline{H}^q(\mathcal{F}))$ is a subpresheaf of $\underline{H}^q(\mathcal{F})^\# = 0$ (see [22, III, Remark 2.2.(c)]), it is zero as promised. □

Remark 9.7. Given an abelian category $\mathcal{A}$ there is an associated category of complexes $\mathrm{Kom}(\mathcal{A})$ defined in the obvious manner, which is also abelian. A functor between abelian categories $F\colon \mathcal{A} \to \mathcal{B}$ induces a functor $\mathrm{Kom}(F)$ between the associated categories of complexes, whose construction is again obvious. For each $q \in \mathbb{Z}$, we have a cohomology functor $H^q\colon \mathrm{Kom}(\mathcal{A}) \to \mathcal{A}$ that takes the

q-th cohomology of the complex. Then, it is a general fact that if $F\colon \mathcal{A} \to \mathcal{B}$ is an exact functor between abelian categories, the following diagrams commutes

$$\begin{array}{ccc} \mathrm{Kom}(\mathcal{A}) & \xrightarrow{\mathrm{Kom}\, F} & \mathrm{Kom}(\mathcal{B}) \\ {\scriptstyle H^q}\downarrow & & \downarrow{\scriptstyle H^q} \\ \mathcal{A} & \xrightarrow[F]{} & \mathcal{B} \end{array}$$

We can also decide whether a sheaf is flabby or not using Čech cohomology instead of derived cohomology.

Proposition 9.8 (Flabbyness using Čech, [22, III. Prop. 2.12])**.** An abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ is flabby if and only if $\underline{\check{H}}^q(\mathcal{F}) = 0$ for all $q > 0$.

Proof. If $\mathcal{F}$ is flabby, then the third spectral sequence in Theorem 9.5 implies $\underline{\check{H}}^q(\mathcal{F}) \cong \underline{H}^q(\mathcal{F}) = 0$ for any $q > 0$. For the converser, assume $\underline{\check{H}}^q(\mathcal{F}) = 0$ for all $q > 0$. We will prove $\underline{H}^q(\mathcal{F}) = 0$ for $q > 0$ by induction. For $q = 1$, this follows from $\underline{H}^1(\mathcal{F}) = \underline{\check{H}}^1(\mathcal{F}) = 0$. Now let $\underline{H}^q(\mathcal{F}) = 0$ for each $0 < q < n$. Then

- $\underline{\check{H}}^n(\underline{H}^0(\mathcal{F})) = \underline{\check{H}}^n(\mathcal{F}) = 0$ by assumption.
- $\underline{\check{H}}^{n-i}(\underline{H}^i(\mathcal{F})) = 0$ for each $1 \leq i \leq n-1$ by the inductive hypothesis.
- $\underline{\check{H}}^0(\underline{H}^n(\mathcal{F})) = 0$ by Proposition 9.6.

Then $\underline{H}^p(\underline{H}^q(\mathcal{F})) \implies \underline{H}^{p+q}(\mathcal{F})$ implies $\underline{H}^n(\mathcal{F}) = 0$ and the induction is complete. □

10 Comparison of cohomologies

10.1 Small vs. big

Theorem 10.1 ([22, III. Prop. 3.1])**.** Let X_E be the big sites corresponding to the fpqc, fppf, étale or Zariski topology and let X_e be the corresponding small site. Let $f\colon X_E \to X_e$ be the morphism of sites induced by the inclusion functor $u\colon X_e \to X_E$. Then

(a) $f_*\colon \mathrm{Sh}(X_E) \to \mathrm{Sh}(X_e)$ is exact and $\mathcal{F} \to f_* f^* \mathcal{F}$ is an isomorphism for any $\mathcal{F} \in \mathrm{Sh}(X_e)$.

(b) The functor $f^*\colon \mathrm{Sh}(X_e) \to \mathrm{Sh}(X_E)$ is fully faithful.

(c) The canonical maps

$$H^i(X_e, f_*\mathcal{F}') \to H^i(X_E, \mathcal{F}') \quad \text{and} \quad H^i(X_e, \mathcal{F}) \to H^i(X_E, f^*\mathcal{F})$$

are isomorphisms for all $i \geq 0$, $\mathcal{F} \in \mathrm{Sh}(X_e)$, $\mathcal{F}' \in \mathrm{Sh}(X_E)$.

Proof. (a) f_* is clearly exact, as there are less objects to cover and the coverings are the same for those objects.

Let $U \to X$ be an E-scheme, i.e, $U \to X \in X_e$. Then $\Gamma(U, f_* f^* \mathcal{F}) = \Gamma(u(U), f^* \mathcal{F})$ where in the right hand side we consider $U \to X$ just as an X-scheme, i.e, an element of X_E. Now, $\Gamma(u(U), u_p \mathcal{F})$ is defined as

$$\varinjlim \mathcal{F}(V)$$

where the limit runs over the category of maps $u(U) \to u(V)$. Such a category has an initial object, namely $(U, u(\mathrm{id}_U))$, so

$$\Gamma(u(U), u_p\mathcal{F}) = \Gamma(U, \mathcal{F}).$$

Since $\mathcal{F}$ is already a sheaf, its sheafification has the same sections over any object in X_E, so

$$\Gamma(U, f_*f^*\mathcal{F}) = \Gamma(u(U), f^*\mathcal{F}) = \Gamma(u(U), u_p\mathcal{F}^\#) = \Gamma(U, \mathcal{F}^\#) = \Gamma(U, \mathcal{F}).$$

This shows that the natural map $\mathcal{F} \to f_*f^*\mathcal{F}$ is an isomorphism.

(b) This follows from part (a) and adjunction, Proposition 5.5.

(c) The first map is an isomorphism since f_* is exact. The second map is an isomorphism since the following composition is the identity

$$H^i(X_e, \mathcal{F}) \to H^i(X_E, f^*\mathcal{F}) \xrightarrow{\sim} H^i(X_e, f_*f^*\mathcal{F}) \xrightarrow{\sim} H^i(X_e, \mathcal{F}).$$

□

10.2 Changing E

Proposition 10.2 (Changing E, [22, III. Prop. 3.13])**.** Let X be a scheme and E,E' be two classes of morphisms inducing an E-topology on (Sch/X) and assume $E' \subset E$. Let $f\colon X_E \to X_{E'}$ be the morphism of sites induced by the inclusion. Assume that every $U \to X$ and every E-cover admits an E'-cover that refines it. Then f_* is exact and hence

$$H^i(X_{E'}, f_*\mathcal{F}) \cong H^i(X_E, \mathcal{F})$$

for any sheaf $\mathcal{F}$ on X_E.

Proof. Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ a surjection of sheaves on X_E and let $s \in \mathcal{G}(U)$. Since the map is E-surjective, there is an E-cover $\{U_i \to U\}$ such that $s|_{U_i} \in \operatorname{im}\varphi(U_i)$ for each i. Let $\{V_j \to U\}$ be an E'-cover that refines $\{U_i \to U\}$. If $V_j \to U$ factors through $U_i \to U$, then the following commutative square shows that $s|_{V_j} \in \operatorname{im}\varphi(V_j)$

$$\begin{array}{ccc} \mathcal{F}(U_i) & \xrightarrow{\varphi(U_i)} & \mathcal{G}(U_i) \\ \downarrow & & \downarrow \\ \mathcal{F}(V_j) & \xrightarrow[\varphi(U_j)]{} & \mathcal{G}(V_j) \end{array}$$

This shows $f_*\mathcal{F} \to f_*\mathcal{G}$ is surjective as a map of sheaves on $X_{E'}$. Thus, f_* is exact and the claim follows. □

Remark 10.3. This proposition can be used in several ways. For instance, the next proposition shows that the classes $E =$ fppf and $E' =$ fppf, locally quasi-finite satisfy the conditions. Moreover, if X is affine, we may use $E' =$ fppf, affine, quasi-finite.

Proposition 10.4. Let $f\colon U \to X$ be an fppf morphism. Then, there is an fppf locally quasi-finite morphism $g\colon U' \to X$ and an X-morphism $U' \to U$. If X is quasi-compact (resp. qcqs), one may further assume that X is affine (resp. X is affine and g is quasi-finite). If f is smooth and surjective, the same conclusions hold with g étale.

Proof. [8, Cor 17.16.2 and Cor. 17.16.3]. □

10.3 Flat cohomology of quasi-coherent sheaaves

Theorem 10.5 (Flat cohomology of quasi-coherent sheaves, [22, III. Prop. 3.7])**.** Let X be a scheme, $\mathcal{F}$ a quasi-coherent sheaf on X and $W(\mathcal{F})$ the corresponding sheaf on X_{Fppf}. Let $f\colon X_{\text{fppf}} \to X_{\text{Zar}}$ be the natural morphism of sites. There are canonical isomorphisms

$$H^i(X_{\text{Zar}}, \mathcal{F}) \to H^i(X_{\text{Fppf}}, W(\mathcal{F})).$$

Proof. The result follows from the Leray spectral sequence for f

$$H^p(X_{\text{Zar}}, R^q f_* W(\mathcal{F})) \implies H^{p+q}(X_{\text{Fppf}}, W(\mathcal{F}))$$

if we prove $R^q f_* W(\mathcal{F}) = 0$ for $q > 0$, since $f_* W(\mathcal{F}) = \mathcal{F}$. Now, $R^q f_* W(\mathcal{F})$ is the sheaf associated to the presheaf $U \mapsto H^q(U_{\text{Fppf}}, W(F)|_U)$ on X_{Zar}. Thus it suffices to show $H^q(U_{\text{Fppf}}, W(F)) = 0$ for U affine and $q > 0$, i.e, to show that $W(F)$ is flabby as sheaf on U_{Fppf}. For flabbyness, we may use Čech cohomology by Proposition 9.8. In fact, it suffices to prove that for any covering $\mathscr{U}$ of U and any $q > 0$, $\check{H}^q(\mathscr{U}, W(\mathcal{F})) = 0$. Indeed, this implies that $\check{H}^q(U_{\text{Fppf}}, W(\mathcal{F})) =$ for each $q > 0$, which in turn implies $\mathcal{F}$ is flabby by the proposition cited.

