Pontificia Universidad Católica de Chile

Master Thesis

# Brauer Groups for Abelian Torsors

*Author:*
*Felipe Hernandez Mella*

*Advisor:*
*Siddharth Mathur*

*Thesis submitted to the Faculty of Mathematics of the Pontificia Universidad Católica, in partial fulfillment of the requirements for the degree of Master in Mathematics.*

Committee:
Giancarlo Lucchini (Universidad de Chile)
Siddharth Mathur (Pontificia Universidad Católica de Chile)
Giancarlo Urzúa (Pontificia Universidad Católica de Chile)

June 30 2025
Santiago, Chile

# Contents

**Acknowledgments** **3**

**Introduction** **5**

**Notations and Conventions** **8**

**1 Preliminaries** **9**

1.1 Torsors . . . 9

1.2 Non Abelian cohomology . . . 11

1.3 Brauer Groups . . . 13

**2 Quotient stacks and Gerbes** **15**

2.1 Algebraic Stacks . . . 15

2.2 Quotient Stacks . . . 22

2.3 Gerbes . . . 32

2.4 Coverings of Stacks . . . 41

**3 Gerbes for Finite Flat Group Scheme** **48**

3.1 Finite Flat Group Schemes . . . 48

3.2 Gerbes are Quotient Stacks . . . 50

3.3 Abelian Torsors . . . 55

3.4 Brauer groups for Abelian Torsors . . . 58

**Bibliography** **69**

# Acknowledgments

This thesis would not have been possible without the guidance, support, and affection of many incredible people. I am deeply grateful to everyone who was part of this master's journey—both those I share the world of mathematics with, and those with whom I connect in other areas of life. It would be impossible for me to name each of you, but to anyone who feels identified with these words, please know that I truly appreciate your support throughout these years.

First and foremost, I would like to express my gratitude to my entire family, especially my mother and grandmother. Their constant support every single day, even in the smallest things, has always been fundamental to me. I would also like to thank my girlfriend for the time we've shared together. She has been an unconditional source of support in recent times.

I would also like to thank all my friends from graduate school. Without a doubt, they are the people I've spent the most time with in recent years—whether solving math problems or talking nonsense. I am very grateful for all those shared moments, and I will always remember our student days with great fondness. I also want to acknowledge my friends outside the academic world, with whom I've shared many adventures. Fridays were always full of unforgettable stories.

Special thanks to my friend Francisco Gallardo, with whom I've shared the journey of learning algebraic geometry. So much of what I've learned is thanks to the countless hours we spent together trying to solve exercises from Vakil's and Hartshorne's books. It has been a challenging but rewarding path.

Finally, the person to whom I owe the most for this thesis is professor Siddharth Mathur.

I truly believe that completing a thesis like this would have been impossible without his supervision. I deeply appreciate the meetings where we worked through Hartshorne exercises, the long hours that he dedicated to explaining to me articles, and his advice on how to do research and being a good mathematician.

The studies that led to the development of this thesis were funded by Sub-direccion de Capital Humano de la ANID (Agencia Nacional de Investigacion y Desarrollo) through the 2023 National Master's Scholarship. The corresponding record number is 22230810

# Introduction

The Brauer group was introduced in 1932 by R. Brauer in the context of fields as an invariant classifying certain types of algebras, [Bra32]. It was studied by Brauer-Hasse-Noether and many others; in particular, they proved that the Brauer group of a field, denoted $\mathrm{Br}(k)$ is isomorphic to the Galois cohomology group $H^2(k, k_s^\times)$, [Roq05, Section 4.3]. Later in 1951, Azumaya extended the notion of the Brauer group to local rings, [Azu51]. Lastly, in 1960, Auslander and Goldman introduced the Brauer group of a commutative ring, [AG60]. All of these works are precursors to the definition of the theory of Brauer groups on schemes.

In 1968 Alexander Grothendieck, in his series of lectures [Gro68a], [Gro95], and [Gro68b], introduced the theory of Brauer groups for a scheme $X$. In it, he describes two groups associated to $X$. One of them, denoted $\mathrm{Br}(X)$, is a generalization of the Brauer group of a field $\mathrm{Br}(k)$, and we refer to it as the Azumaya-Brauer group of $X$. The second group, inspired by the isomorphism $\mathrm{Br}(k) \simeq H^2(k, k_s^\times)$, is defined as the torsion subgroup of the étale cohomology group $H^2(X_{\text{ét}}, \mathbb{G}_m)$, usually known as the cohomological Brauer group of $X$, and denoted $\mathrm{Br}'(X)$. In an attempt to extend the isomorphism $\mathrm{Br}(k) \simeq H^2(k, k_s^\times)$ to an isomorphism $\mathrm{Br}(X) \to \mathrm{Br}'(X)$, is that Grothendieck describes a morphism $\mathrm{Br}(X) \to \mathrm{Br}'(X)$ that is always injective, we refer to it as the Brauer map of $X$. Consequently, he introduces the following problem:

**Question.** *When is the Brauer map* $\mathrm{Br}(X) \to \mathrm{Br}'(X)$ *an isomorphism?*

This is known as the $\mathrm{Br} = \mathrm{Br}'$ problem. Since its introduction, it has given rise to a lot of new mathematics, and both $\mathrm{Br}(X), \mathrm{Br}'(X)$ have several applications in algebraic and arithmetic geometry (the book [CTS21] contains many of the applications of the theory of Brauer groups).