Before proceeding, we change $E = $ Fppf, as in the previous section. In fact, we use exactly $E' =$ fppf, affine, quasi-finite. Thus, we must prove $\check{H}^q(\mathscr{U}/U, W(\mathcal{F})) = 0$ for $q > 0$ and $\mathscr{U} = \{U_i \to U\}$ is an E'-cover. By changing $\{U_i \to U\}$ by $\coprod_i U_i \to U$ we may assume that $\mathscr{U}$ is just a finite type, faithfully flat morphism between affine schemes $V \to U$. In this case, the Čech complex translates into the complex

$$M \otimes_A B \to M \otimes_A B^{\otimes 2} \to M \otimes_A B^{\otimes 3} \to \cdots$$

where $U = \operatorname{Spec} A$, $V = \operatorname{Spec} B$ and $W(F) = \tilde{M}$. Also, the boundary maps $d^{i-1}\colon M \otimes_A B^{\otimes i} \to M \otimes_A B^{\otimes i+1}$ are defined by

$$(1 \otimes b_1 \otimes \cdots \otimes b_i) \mapsto \sum_{k=0}^{i-1} (-1)^k (b_0 \otimes \cdots b_{k-1} \otimes 1 \otimes b_k \otimes \cdots \otimes b_{i-1}).$$

We write $\mathscr{C}(A \to B)$ for this complex.

If $\varphi\colon A \to B$ admits a section $\psi\colon B \to A$ then the morphisms $k_i\colon B^{\otimes i+2} \to B^{\otimes i+1}$ defined by

$$b_0 \otimes \cdots \otimes b_{i+1} \mapsto \psi(b_0) b_1 \otimes b_2 \otimes \cdots \otimes b_{i+1}, \qquad i \geq -1$$

form a contracting homotopy, i.e, $k^{i+1} d^{i+1} + d^i k^i = 1$. This shows the complex is exact if φ has a section. We may base change the entire complex by B and the resulting Čech complex is now the Čech complex for the morphism $B \to B \otimes B$, $b \mapsto b \otimes 1$, i.e,

$$\mathscr{C}(A \to B) \otimes_A B = \mathscr{C}(B \to B \otimes_A B).$$

If we prove that this new complex is exact, then the same will be true for the Čech complex associated to $A \to B$, as $A \to B$ is faithfully flat. Since $B \to B \otimes B$ admits a section, namely, $b \otimes b' \to bb'$, $\mathscr{C}(B \to B \otimes_A B)$ is exact by the last paragraph, as required. □

10.4 Cohomology of smooth commutative group schemes

Before stating the result, we start recalling the following lemma.

Proposition 10.6 (Smooth schemes over henselian local rings)**.** Let A be a henselian local ring with residue field k. Then for every smooth morphism $X \to \operatorname{Spec} A$, the map $X(A) \to X(k)$ is surjective.

Proof. [55, Theorem 20.12]. □

Corollary 10.7. Let A be a strictly henselian local ring with residue field k. Then for every smooth algebraic space $X \to \operatorname{Spec} A$, the map $X(A) \to X(k)$ is surjective.

Proof. Take $S \to X$ a smooth atlas and consider the square

$$\begin{array}{ccc} S(A) & \longrightarrow & S(k) \\ \downarrow & & \downarrow \\ X(A) & \longrightarrow & X(k) \end{array}$$

The top arrow is surjective by the previous proposition and the right arrow is surjective since k is separably closed and $S \to X$ is étale-surjective. Thus, if we take $t \in X(k)$, it has a preimage $s \in S(A)$ and the image of such point in $X(A)$ maps to t. □

Theorem 10.8 (Flat and étale cohomology coincides for smooth commutative group schemes, [22, III. Thm. 3.9])**.** Let G be a smooth commutative group S-scheme. Then the canonical maps

$$H^i(X_{\text{ét}}, G) \to H^i(X_{\text{fppf}}, G)$$

are isomorphisms for all $i \geq 0$.

Proof. Let $\sigma\colon X_{\text{fppf}} \to X_{\text{ét}}$ be the natural inclusion of sites. It suffices to show that $R^i\sigma_*G = 0$ for $i > 0$, and by looking at stalks (in the étale topology), it suffices to show that $H^i(X_{\text{fppf}}, G) = 0$ for any strictly local scheme X.

To do this, we will change fppf by the E-topology, where E is the class of finite flat morphisms. Certainly, the fppf topology is generated by quasi-finite flat morphisms, and in the particular case of strictly local schemes, it is generated by finite flat morphisms, as any quasi-finite A-algebra S over a henselian local ring A is isomorphic to $B \times C$ where C is finite over A. By Proposition 10.2, we have

$$H^i(X_E, G) \cong H^i(X_{\text{fppf}}, G).$$

We must show G is flabby for this topology, i.e, for any finite faithfully flat morphism $X' \to X$ and every $i \geq 1$ we must show $\check{H}^i(X'/X, G) = 0$. The key part of the argument is to show that $\check{H}^i(X'/X, G) \cong \check{H}^i(X'_0/X_0, G)$ where X_0 is the spectrum of the residue field of A and X'_0 is just the special fiber $X' \times_X X_0$.

Let us prove this key argument (don't be concerned if we prove some random map is smooth, we will use this and the previous corollary for the key argument). Define, for each $i \geq 0$, the following Čech functor associated to the cover $X' \to X$ and the fppf abelian sheaf G:

$$\mathcal{C}^i(X'/X, G)\colon (\mathrm{Sch}/X) \to \mathrm{Ab}, \qquad Y \mapsto C^i(Y'/Y, G)$$

where $Y' := Y \times_X X'$ and $C^i(Y'/Y, G)$ is the i-th part of the Čech complex for the cover $Y' \to Y$ with respect to G. This definition works for any fppf abelian sheaf of course. Define also $\mathcal{Z}^i(X'/X, G)$ as the kernel of the map

$$d^i\colon \mathcal{C}^i(X'/X, G) \to \mathcal{C}^{i+1}(X'/X, G)$$

We will prove that $d^{i-1}\colon \mathcal{C}^{i-1}(X'/X, G) \to \mathcal{Z}^i(X'/X, G)$ is a smooth morphism of algebraic spaces. Firstly, notice that $\mathcal{C}^i(X'/X, G)$ is the Weil restriction of G along $(X')^i \to X$, since it is given by the rule

$$Y \mapsto G(Y' \times_Y Y' \times \cdots) = G(Y \times_X X' \times_X X' \times \cdots)$$

Since $(X')^i \to X$ is finite flat, $\mathcal{C}^i(X'/X,G)$ is represented by an algebraic space of finite presentation by by [53, Theorem 6.2]. Then $\mathcal{Z}^i(X'/X,G)$ is also a locally finitely presented algebraic space, as it is the kernel of a map of algebraic spaces. To check it is smooth, it suffices to check it is formally smooth. For this, we must prove that for any affine scheme T, and $T_0 \to T$ defined by square-zero ideal sheaf $\mathcal{I}$, any commutative square

$$\begin{array}{ccc} T_0 & \longrightarrow & \mathcal{C}^{i-1}(X'/X,G) \\ \downarrow & & \downarrow \\ T & \longrightarrow & \mathcal{Z}^i(X'/X,G) \end{array}$$

admits at least one arrow $T \to C^{i-1}(X'/X,G)$ such that the resulting diagram still commutes. For that consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^{i-1}(T'/T,N) & \xrightarrow{\alpha} & \mathcal{C}^{i-1}(X'/X,G)(T) & \xrightarrow{\beta} & \mathcal{C}^{i-1}(X'/X,G)(T_0) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d^{i-1}} & & \downarrow{\scriptstyle d^{i-1}} & & \downarrow{\scriptstyle d^{i-1}} & & \\ 0 & \longrightarrow & Z^i(T'/T,N) & \xrightarrow[\gamma]{} & \mathcal{Z}^i(X'/X,G)(T) & \xrightarrow[\delta]{} & \mathcal{Z}^i(X'/X,G)(T_0) & & \end{array}$$

where N is the functor on T-schemes, defined by $Y \mapsto \ker(G(Y) \to G(Y \times_T T_0))$. What matters about N is that it is the restriction to the flat site of the coherent $\mathcal{O}_X$-module $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(e^*_{G_T}(\Omega^1_{G_T/T}),\mathcal{I})$, where $e_T\colon T \to G_T$ is the identity section of G_T, see [12, I, §4, n°2, 2.4]. In terms of the last diagram, the formal smoothness question translates as follows: given $x \in \mathcal{C}^{i-1}(X'/X,G)(T_0)$ and $y \in \mathcal{Z}^i(X'/X,G)(T)$ with common image in $\mathcal{Z}^i(X'/X,G)(T_0)$, there is $z \in \mathcal{C}^{i-1}(X'/X,G)(T)$ that maps to x (horizontally) and to y (vertically). The surjectivity of the leftmost vertical arrow suffices to solve this problem. Indeed, since G is smooth, β is surjective, so take b such that $\beta(b) = x$. Naturally, $d^{i-1}(b)$ is not necessarily equal to y. Nonetheless, there is $c \in Z^i(T'/T,N)$ with $\gamma(c) = y - d^{i-1}(b)$. The assumed surjectivity gives $a \in C^{i-1}(T'/T,N)$ with $d^{i-1}(a) = c$. But then $\alpha(a) + b$ is the required z, solving the problem. The assumed surjectivity follows from the vanishing of $\check{H}^i(T'/T,N)$, which is guaranteed by the following three facts: N is the fppfication of a coherent sheaf on T, Proposition 10.5 and Serre's vanishing theorem of quasi-coherent sheaves on affine schemes [18, III. Thm. 3.7]

Going back to the key argument, notice that even $X' \times_X \cdots \times_X X'$ (i times) is not strictly local, it is a disjoint union of strictly local schemes, hence the map

$$C^i(X'/X,G) \to C^i(X'_0/X_0,G)$$

is surjective by the previous corollary. Thus we have an exact sequence of complexes

$$0 \longrightarrow K^\bullet \longrightarrow C^\bullet(X'/X,G) \longrightarrow C^\bullet(X'_0/X_0,G) \longrightarrow 0$$

where

$$K^i := \ker(C^i(X'/X,G) \to C^i(X'_0/X_0,G)).$$

This induces a long exact sequence in cohomology, and the key argument would follow if we prove $K^\bullet$ is exact. Take $c \in Z^i(X'/X,G) \cap K^i = \ker(K^i \to K^{i+1})$. We are looking for $c \in C^{i-1}(X'/X,G) \cap K^{i-1}$ such that $d^{i-1}(c) = z$. Since d^{i-1} is smooth, $d^{i-1}(z)$ is a smooth subscheme of $\mathcal{C}^i(X'/X,G)$. It has a section over X_0 (the zero section for instance) and by the previous corollary, it lifts to a section over X. This gives the desired c.

So finally, the proof reduces to show that $\check{H}^i(X'/X,G) = 0$, for $i \geq 1$, where X is the spectrum of a separably closed field. In fact, it suffices to show this assuming X is the spectrum of an algebraically closed field (since d^{i-1} is smooth). But in this case $X' \to X$ has a section, so the groups in question are 0 (construct a contracting homotopy). $\square$

10.5 Hilbert 90

The following theorem may be considered as a generalization of Hilbert's theorem 90.

Theorem 10.9 (Hilbert 90)**.** Let X be any scheme. Then we have isomorphisms

$$H^1(X_{\text{zar}}, \mathrm{GL}_n) \cong H^1(X_{\text{ét}}, \mathrm{GL}_n) \cong H^1(X_{\text{fppf}}, \mathrm{GL}_n).$$

In particular, $H^1(X_{\text{ét}}, \mathbb{G}_m) \cong \mathrm{Pic}(X)$.

Proof. We only prove $H^1(X_{\text{fppf}}, \mathrm{GL}_n) \cong H^1(X_{\text{zar}}, \mathrm{GL}_n)$, since the proof for the étale site is the same.

Let $\sigma\colon X_{\text{fppf}} \to X_{\text{zar}}$ be the canonical morphisms of sites. It suffices to show that $R^1\sigma_* \mathrm{GL}_n = 0$. This is a Zariski sheaf, so it suffices to prove $H^1(\operatorname{Spec} A, \mathrm{GL}_n)$ where A is a local ring (by looking at stalks). In fact, it suffices to prove that for a flat morphism $A \to B$, we have $\check{H}^1(B/A, \mathrm{GL}_n) = 0$ by Prop. 9.6, or in other words, that if M is a locally free A-module of rank n with $M \otimes_A B$ free, then M is free. Being finitely presented is Zariski-local [41, Tag 00EO(8)], so M finite locally free implies M is finitely presented, and hence so is $M \otimes_A B$. Since $M \otimes_A B$ is free, it is in particular flat, hence so is M as flatness descends [41, Tag 0584]. The following lemma implies M is free, concluding the proof. □

Lemma 10.10. Let $(R, \mathfrak{m}, k)$ be a local ring and M a finitely presented flat module. Then M is free.

Proof. Let $f'\colon k^n \to M/\mathfrak{m}M$ be an isomorphism of k-vector spaces. By Nakayamma's Lemma [41, Tag 00DV(8)] we can lift generators so that we have a surjective map $f\colon R^n \to M$ whose base change with $R \to k$ is f'. If the $K := \ker f$ is finitely generated, then $K \otimes_R k = \ker f' = 0$ implies $K = 0$ and we are done.

To prove K is finitely generated consider a finite presentation of M

$$R^b \longrightarrow R^a \longrightarrow M \longrightarrow 0.$$

Since R^n is projective, the presentation fits into a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^b & \longrightarrow & R^a & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{id}_M} & & \\ 0 & \longrightarrow & K & \longrightarrow & R^n & \underset{f}{\longrightarrow} & M & \longrightarrow & 0 \end{array}$$

By the snake lemma $\operatorname{coker} \alpha \cong \operatorname{coker} \beta$. The latter one is finitely generated, hence so is $\operatorname{coker} \beta$. But then K is finitely generated, as it is an extension of finitely generated modules:

$$0 \longrightarrow \operatorname{im} \beta \longrightarrow K \longrightarrow \operatorname{coker} \beta \longrightarrow 0.$$

□

11 Fundamental theorems of étale cohomology

The theorems in this section won't be proven. These theorems establish relations between cohomology and base change in étale cohomology, just as in the Zariski case.

Theorem 11.1 (Proper base change)**.** Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[g]{} & S \end{array}$$

be a cartesian diagram of schemes. If f is proper and $\mathscr{F}$ is a torsion abelian sheaf in X_{et}. Then the base change morphism

$$g^*(R^i f_* \mathscr{F}) \to R^i f'_*(g'^* \mathscr{F})$$

is an isomorphism for any $i \geq 0$.

Proof. [22, VI, Corollary 2.3.] □

In fact, proper base change is a corollary of the stronger result:

Theorem 11.2. Let $f\colon X \to S$ proper and $\mathscr{F}$ constructible on $X_{\text{Ét}}$. Then $R^i f_* \mathscr{F}$ is constructible on $S_{\text{Ét}}$ for all $i \geq 0$.

Proof. [22, VI, Theorem 2.1]. □

Corollary 11.3. Let $S = \operatorname{Spec}(A)$ of a henselian ring A and let S' be the spectrum of the residue field of its closed point. Let $f\colon X \to S$ be proper and $\mathscr{F}$ a torsion abelian sheaf on X_{et}. Then there is a canonical isomorphism

$$H^i(X, \mathscr{F}) \cong H^i(X', \mathscr{F}')$$

where $\mathscr{F}' = \mathscr{F}|_{X'}$, for all $i \geq 0$.

Proof. [22, VI, Corollary 2.7]. □

To state smooth base change we make a definition.

Definition 11.4. Given a scheme X we define

$$\operatorname{char}(X) := \{\operatorname{char} k(x) \colon x \text{ a point of } X\}.$$

We say that a torsion sheaf $\mathscr{F}$ on X is prime to $\operatorname{char}(X)$ if multiplication by p is injective on $\mathscr{F}$ for all $p \in \operatorname{char} X$.

Theorem 11.5 (Smooth base change)**.** Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[g]{} & S \end{array}$$

be a cartesian diagram of schemes. Suppose f is quasi-compact, g is smooth and $\mathscr{F}$ is a torsion sheaf on X_{et} with torsion prime to $\operatorname{char}(S)$. Then

$$g^*(R^i f_* \mathscr{F}) \to R^i f'_*(g'^* \mathscr{F})$$

is an isomorphism for all $i \geq 0$.

Proof. [22, VI, Theorem 4.1]. □

Recall that a *locally constant sheaf* on an object U of a site $\mathcal{C}$ is a sheaf $\mathscr{F}$ on U for which there exists a cover $\{U_i \to U\}$ such that $\mathscr{F}|_{U_i}$ is constant for each i.

Corollary 11.6 (Smooth specialization)**.** Let $f\colon X \to S$ proper and smooth and $\mathscr{F}$ a constructible locally constant sheaf on X_{et} with torsion prime to char S. Then $R^i f_* \mathscr{F}$ is constructible and locally constant for any $i \geq 0$.

Proof. [22, VI, Corollary 4.2]. □

12 The Picard functor

Definition 12.1 (Picard functor)**.** Let $f\colon X \to S$ be a morphism of schemes. The *Picard functor* of f is the sheafification of the presheaf

$$\operatorname{Pic}_{X/S}\colon (\mathrm{Sch}/S) \to \mathrm{Ab}, \quad (T \to S) \mapsto \operatorname{Pic}(X \times_S T)$$

on the étale topology. Since $H^1(T, \mathbb{G}_{m,T}) \cong \operatorname{Pic}(T)$, we have that $\operatorname{Pic}_{X/S} = R^1 f_* \mathbb{G}_m$ by Proposition 6.3.