The earliest progress concerning this question belongs to Grothendieck himself, who showed that cohomological Brauer classes on regular schemes are representable by Azumaya algebras away from a codimension $\geq 3$ subscheme, therefore establishing the result for regular surfaces, [Gro95, Theorem 2.1]. After this, R. Hoobler proved that $\mathrm{Br} = \mathrm{Br}'$ for smooth affine varieties over a field, [Hoo80]. Later, Ofer Gabber significantly improved Hoobler's result in his thesis. By a clever use of the $K$-theory of rings, Gabber proved $\mathrm{Br} = \mathrm{Br}'$ for any affine scheme, and also for separated unions of two affine schemes, [Gab78, Chapter II, Theorem 1]. Berkovich and Hoobler also established $\mathrm{Br} = \mathrm{Br}'$ for Abelian varieties in $\mathrm{char}\, k = 0$, [Ber72] and [Hoo72]. Using elementary transformations, Schroer proved that all separated geometrically normal surfaces satisfy $\mathrm{Br} = \mathrm{Br}'$, [Sch01]. A striking progress came in the 90s, when Gabber announced a proof of $\mathrm{Br} = \mathrm{Br}'$ for any scheme $X$ admitting an ample line bundle.

Unfortunately, Gabber's proof is still unpublished. However, in the early 2000s, de Jong found a proof using the theory of twisted sheaves, [dJ03]. This new perspective shifted the research away from Azumaya algebras and towards vector bundles on stacks. A lot of these ideas are present in the work of Edidin, Hassett, Kresch, and Vistoli in [EHKV01], where they found an example of a non-separated singular surface such that $\mathrm{Br}$ and $\mathrm{Br}'$ differ. The work of [EHKV01] also makes explicit connections between the problem $\mathrm{Br} = \mathrm{Br}'$ and determining whether a stack is a quotient stack.

Among stacks, quotient stacks have several good properties that make them interesting objects of study. For example, it is a classical result that étale locally, Deligne-Mumford stacks are quotient stacks [LMB00, Theorem 6.2]. Recently, Alper, Hall, and Rydh proved that the same holds for algebraic stacks with affine linearly reductive stabilizers, [AHR20]. However, these results do not imply the property of being a global quotient stack. As a consequence of $\mathrm{Br} = \mathrm{Br}'$ for quasi-projective schemes, Kresch and Vistoli showed that every smooth, generically tame, separated Deligne-Mumford stack over a field is a quotient stack, [KV04].

Later in 2002, Totaro showed that being a global quotient stack is deeply related to coherent sheaves being quotients of vector bundles, this is known as the resolution property, [Tot04]. Several research studies have been done in this direction. For instance, Mathur showed that finite type, normal algebraic stacks of dimension 2 over a field have the resolution property, implying that they are quotient stacks, [Mat21] and [Mat18], he also established

$\mathrm{Br} = \mathrm{Br}'$ for separated surfaces, improving the result of Schroer, [Mat22], for an application of these methods to classical geometry, see [Mat23]. In addition, Bragg, Hall, and Mathur showed that stacks that have a unipotent morphism to a stack which satisfies the resolution property are quotient stacks, [BHM25]. Mathur and Schroer also show that under mild hypotheses, the resolution property holds away from a closed subset of codimension $\geq 3$, [MS23].

The purpose of this thesis is to study the question $\mathrm{Br}(X) = \mathrm{Br}'(X)$ when $X$ is a torsor under an abelian scheme. To do this, we follow the approach of [EHKV01] in relating the question to gerbes being quotient stacks. A consequence of $\mathrm{Br} = \mathrm{Br}'$ for quasi-projective schemes is that every $\mu_n$-gerbe over a quasi-projective scheme is a quotient stack. And by a clever argument, we can reduce the $\mathrm{Br} = \mathrm{Br}'$ problem for abelian torsors, to the problem of $A[n]$-gerbes being quotient stacks. In particular, we can prove the following:

**Theorem.** *Let $S$ be a quasi-projective scheme over a field $k$, $A$ an abelian scheme over $S$, and $X$ an $A$-torsor over $S$.*