The Picard scheme goes back to Grothendieck, see [26], for an expository treatment see [33]. In some circumstances, this functor admits a simpler description.

Definition 12.2. For a morphism of schemes $f\colon X \to S$ we say $f_*\mathcal{O}_X \cong \mathcal{O}_S$ holds *universally* if $f_*\mathcal{O}_X \cong \mathcal{O}_S$ and $f^*_{S'}\mathcal{O}_{X \times_S S'} \cong \mathcal{O}_{S'}$ for any base change $S' \to S$.

Proposition 12.3. If $f\colon X \to S$ is such that $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ universally and f admits a section, then $\operatorname{Pic}_{X/S}$ is the sheaf on $S_{\text{Ét}}$ that sends an S-scheme T to the abelian group

$$\frac{\operatorname{Pic}(X \times_S T)}{f_T^* \operatorname{Pic}(T)}$$

where we are taking quotient by the image of the natural pullback morphism $(f \times_S \mathrm{id}_T)^*\colon \operatorname{Pic}(T) \to \operatorname{Pic}(X \times_S T)$.

Proof. [33, Thm. 2.5] □

If a scheme represents the Picard functor, is called the *Picard scheme* and is denoted by $\operatorname{Pic}_{X/S}$ too. Regarding the representability of the Picard functor we have the following results.

Definition 12.4 (Nice morphism)**.** A *nice morphism* of schemes $f\colon X \to S$ is one that is proper, flat, of finite presentation with geometrically connected and geometrically reduced fibers. If we further impose geometrically irreducible fibers (so it has geometrically integral fibers), then we say f is *very nice*.

Theorem 12.5 (Representability)**.** Let $f\colon X \to S$ be nice. Then $\operatorname{Pic}_{X/S}$ is an S-algebraic space locally of finite presentation.

If f is very nice, then $\operatorname{Pic}_{X/S}$ is also separated.

Proof. [55, Thm. 27.117 and Prop. 27.118]. □

Since group algebraic spaces that are locally of finite type and separated over a field are schemes [41, Tag 0B8F] we have the following corollary:

Theorem 12.6 (Representability over fields)**.** Let X be a proper, geometrically integral k-scheme. Then $\operatorname{Pic}_{X/k}$ is a separated locally of finite type k-scheme.

Chapter 3

Group schemes

1 Definition and examples

In this chapter we review basic facts about group schemes that we will require later.

Definition 1.1. Let S be a scheme. A *group S-scheme* is an S-scheme G whose functor of points h_G, factors through the forgetful functor Grp $\to$ Sets.

We also write G for h_G.

Definition 1.2. A *morphism of group schemes* is a morphism of group presheaves. For instance the natural inclusion $\mathbb{G}_m \to \mathbb{G}_a$ (see next example) is not a morphism of group schemes, basically because $xy = x + y$ does not hold in rings. Since group schemes are sheaves for the fppf-topology, we may talk about kernel, cokernel, exact sequences and so on. If G, H are S-group schemes, then so is $G \times_S H$. In particular, the kernel of a morphism of group schemes is a group scheme. But the cokernel may not be representable. Thus, whenever we talk about exact sequence of group schemes, we mean exactness as fppf group sheaves. We will specify if we use only exactness as étale sheaves.

Example 1.3. (a) Let Sch $\to$ Ab be defined by $X \mapsto \Gamma(X, \mathcal{O}_X)$. This functor is representable by $\mathbb{A}^1_{\mathbb{Z}}$, so $\mathbb{A}^1_{\mathbb{Z}}$ is a commutative group scheme. Indeed, to give a global section of a scheme X, is the same as giving a ring morphism $\mathbb{Z}[t] \to \Gamma(X, \mathcal{O}_X)$, which is equivalent to giving a morphism of schemes $X \to \mathbb{A}^1_{\mathbb{Z}}$. We write $\mathbb{G}_a$ for the functor and the representing object. Of course, $\mathbb{A}^1_S \cong \mathbb{A}^1_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} S$ is also a commutative group S-scheme, denoted by $\mathbb{G}_{a,S}$ that represents the same functor but starting at (Sch$/S$).

(b) The functor Sch $\to$ Ab, $X \mapsto \Gamma(X, \mathcal{O}_X)^\times$ also defines a commutative group scheme. It is represented by $\operatorname{Spec}\mathbb{Z}[t, t^{-1}] = \operatorname{Spec}\mathbb{Z}[x, t]/(xt - 1)$. We write $\mathbb{G}_m$ for this group scheme. Of course, $\mathbb{G}_{m,S}$ is also defined as in the previous example.

(c) Generaling (b), we get the functor $X \mapsto \mathrm{GL}_n(\Gamma(X, \mathcal{O}_X))$. It is representable by the scheme $\operatorname{Spec}\mathbb{Z}[x_1, \ldots, x_n, y_n, \ldots, y_n, t]/(t\det([x_i y_j]_{ij}) - 1)$. We denote this group scheme by GL_n. We can also define SL_n and PGL_n and they are representable. They are represented by

$$\operatorname{Spec}\frac{\mathbb{Z}[x_1, \ldots, x_n, y_n, \ldots, y_n, t]}{(t\det([x_i y_j]_{ij}) - 1)} \quad \text{and} \quad \operatorname{Spec}\left(\frac{\mathbb{Z}[x_1, \ldots, x_n, y_n, \ldots, y_n, t]}{(t\det([x_i y_j]_{ij}) - 1)}\right)_0$$

respectively (the 0 subscript stands for elements of degree 0). Furthermore, we have exact sequences of presheaves

$$0 \longrightarrow \mu_{n,X} \longrightarrow \mathrm{SL}_{n,X} \longrightarrow \mathrm{PGL}_{n,X} \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{G}_{m,X} \longrightarrow \mathrm{GL}_{n,X} \longrightarrow \mathrm{PGL}_{n,X} \longrightarrow 0$$

for any scheme X, and any n.

(d) There is a multiplication by n map, $[n]\colon \mathbb{G}_m \to \mathbb{G}_m$ given by $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X)$, $x \mapsto x^n$ on points. The kernel of this map is called μ_n and we have

$$\mu_n(T) := \{t \in \Gamma(T, \mathcal{O}_T)^\times \mid t^n = 1\}.$$

It is represented by $\operatorname{Spec} \mathbb{Z}[t]/(t^n - 1)$.

(e) Given a finite (ordinary) group G, there is an associated group S-scheme for any scheme S. It is the sheaf defined as

$$T \mapsto \{\text{locally constant maps } T \to G\}.$$

It is represented by $\coprod_{g \in G} S$ and we denote this group S-scheme by G_S. For example, if X is an $\mathbb{F}_p$-scheme, then since

$$\operatorname{Spec} \mathbb{F}_p[t]/(t^p - t) = \operatorname{Spec} \frac{\mathbb{F}_p[t]}{\prod_{i=0}^{p-1} t - i} = \coprod_{i=1}^{p-1} \operatorname{Spec} \mathbb{F}_p[t]/(t - i) \cong \coprod_{i=0}^{p-1} \operatorname{Spec} \mathbb{F}_p \cong (\mathbb{Z}/p\mathbb{Z})_{\operatorname{Spec} \mathbb{F}_p}$$

we have

$$(\mathbb{Z}/p\mathbb{Z})_X(T) = \{s \in \Gamma(T, \mathcal{O}_T) \mid s^p - s = 0\}$$

for any X-scheme T.

(f) Let X be an $\mathbb{F}_p$-scheme. Then $\mathbb{G}_{a,X} \overset{x \mapsto x^p}{\longrightarrow} \mathbb{G}_{a,X}$ is a morphism of group schemes since we are in $\operatorname{char} X = p$. We define $\alpha_{p,X}$ as the kernel of this map. It is the functor defined by

$$(T \to X) \mapsto \{s \in \Gamma(T, \mathcal{O}_T) \mid s^p = 0\}.$$

Over $\mathbb{F}_p$, it is represented by $\operatorname{Spec} \mathbb{F}_p[t]/(t^p)$.

Proposition 1.4. Let X be a scheme. For any n, we have ane Fppf-exact sequence of group schemes

$$0 \longrightarrow \mu_{n,X} \longrightarrow \mathbb{G}_{m,X} \overset{(\cdot)^n}{\longrightarrow} \mathbb{G}_{m,X} \longrightarrow 0.$$

If n is invertible in $\mathcal{O}_X$, then it is also Ét-exact. This is the *Kummer sequence.*

Proof. For the étale topology, it suffices to show that $[n]\colon \mathbb{G}_{m,X} \to \mathbb{G}_{m,X}$ is surjective on stalks. So let A be strictly henselian local ring and let $a \in A$. We want to find, locally, in the étale topology an nth-root of A. But $B := A[t]/(t^n - a)$ is an étale A-algebra for any $a \in A^\times$, as long as n is invertible on A (for then $t^n - a$ is separable). This implies the assertion, as $t \mapsto a$ after tensoring with the étale map $A \to B$ (i.e restricting to $\operatorname{Spec} B \to \operatorname{Spec} A$).

For the flat topology, take $a \in \Gamma(X, \mathcal{O}_X)$ and a cover of X by open affines $X = \bigcup_i U_i$ with $U_i = \operatorname{Spec} A_i$. Now cover each U_i by $U_i' = \operatorname{Spec} A_i[t]/(t^n - a)$. Then $\{U_i' \to X\}$ is a flat cover of X and a lies in the image of $\mathbb{G}_m(U_i') \to \mathbb{G}_m(U_i')$ for each i. Thus, $[n]\colon \mathbb{G}_{m,X} \to \mathbb{G}_{m,X}$ is fppf-surjective as promised. □

2 Cartier duality

Theorem 2.1 (Cartier duality)**.** Let G be a commutative finite locally free group S-scheme. Then the functor of characters

$$\begin{aligned}(\mathrm{Sch}/S) &\to \mathrm{Ab}\\ (T \to S) &\mapsto \mathrm{Hom}_{T\text{-Grp}}(G_T, \mathbb{G}_{m,T})\end{aligned}$$

is representable by a commutative finite locally free group S-scheme G^D whose formation is compatible with base change, i.e, $(G_T)^D \cong (G^D)_T$. We have $G \cong (G^D)^D$ functorially.

Proof. [55, Prop. 27.81, Rem. 27.83]. □

The following technical result will be used in the proof of the theorem of the cube for Brauer classes.

Theorem 2.2 (Grothendieck duality)**.** Let S be a connected scheme and $G \to S$ be a commutative finite locally free group S-scheme. Let $f\colon X \to S$ be a proper S-scheme with section $e\colon S \to X$ and such that $f_*\mathcal{O}_X \cong \mathcal{O}_S$ universally. Then for any $T \to S$ there is a natural isomorphism

$$H^1_N(X_T, G_X) := \ker(H^1(X_T, G_X) \xrightarrow{e_T^*} H^1(T, G_X)) \cong \mathrm{Hom}_{\tilde{T}_{\text{ét}}}(G^D, \mathrm{Pic}_{X/S}).$$

At the level of sheaves,

$$R^1 f_* G \cong \underline{\mathrm{Hom}}_{\tilde{S}}(G^D, \mathrm{Pic}_{X/S})$$

is an isomorphism in $\tilde{S}_{\mathrm{Fppf}}$.

Proof. See [22, III. Prop. 4.16] and [16, Thm. 2.3]. □

Chapter 4

The Brauer group of a scheme

1 Brauer and cohomological Brauer group of a scheme

Theorem 1.1 (Azumaya algebras over a scheme)**.** Let X be a scheme and $\mathcal{A}$ an $\mathcal{O}_X$-algebra (not necessarily commutative) that is finite locally free as an $\mathcal{O}_X$-module. The following conditions are equivalent:

(a) The map $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^{\mathrm{op}} \to \underline{\mathrm{End}}_{\mathrm{Mod}_{\mathcal{O}_X}}(\mathcal{A})$ is an isomorphism;

(b) For every $x \in X$, $A(x) := \mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} k(x)$ is an Azumaya algebra over $k(x)$ (as defined in Chapter I);

(c) There is a covering $\{U_i \to X\}_{i \in I}$ for the étale topology such that, for each $i \in I$, there is an $r_i \geq 1$ such that $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \cong M_{r_i}(\mathcal{O}_{U_i})$;

(d) There is a covering $\{U_i \to X\}_{i \in I}$ for the fppf topology such that, for each $i \in I$, there is an $r_i \geq 1$ such that $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \cong M_{r_i}(\mathcal{O}_{U_i})$;

Such algebra is called an *Azumaya algebra* over X. The *degree* of $\mathcal{A}$ at x is the degree of $A(x)$ over $k(x)$. $\mathcal{A}$ has degree n, if it has degree n at every point of X.

Proof. [22, IV, Prop. 2.1]. □

Definition 1.2 ([22, IV, p. 141])**.** Two Azumaya algebras $\mathcal{A}, \mathcal{B}$ over X are *equivalent* if there are finite locally free $\mathcal{O}_X$-modules $\mathcal{E}$ and $\mathcal{E}'$ such that

$$\mathcal{A} \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}_{\mathrm{Mod}_{\mathcal{O}_X}}(\mathcal{E}) \cong \mathcal{B} \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}_{\mathrm{Mod}_{\mathcal{O}_X}}(\mathcal{E}').$$

This is clearly an equivalence relation. By part (a) of last theorem, the set of equivalence classes of Azumaya algebras is a group with the tensor operation. This is the *Brauer group* of X.

Of course, $\mathrm{Br}(\operatorname{Spec} k) = \mathrm{Br}(k)$ as defined in 1.9.

Definition 1.3. Let X be a scheme. The cohomological Brauer group of X is defined as

$$\mathrm{Br}'(X) := H^2(X_{\text{ét}}, \mathbb{G}_{m,X})_{\mathrm{tors}}.$$

Again, $\mathrm{Br}'(\operatorname{Spec} k) = H^2(k, \mathbb{G}_m(k_s))$ by Theorem 7.1.

Remark 1.4. By Theorem 10.8, we could also define $\mathrm{Br}'(X) = H^2(X_{\mathrm{fppf}}, \mathbb{G}_m)_{\mathrm{tors}}$

2 The Brauer map

We will state the existence of the Brauer map as a proposition.

Proposition 2.1. For any scheme X, there is an injective morphism of groups $\mathrm{Br}(X) \to \mathrm{Br}'(X)$, called *the Brauer map*.

Proof. [14, V. 4.4]. $\square$

Remark 2.2. To define the Brauer map one can proceed in two different manners. The first one uses a generalization of the non-abelian cohomology used in the first chapter, which does not hold in total generality. The second one uses the language of stacks. More precisely, that any elements of $\mathrm{Br}'(X)$ are represented by μ_n-gerbes. This is perhaps the most geometric and natural one and the one cited above.

There are criteria for a class $\alpha \in \mathrm{Br}'(X)$ to be in the image of the Brauer map $\mathrm{Br}(X) \to \mathrm{Br}'(X)$. The following is enough for our purposes.

Theorem 2.3. Let X be qcqs and let $\alpha \in \mathrm{Br}'(X)$. If there is a finite flat cover $f\colon Y \to X$ such that $f^*\alpha = 0 \in \mathrm{Br}(Y)$, then α is in the image of the Brauer map.

Proof. [28, Theorem 3.6]. $\square$

3 The Kummer sequence and cohomology

Proposition 3.1. Let X be a scheme and $n \geq 1$. Then there is an exact sequence of abelian groups functorial in X

$$0 \longrightarrow \frac{\mathrm{Pic}(X)}{n\,\mathrm{Pic}(X)} \longrightarrow H^2_{\mathrm{fppf}}(X, \mu_n) \longrightarrow \mathrm{Br}'(X)[n] \longrightarrow 0.$$

If n is invertible on X, then the middle term can be calculated in the étale site.

Proof. This follows from taking the long exact sequence in cohomology from the Kummer sequence 1.4

$$0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{(\cdot)^n} \mathbb{G}_m \longrightarrow 0.$$

The functoriality comes from the functoriality of the Kummer sequence. $\square$

Definition 3.2. If X is an S-scheme with a section $e\colon S \to X$, we define for any functor $F\colon (\mathrm{Sch}/S) \to \mathrm{Ab}$ the group $F_N(X) = \ker(F(e)\colon F(X) \to F(S))$.

Proposition 3.3. Let $f\colon X \to S$ be an S-scheme with section $e\colon X \to S$ and let $n \geq 1$. Then then maps $H^2_N(X, \mu_n) \to \mathrm{Br}'_N(X)[n]$ are also surjective.

Proof. The Kummer sequence induces the following diagram with exact rows by Prop. 3.1

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}(X)/n\,\mathrm{Pic}(X) & \longrightarrow & H^2(X, \mu_n) & \longrightarrow & H^2(X, \mathbb{G}_m)[n] & \longrightarrow & 0 \\
 & & f^* \uparrow \downarrow e^* & & f^* \uparrow \downarrow e^* & & f^* \uparrow \downarrow e^* & & \\
0 & \longrightarrow & \mathrm{Pic}(S)/n\,\mathrm{Pic}(S) & \longrightarrow & H^2(S, \mu_n) & \longrightarrow & H^2(S, \mathbb{G}_m)[n] & \longrightarrow & 0
\end{array}$$

Since e is a section of f, f^* is a section of e^*, hence e^* is surjective. This fact and the snake lemma shows that $H^2_N(X, \mu_n) \to H^2_N(X, \mathbb{G}_m)[n]$ is surjective. $\square$

4 Abelian schemes

Definition 4.1. An *abelian scheme* $A \to S$ is a nice group S-scheme. Equivalently, it is a group S-scheme $A \to S$ which is proper, smooth with geometrically connected fibers, see [55, Prop. 27.92].

Abelian schemes are families of abelian varieties and are well-studied objects. As such, thery are well-suited for testing conjectures and open problems. In fact, many important results in arithmetic geometry are first stated for abelian schemes, see [7, 23, 50, 62]. One feature of abelian schemes that is used sometimes is the following:

Proposition 4.2. Let $f: A \to S$ be an abelian scheme. Then $\mathcal{O}_S \cong f_*\mathcal{O}_A$ universally (see Definition 12.2).