*(i) If* $\mathrm{char}\, k = 0$*, then* $\mathrm{Br}(X) = \mathrm{Br}'(X)$*.*

*(ii) If* $\mathrm{char}\, k = p > 0$*, the group $A[p]$ is isotrivial, and $S$ is reduced, then* $\mathrm{Br}(X) = \mathrm{Br}'(X)$*.*

*(iii) If* $\mathrm{char}\, k = p > 0$*, and $A$ is an ordinary abelian scheme, then* $\mathrm{Br}(X) = \mathrm{Br}'(X)$*.*

We would like to remark that the $\mathrm{Br} = \mathrm{Br}'$ problem has been related to many other problems, and it is still an important source of research. For example, Toen studied the existence of derived Azumaya algebras, [Toe12]. More recently, de Jong, Lieblich, and Shin have studied twisted sheaves of infinite rank, [dJLS23]. Schroer [Sch05], and Huybrechts-Schroer [HS03] have studied the analog problem for complex analytic surfaces, and Heinloth-Schroer [HS09] have studied representability by non-Azumaya algebras. Shin has also studied the question for stacks [Shi21],[Shi23], and [Shi24]. The problem is still wide open for smooth separated schemes, in particular, little is known for smooth algebraic threefolds.

# Notations and Conventions

- Given a morphism of schemes $T \to S$, and an object $X$ over $S$ (for example a scheme, a sheaf, or a cohomology class) we denote $X_T$ the pullback of the object to $T$. Some other authors use $X|_T$ or $X \times_S T$ when $X$ is a sheaf or a scheme.
- An action of a group $G$ on a set $X$ is always a left action. So $G$-torsors are always assumed to be left $G$-torsors.
- $G$-torsors and stacks are considered to be defined in the big fppf site of a base scheme $S$ unless other things are specified.
- Given a morphism of schemes $T \to S$, we denote by $p_1, p_2 : T \times_S T \to T$ the first and second projection. We also denote $p_{12}, p_{13}, p_{23} : T \times_S T \times_S T \to T \times_S T$ the corresponding projections.
- An fppf group scheme $G$ over $S$, is a group such that its structure morphism to $S$ is faithfully flat and locally of finite presentation.
- When we say a quotient stack, we mean a stack of the form $[X/G]$ where $X$ is an algebraic space and $G$ is a subgroup scheme of $\mathrm{GL}_n$ for some $n$.
- Given two sheaves $\mathscr{F}, \mathscr{G}$ over $S$, we denote by $\underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G})$ the sheaf that for any $T \to S$ associates the set $\mathrm{Hom}(\mathscr{F}_T, \mathscr{G}_T)$.
- Given a stack $F$ over $S$ and two objects $x, y \in F(S)$, we denote by $\underline{\mathrm{Isom}}_F(x, y)$ the sheaf that for any $T \to S$ associates the group of isomorphisms between $x_T$ and $y_T$. Consequently, we denote $\underline{\mathrm{Aut}}_F(x)$ the sheaf $\underline{\mathrm{Isom}}_F(x, x)$.
- The cohomology groups $H^i(S, -)$ are assumed to be fppf cohomology groups. Some authors use $H^i_{\mathrm{fppf}}(S, -)$ or $H^i(S_{\mathrm{fppf}}, -)$.

# Chapter 1

# Preliminaries

In this first chapter, we introduce preliminaries that will be necessary for the following chapters. Although the theory presented here is not particularly complicated, we simply provide references for the proofs. We begin by reviewing the basic theory of $G$-torsors, then state results we will use from non-abelian cohomology theory, and finally go over the basics of Brauer groups for schemes.

## 1.1 Torsors

**Definition 1.1.1.** *[Vis07, Proposition 4.43] Let $S$ be a scheme and $G$ be a sheaf of groups on $S$. A $G$-torsor over $S$ is a sheaf $X$ on $S$ with a simply transitive $G$-action, such that for every $T$, there exists a covering $\{U_i \to T\}$ with $X(U_i) \neq \emptyset$. A morphism of $G$-torsors is a morphism of sheaves that is $G$-equivariant.*

**Example 1.1.2.** *Let $G$ be a group sheaf. Then $G$ acts on itself by left multiplication. This action is simply transitive, therefore, $G$ is a $G$-torsor. It is called the trivial $G$-torsor.*

**Example 1.1.3.** *[Vis07, Example 4.45] Let $G = \mathbb{Z}/2$ be the constant group scheme over $\operatorname{Spec}\mathbb{R}$ associated to the abstract group $\mathbb{Z}/2$. Then $G$ acts on $\operatorname{Spec}\mathbb{C}$ by conjugation. With this action $\operatorname{Spec}\mathbb{C}$ is a $G$-torsor. Observe that $\operatorname{Spec}\mathbb{C}$ is not isomorphic to $\mathbb{Z}/2$ because $\operatorname{Spec}\mathbb{C}$ is connected.*

If $X, Y$ are two $G$-torsors, we denote by $\underline{\operatorname{Hom}}_S^G(X, Y)$ the sheaf such that to any $U \to S$

associates the set of $G_U$-equivariant morphisms $X_U \to Y_U$, and $\underline{\mathrm{Isom}}_S^G(X,Y)$, the sheaf which associates the set of $G$-isomorphisms.