Proof. First, as the fibers $A_s \to k(s)$ are proper, geometrically connected and geometrically reduced, we have $H^0(A_s, \mathcal{O}_{A_s}) = k(s)$ for each $s \in S$. For the rest we use cohomology and base change and the related notation. In particular, we have $\beta^0(k(s)): f_*\mathcal{O}_A \otimes k(s) \to H^0(A_s, \mathcal{O}_{A_s}) = k(s)$ is surjective, as it is a morphism of $k(s)$-algebras. Since $\beta^{-1}(k(s))$ is trivially surjective, we get $f_*\mathcal{O}_A$ is a locally free $\mathcal{O}_S$-module of rank 1. But both are $\mathcal{O}_S$-algebras, so we have $f_*\mathcal{O}_A \cong \mathcal{O}_S$. Since the hypothesis are stable under arbitrary base change, the isomorphism $\mathcal{O}_S \cong f_*\mathcal{O}_A$ holds universally. □

Lemma 4.3. Let $f: X \to Y$ be a surjective closed map of topological spaces with connected fibers. If Y is connected, then so is X. In particular, if $f: A \to S$ is an abelian scheme with S connected. Then A and $A \times_S A$ are connected.

Proof. Suppose X is not connected and write $X = X_1 \cup X_2$ for X_1, X_2 two disjoint open subsets of X. Consider $Y_i = \{y \in Y : \emptyset \neq f^{-1}(y) \subset X_i\}$ for $i = 1, 2$. Since f is surjective and has connected fibers, we have $Y = Y_1 \cup Y_2$ and $Y_1 \cap Y_2 = \emptyset$. So it suffices to show they are closed. But $f(X_i) = Y_i$ and f is closed so we are done. By the previous proposition, $\mathcal{O}_S \cong f_*\mathcal{O}_A$ universally, so both $A \to S$ and $A \times_S A \times_S A$ are surjective with connected fibers by Zariski's connectedness theorem [44, Thm. 12.63]. Both are also proper, hense (topologically) closed and the claim follows. □

The other fact we will use about abelian schemes, is that the multiplication by n map is flat, and sometimes étale. We will prove the latter case which is stronger but admits an easier argument.

Proposition 4.4. Let $A \to S$ be an abelian scheme and n an integer. Then the multiplication by n map, $[n]: A \to A$, is finite flat and surjective. If n is invertible on S, then $[n]$ is finite étale.

Proof. We will only prove this fact when n is invertible on S, since it is easier. For the general case see [55, Prop. 27.186]. Since étaleness may be checked on fibers [55, Prop. 18.45], we may assume S is the spectrum of a field and using faitfully flat descent we may further assume that k is algebraically closed [8, 2.7.1.(xv)], [8, 17.7.3.(ii)]. In this context, to check $[n]$ is étale, it suffices to check isomorphism on tangent spaces [55, Thm. 18.74.(iv)] and, by homogeneity (translation), it suffices to check this just at the tangent space at the identity

$$\mathrm{Lie}(A) := T_e(A/S).$$

But multiplication by n on A induces multiplication by n on $\mathrm{Lie}(A)$ which is then an isomorphism since $n \in k^\times$. □

The following results about the Picard scheme of an abelian scheme will be important for the next section.

Theorem 4.5. Let $A \to S$ be an abelian scheme. Then $\mathrm{Pic}^0_{A/S}$ is an abelian scheme. We have $\mathrm{Pic}^0_{A/S}[n] = \mathrm{Pic}_{A/S}[n]$ for any $n \geq 1$ and the map

$$\alpha \colon \mathrm{Pic}_{A/S}[n] \times \mathrm{Pic}_{A/S}[n] \to \mathrm{Pic}_{A\times A/S}[n]$$
$$(\mathcal{L}, \mathcal{M}) \mapsto \mathrm{pr}_1^* \mathcal{L} \otimes \mathrm{pr}_2^* \mathcal{M}$$

is an isomorphism. In particular, $\mathrm{Pic}^0_{A/S}[n]$ is a finite flat group scheme.

Proof. The fact that it is an abelian scheme follows from [55, Cor. 27.207, Prop. 27.212]. The second fact follows from [55, Remark 27.233(2)] and the third fact follows from [55, Remark 27.233(1)]. The last assertion follows from the previous proposition. □

The smoothness and structure of Pic^0 is a delicate matter, see [56, Prop. 4.8 and Rem. 4.9] and [58, Prop. 2.2] for instance.

5 Theorem of the cube for $H^2(-, \mu_n)$

In this section we prove a variant of the theorem of the cube [16, Thm. 2.4], where we change $\mathbb{G}_m$ by μ_{l^n} and assume the $f_i \colon X_i \to S$ are abelian schemes, in fact the same abelian scheme. We will use different indices for notational purposes.

Let $f \colon A \to S$ be an abelian scheme with section $e \colon S \to A$. We denote by $s_{ij} \colon A^2 \to A^3$ the natural sections of pr_{ij} given by the identities $X_k \to X_k$ for $k = i, j$ and e_k for the remaining index, e.g,

$$s_{13} = (\mathrm{id}_{X_1}, e, \mathrm{id}_{X_3}) : A \times S \times A \to A \times A \times A.$$

Theorem 5.1 (Theorem of the cube, [16, Thm. 2.4])**.** *Let $f \colon A \to S$ be an abelian scheme and l a prime invertible in $\mathcal{O}_S$. Then the natural map*

$$(s_{12}^*, s_{13}^*, s_{23}^*) \colon H^2(A^3, \mu_{l^n}) \to \prod_{1 \leq i < j \leq 3} H^2(A^2, \mu_{l^n})$$

injective for each n.

Proof. Clearly we can assume S is connected by working with each component separately, hence A and A^2 are connected by Lemma 4.3. The Leray spectral sequence 6.5 for $\mathrm{pr}_{12} \colon A^3 \to A \times A \times S$ gives a commutative diagram with exact row:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & H^2(A^2, \mu_{l^n}) & \xrightarrow{\mathrm{pr}_{12}^*} & F^1 H^2(A^3, \mu_{l^n}) & \xrightarrow{t} & H^1(A^2, \mathrm{Pic}_{A/S}[l^n]) & \longrightarrow & 0 \\
 & & & \xleftarrow[s_{12}^*]{} & \downarrow & & & & \\
 & & & & H^2(A^3, \mu_{l^n}) & & & & \\
 & & & & \downarrow{\scriptstyle \pi_{12}} & & & & \\
 & & & & H^0(A \times A \times S, R^2 f \mu_{l^n}) & & & &
\end{array}$$

where there are zeroes on both sides of the row since the maps $\mathrm{pr}_{12}^* \colon H^i(A^2, \mu_{l^n}) \to H^i(A^3, \mu_{l^n})$ have retractions s_{12}^*, in particular, the horizontal sequence splits. Notice we wrote $\mathrm{Pic}_{A/S}$ instead of $\mathrm{Pic}_{A^3/A\times A\times S}$ and $R^2 f_*\mu_{l^n}$ instead of $R^2\,\mathrm{pr}_{12*}\,\mu_{l^n}$, just because the latter ones are the restrictions of the first ones to $A\times A\times S$, by base changing $A\to S$ by $A^2\to S$. The same diagram above can be done for $\mathrm{pr}_1\colon A\times A\to A$ using its Leray spectral sequence it is related to the diagram above by functorialty of spectral sequences.

Take $y\in H^2(A^3,\mu_{l^n})$ such that $s_{ij}^*(y)=0$ for $1\le i<j\le 3$. The following commutative square comes from functoriality of spectral sequences

$$\begin{array}{ccc} H^2(A^2,\mu_{l^n}) & \xleftarrow{\;s_{13}^*\;} & H^2(A^3,\mu_{l^n}) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_{12}} \\ H^0(A,R^2 f_*\mu_{l^n}) & \xleftarrow{\;s_1^*\;} & H^0(A^2,R^2 f_*\mu_{l^n}) \end{array}$$

so $s_1^*\pi_{12}(y)=\pi_1 s_{13}^*(y)=0\in H^0(A,R^2 f_*\mu_{l^n})$ (here π_1 comes from the Leray spectral sequence for $\mathrm{pr}_1\colon A^2\to A$). Theorem 11.6 says $R^2 f_*\mu_{l^n}$ is locally constant so s_1^* is injective, hence $\pi_{12}(y)=0$ and

$$y\in F^1H^2(A^3,\mu_{l^n})\cong \mathrm{pr}_{12}^*(H^2(A^2,\mu_{l^n}))\oplus H^1(A^2,\mathrm{Pic}_{A/S}[l^n]).$$

The value of y on the first component is $\mathrm{pr}_{12}^*(s_{12}^*(y))=\mathrm{pr}_{12}^*(0)=0$ so it only remains to prove $t(y)=0$. By functoriality arguing as above we can prove $s_1^*(t(y))=s_2^*(t(y))=0$. If we further pullback $t(y)$ to S, we get $t(y)\in H^1_N(A^2,\mathrm{Pic}_{A/S}[l^n])$ where the N stands for the kernel along the section $S\to A\times A$. By Theorem 4.5, $H:=\mathrm{Pic}_{A/S}[l^n]=\mathrm{Pic}^0_{A/S}[l^n]$ is a finite locally free group scheme, annihilated by l^n and hence Grothendieck's duality 2.2 shows there are functorial isomorphisms (in X_{23})

$$H^1_N(X_{12},H)\to \mathrm{Hom}_{\tilde S}(H^D,\mathrm{Pic}_{X_{12}/S}[l^n]),$$

where we have observed that l^n annihilates H^D. But then using Theorem 4.5 we get

$$\mathrm{Hom}_{\tilde S}(H^D,\mathrm{Pic}_{A^2/S}[l^n])\cong \mathrm{Hom}_{\tilde S}(H^D,\mathrm{Pic}_{A/S}[l^n])\times \mathrm{Hom}_{\tilde S}(H^D,\mathrm{Pic}_{A/S}[l^n]).$$

Using Grothendieck duality as above, we conclude that

$$\{z\in H^1_N(X_{12},H)\mid s_1^*(z)=s_2^*(z)=0\}=0,$$

so $t(y)=0$. □

Remark 5.2. The only part of the proof where we used that l was invertible in $\mathcal{O}_S$ was to show s_1^* is injective, using $R^2 f_*\mu_{l^n}$ was locally constant.

Definition 5.3 ([16, p. 58])**.** An abelian scheme $A\to S$ *satisfies the generalized theorem of the cube* for a prime l if the map

$$\phi\colon\ \mathrm{Br}'(A^3)[l^\infty]\to \prod_{1\le i<\le 3}\mathrm{Br}'(A^2)[l^\infty]$$

is injective.

For the rest of this section, we write $\prod_{123}$ instead of $\prod_{1\le i<j\le 3}$ for aesthetic purposes. In [16, Thm. 3.3], Hoobler assumes that $A\to S$ satisfies the generalized theorem of the cube for a prime l invertible on $\mathcal{O}_S$. We argue that this is superfluous, i.e, always holds.

Proposition 5.4. Let $A \to S$ be an abelian scheme and l a prime invertible in $\mathcal{O}_S$. Then A satisfies the generalized theorem of the cube for l.

Proof. Let $T\colon (\mathrm{Sch}/S) \to \mathrm{Ab}$ be any contravariant functor that vanishes on S. Then the maps e_{ij} for $1 \leq i < j \leq 3$ induce a map

$$\beta\colon T(A^3) \to \prod_{123} T(A^2).$$

Analogously, pullback along the different projections pr_{ij} induce a map

$$\alpha\colon \prod_{123} T(A^2) \to T(A^3).$$

These maps induce a splitting $T(A^3) = \ker\beta \oplus \operatorname{im}\alpha$, as suggesteed by Mumford in [13, p. 55] (see this answer for a proof), so α is surjective if and only if β is injective. With this in mind, consider the following commutative diagram with surjective horizontal maps from Prop. 3.3

$$\begin{array}{ccc} \prod_{123} H^2_N(A^2_{\text{ét}}, \mu_{l^n}) & \twoheadrightarrow & \prod_{123} \mathrm{Br}'(A^2)[l^n] \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} \\ H^2_N(A^3_{\text{ét}}, \mu_{l^n}) & \twoheadrightarrow & \mathrm{Br}'(A^3)[l^n] \end{array}$$

By the first paragraph and the theorem of the cube for μ_{l^n} 5.1 the map α is surjective. A simple diagram chase shows α' is surjective, but then again, the first paragraph implies that the map

$$\mathrm{Br}'(A^3)[l^n] \to \prod_{123} \mathrm{Br}'(A^2)[l^n]$$

is injective for each n and we are done. $\square$

Definition 5.5. Suppose we have an S-scheme $t\colon T \to S$ and $x_1, x_2, x_3 \in A(T)$ S-morphisms. If $I \subset \{1, 2, 3\}$, we define

$$x_I = \sum_{i \in I} x_i := + \circ (y_1, y_2, y_3)$$

where $y_i = x_i$ if $i \in I$, $y_i = e_T := e \circ t$ otherwise, (y_1, y_2, y_3) is the natural morphism $T \to A^3$ and $+\colon A^3 \to A$ is the sum morphism.

Remark 5.6. There is a canonical object with three morphisms to A, namely, A^3 and the three projections $\mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3$ and we can prove that

$$x_I = (\mathrm{pr})_I \circ (x_1, x_2, x_3).$$

It is also useful to note that $(\mathrm{pr})_I = y_1 \times y_2 \times y_3\colon A^3 \to A^3$ where $y_i = \mathrm{id}_A$ if $i \in I$ and $y_i = e_A$ otherwise, e.g, $(\mathrm{pr})_{13} = \mathrm{id}_A \times e_A \times \mathrm{id}_A$.

As in the line bundle case [13, II,§6, Cor. 3] (see [55, Prop. 27.184]), there is a numerical formula that explains the behaviour of cohomological Brauer classes under pullback by multiplication by n maps.

Proposition 5.7. Let $y \in H^2(A, \mu_{l^m})$, l invertible in $\mathcal{O}_S$.

(a) Let T be an S-scheme and $x_1, x_2, x_3 \in A(T)$. Then, the class

$$\theta_{T,x}(y) := x_{123}^* y - x_{12}^* y - x_{13}^* y - x_{23}^* y + x_1^* y + x_2^* y + x_3^* y - e_T^* y$$

is trivial in $H^2(T, \mu_{l^m})$.

(b) Let $n \in \mathbb{Z}$. Then

$$[n]^* y = \left(\frac{n^2+n}{2}\right) y + \left(\frac{n^2-n}{2}\right) [-1]^* y + (1-n^2)[0]^* y.$$

Proof. (a) By the previous remark, we notice that $x_{123}^* y = (x_1, x_2, x_3)^* (\mathrm{pr})_I^* y$, so

$$\theta_{T,x}(y) = (x_1, x_2, x_3)^* \theta_{A^3,\mathrm{pr}}(y)$$

and hence it suffices to show $\theta(y) := \theta_{A,\mathrm{pr}}(y)$ is trivial on $H^2(A^3, \mu_{l^m})$.

By the theorem of the cube, it is enough to prove that $s_{ij}^* \theta(y)$ is trivial for each $1 \leq i < j \leq 3$. We just do the case for s_{12}, as the proof for the others is identical.

Very explicitly we have

- $s_{12}^* (\mathrm{pr})_{123}^* y = (\mathrm{id}_A \times \mathrm{id}_A \times e)^* (\mathrm{id}_A \times \mathrm{id}_A \times \mathrm{id}_A)^* y = (\mathrm{id}_A \times \mathrm{id}_A \times e)^* y$.
- $-s_{12}^* (\mathrm{pr})_{12}^* y = -(\mathrm{id}_A \times \mathrm{id}_A \times e)^* (\mathrm{id}_A \times \mathrm{id}_A \times e_A)^* y = -(\mathrm{id}_A \times \mathrm{id}_A \times e)$.
- $-s_{12}^* (\mathrm{pr})_{13}^* y = -(\mathrm{id}_A \times \mathrm{id}_A \times e)^* (\mathrm{id}_A \times e_A \times \mathrm{id}_A)^* y = -(\mathrm{id}_A \times e_A \times e)^* y$.
- $-s_{12}^* (\mathrm{pr})_{23}^* y = -(\mathrm{id}_A \times \mathrm{id}_A \times e)^* (e_A \times \mathrm{id}_A \times \mathrm{id}_A)^* y = -(e_A \times \mathrm{id}_A \times e)^* y$.
- $s_{12}^* (\mathrm{pr})_1^* y = (\mathrm{id}_A \times \mathrm{id}_A \times e)^* (\mathrm{id}_A \times e_A \times e_A)^* y = (\mathrm{id}_A \times e_A \times e)^* y$.
- $s_{12}^* (\mathrm{pr})_2^* y = (\mathrm{id}_A \times \mathrm{id}_A \times e)^* (e_A \times \mathrm{id}_A \times e_A)^* y = (e_A \times \mathrm{id}_A \times e)^* y$.
- $s_{12}^* (\mathrm{pr})_3^* y = (\mathrm{id}_A \times \mathrm{id}_A \times e)^* (e_A \times e_A \times \mathrm{id}_A)^* y = (e_A \times e_A \times e)^* y$.
- $-s_{12}^* e_{A^3}^* y = -(\mathrm{id}_A \times \mathrm{id}_A \times e)^* (e_A \times e_A \times e_A)^* y = -(e_A \times e_A \times e)^* y$.

where we used $e_A \circ e = e$. If we sum all terms we get $s_{1,2}^* \theta(y) = 0 \in H^2(A^3, \mu_{l^m})$ as we wanted.

(b) Define $f\colon \mathbb{Z} \to H^2(A, \mu_{l^m})$ by $n \mapsto [n]^*(y)$. By the first part, using $x_1 = (n+1)$, $x_2 = 1$ and $x_3 = -1$ we get

$$f((n+1)+1-1) = f(n+2) + f(n) + f(0) - f(n+1) - f(1) - f(-1) + f(0).$$

where we have used $e_A = [0]$. Rearranging we obtain

$$f(n+2) = 2f(n+1) - f(n) + f(1) + f(-1) - 2f(0).$$

This recurrence formula shows $f(n)$ is of the form $A_n f(1) + B_n f(-1) + C_n f(0)$, we just need to figure what this coefficients are. The recurrence relation for f implies the following recursions for the coefficients

$$\begin{aligned} A_{n+2} &= 2A_{n+1} - A_n + 1 \\ B_{n+2} &= 2B_{n+1} - B_n + 1 \\ C_{n+2} &= 2C_{n+1} - C_n - 2. \end{aligned}$$

We claim that

$$A_n = \frac{n^2+n}{2}, \quad B_n = \frac{n^2-n}{2} \quad \text{and} \quad C_n = 1-n^2.$$

Indeed, the values of A_n, B_n and C_n for $n = -1, 0, 1$ coincide with the formulas and for the general case we use induction:

$$\begin{aligned} A_{n+2} = 2A_{n+1} - A_n + 1 = &= (n+1)^2 + (n+1) - \frac{n^2+n}{2} + 1 \\ &= n^2 + 2n + 1 + n + 1 + 1 - \frac{n^2}{2} - \frac{n}{2} \\ &= \frac{2n^2+6n+6-n^2-n}{2} \\ &= \frac{n^2+5n+6}{2} \\ &= \frac{(n+2)^2+(n+2)}{2}. \end{aligned}$$

The calculations for B_n and C_n similar and our claim is proven.

□

Finally, Hoobler's result without hypotheses on S.

Theorem 5.8. Let $f\colon A \to S$ be an abelian scheme with section $e\colon S \to A$, and let l be a prime invertible in $\mathcal{O}_S$. Then

$$\mathrm{Br}'_N(A)[l^\infty] \subset \mathrm{Br}(A).$$

If $n \geq 1$ is invertible in $\mathcal{O}_S$ the same conclusion holds for the n torsion of $\mathrm{Br}'_N(A)$.

Proof. Let $\alpha \in \mathrm{Br}'_N(A)[l^n]$. By Theorem 2.3 and Prop. 4.4, it suffices to show that $[k]^*\alpha = 0$ for some integer k. But then it suffices to kill a preimage $\beta \in H^2_N(A, \mu_{l^n})$ of α, which exists since $H^2_N(A, \mu_{l^n}) \to \mathrm{Br}'_N(A)[l^n]$ is surjective by Proposition 3.3. If l is odd, then so is $m = l^n$ and $\frac{m^2 \pm m}{2}$ is divisible by m, since $m \pm 1$ is even. But $H^2(A, \mu_m)$ is m-torsion, so

$$[m]^*\beta = \left(\frac{m^2+m}{2}\right)\beta + \left(\frac{m^2-m}{2}\right)[-1]^*\beta + (1-n^2)[0]^*\beta = 0,$$

where we also used $[0]^*\alpha = 0$ as $y \in H_N(A, \mu_{l^n})$. If $l = 2$, then

$$\frac{2^{n+1}(2^{n+1}-1)}{2}$$

is divisible by 2^n, so $[l^{n+1}]^*\beta = 0$ and we are done.

The second assertion in the theorem follows using the first part of the proof and the next lemma. □

Lemma 5.9. Suppose m, n are such that classes in $H^2_N(A, \mu_m)$ and $H^2_N(A, \mu_n)$ are killed by multiplication by integers maps, then classes in $H^2_N(A, \mu_{mn})$ are killed by multiplication by integer maps.

Proof. Consider the exact sequence of sheaves on A

$$0 \longrightarrow \mu_m \longrightarrow \mu_{mn} \xrightarrow{(\cdot)^m} \mu_n \longrightarrow 0$$

and the induced exact sequence of groups

$$H^2(X, \mu_m) \xrightarrow{h_1} H^2(X, \mu_{mn}) \xrightarrow{h_2} H^2(X, \mu_n).$$

We claim that

$$H^2_N(X, \mu_m) \xrightarrow{h_1} H^2_N(X, \mu_{mn}) \xrightarrow{h_2} H^2_N(X, \mu_n)$$

is also exact. Indeed, consider the following commutative diagram

$$
\begin{array}{ccccccc}
 & & H^2_N(A,\mu_n) & \xrightarrow{a} & H^2_N(A,\mu_{mn}) & \xrightarrow{b} & H^2_N(A,\mu_m) \\
 & & \downarrow & & \downarrow & & \downarrow \\
H^1(A,\mu_m) & \xrightarrow{\delta} & H^2(A,\mu_n) & \xrightarrow{a'} & H^2(A,\mu_{mn}) & \longrightarrow & H^2(A,\mu_m) \\
\downarrow{\scriptstyle e^*} & & \downarrow{\scriptstyle e^*} & & \downarrow & & \downarrow \\
H^1(S,\mu_m) & \xrightarrow{\delta} & H^2(S,\mu_n) & \xrightarrow{a''} & H^2(S,\mu_{mn}) & \longrightarrow & H^2(S,\mu_m) \\
\downarrow & & & & & & \\
0 & & & & & &
\end{array}
$$

with exact columns and the two bottom rows are exact, and notice that the vertical map on the bottom left corner is surjective since it has a section f^*. We take $y \in \ker b$. Then $y \in \operatorname{im} a'$, say $a'(y') = y$, but not necessarily in $\operatorname{im} a$, which is what we want. But since $e^* y = 0$, we have $a''(e^* y') = e^* y = 0$ and there is $z \in H^1(S, \mu_m)$ with $\delta(z) = e^* y'$. By surjectivity, there is $z' \in H^1(A, \mu_m)$ mapping to z, so $e^*(y' - \delta(z')) = 0$. But then $y' - \delta(z') \in H^2_N(A, \mu_n)$ and maps to y, proving the desired exactness.

Now take a class $\alpha \in H^2_N(X, \mu_{mn})$. Then there is $k_2 \geq 1$ such that

$$h_2([k_2]^* \alpha) = [k_2]^*(h_2(\alpha)) = 0$$

since classes in $H^2_N(A, \mu_n)$ are killed by multiplication by integers maps, so $[k_2]^* \alpha \in \ker h_2 = \operatorname{im} h_1$, say $h_1(\beta) = [k_2]^* \alpha$. Now again there is an integer $k_1 \geq 1$ such that $[k_1]^* \beta = 0$, so

$$[k_1 k_2]^* \alpha = [k_1]^* [k_2]^* \alpha = [k_1]^* h_1(\beta) = h_1([k_1]^* \beta) = 0$$

proving the claim. □

6 A non-projective abelian scheme

The following is an example of a non-projective abelian scheme due to Raynaud [6, XII.2]. This shows that Theorem 5.8 really covers cases that are not considered in Gabber's result.

Proposition 6.1. There is an abelian scheme $A \to S$ where S is a local integral scheme of dimension 1 and A is not projective.

Proof. Let k be an algebraically closed field and S be the spectrum of the local ring at the node of the nodal cubic $V(y^2z - x^2(x+1)) \subset \mathbb{P}^2_k$. Let B_k be a non-trivial abelian variety over k and let $B := B_k \times_k S$ be the constant abelian variety over S. Consider $A = B \times_S B$ together with the automorphism (of infinite order) given on points by $(b, b') \mapsto (b, b + b')$. We call this automorphism u, and we also note that $\mathrm{NS}_{A/S}$ is a constant group scheme.

Now, there is an étale cover $S' \to S$, connected, Galois and with Galois group $\mathbb{Z}$ generated by the translation that maps a line isomorphically into the next, see Figure 4.1

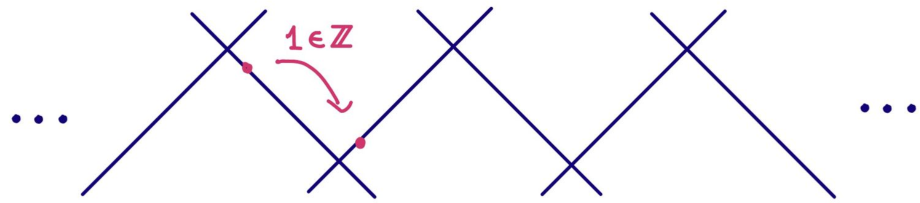


Figure 4.1: Étale cover of the node

We consider the action of $\mathbb{Z}$ on A' given by $n \mapsto u^n \times n$. The structure morphism (projection onto the second factor) $A' \to S'$ is $\mathbb{Z}$-equivariant, and hence we get a descent datum on A' relative to $S' \to S$ by Proposition 3.8. The descent datum is effective in the category of algebraic spaces by [27, Cor. 1.6.4], so there is an algebraic space C over S with $C \times_S S' = A'$. Since being proper, smooth with geometrically irreducible fibers is local for the fppf topology [41, Tag 041K], C is an abelian space and hence is representable by a scheme [25, Thm. 1.9.(a)].

Assume C has an ample line bundle $\mathcal{L}$ which then defines a global section of the Nerón-Severi scheme $c \in \mathrm{NS}_{C/S}(S)$. Let $c' \in \mathrm{NS}_{A/S}(A')$ be its pullback along $S' \to S$. Since $\mathrm{NS}_{A/S}$ is a constant group scheme and S' is connected, the restriction map $\mathrm{NS}_{A/S}(A) \to \mathrm{NS}_{A/S}(A')$ is bijective, so c' comes from an element $a \in \mathrm{NS}_{A/S}(A)$ which defines a polarisation of A, since $\mathcal{L}$ is ample. Moreover a is invariant under the action of the automorphism u, since $\mathrm{NS}_{C/S}$ is obtained from $\mathrm{NS}_{A/S}$ using the same descent datum. Therefore, the automorphism u is an infinite order automorphism of the polarisation $(A, \mathcal{L})$ which contradicts Matsusaka's theorem [4, I, 4.4, Prop. 17]. We conclude C is not projective.

□

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