**Proposition 1.1.4.** *[Gir71, Théorème 1.4.5] Let $X$ and $Y$ be $G$-torsors over $S$, the following propositions hold:*

*(i) There is a natural isomorphism $X \to \underline{\mathrm{Hom}}_S^G(G,X)$.*

*(ii) The natural inclusion $\underline{\mathrm{Isom}}_S^G(X,Y) \to \underline{\mathrm{Hom}}_S^G(X,Y)$ is an isomorphism of sheaves over $S$.*

In particular, every morphism of $G$-torsors is an isomorphism, and a $G$-torsor $X$ is trivial if and only if $X(S) \neq \emptyset$.

**Definition 1.1.5.** *[Mil80, Page 134] Let $S$ be a scheme and $B$ a sheaf on $S$. A twisted form of $B$ is a sheaf $T$ on $S$ such that there exists a cover $S' \to S$ with $T_{S'} \simeq B_{S'}$.*

Let $\mathrm{Tors}_S(G)$ be the set of isomorphism classes of $G$-torsors over $S$ and $\mathrm{Twist}_S(B)$ be the set of isomorphism classes of twisted forms of $B$.

**Theorem 1.1.6.** *[Gir71, III Théorème 2.5.1] Let $S$ be a scheme, $B$ a sheaf on $S$ and $G := \underline{\mathrm{Aut}}_S(B)$. There is a bijective correspondence between $\mathrm{Tors}_S(G)$ and $\mathrm{Twist}_S(B)$ given by*

$$\begin{aligned} \mathrm{Tors}_S(G) &\longrightarrow \mathrm{Twist}_S(B) \\ P &\longmapsto (P \times_S B)/G \\ (T \to S) &\longleftarrow \underline{\mathrm{Isom}}_S(B,T) \end{aligned}$$

Recall that $\mathbb{G}_m$ is the group scheme that for any scheme $T$ associates the group $\mathscr{O}_T^\times(T)$.

**Corollary 1.1.7.** *There is a functorial bijection between $\mathrm{Tors}_S(\mathbb{G}_m)$, invertible sheaves on $S$ and line bundles on $S$.*

Recall that $\mathrm{GL}_n$ is the group scheme which to any scheme $T$ associates the group of invertible matrices with coefficients in $\mathscr{O}_T(T)$. The following is a general version of the previous corollary.

**Corollary 1.1.8.** *There is a functorial bijection between $\mathrm{GL}_n$-torsors over $S$, locally free sheaves of rank $n$ on $S$ and vector bundles of rank $n$ in $S$.*

**Definition 1.1.9.** *[Mil80, IV Proposition 2.1] Let $X$ be a scheme. An $\mathscr{O}_X$-algebra $A$ (possibly non-commutative) is called an Azumaya algebra over $X$ if it is of finite type as an $\mathscr{O}_X$-module and there is an étale covering $\{U_i \to X\}$ such that for all $i$, there is an $r_i$, for which $A_{U_i} \simeq M_{r_i}(\mathscr{O}_{U_i})$, the $\mathscr{O}_{U_i}$-algebra of $r_i \times r_i$ matrices over $\mathscr{O}_{U_i}$. The locally constant function $x \mapsto r_i^2$ is the rank of $A$.*

**Example 1.1.10.** *If $X = \operatorname{Spec} \mathbb{R}$, then $\mathbb{H}$ the Hamilton quaternion algebra over $\mathbb{R}$, is an Azumaya algebra of rank 4. Indeed, one can check that there is an isomorphism $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq M_2(\mathbb{C})$.*

**Definition 1.1.11.** *[Mil80, Page 134] Let $P \to X$ be a morphism of schemes, we say that $P \to X$ is a Brauer-Severi scheme of relative dimension $n$ if there is a cover $U \to X$ and an isomorphism $P_U \simeq \mathbb{P}^n_U$.*

**Example 1.1.12.** *If $X = \operatorname{Spec} \mathbb{R}$, then $\operatorname{Proj} \mathbb{R}[x, y, z]/(x^2+y^2+z^2) \to \operatorname{Spec} \mathbb{R}$ is a Brauer-Severi scheme of relative dimension one. Indeed, after base change to $\operatorname{Spec} \mathbb{C}$, we note that $\operatorname{Proj} \mathbb{C}[x, y, z]/(x^2 + y^2 + z^2) \simeq \mathbb{P}^1_{\mathbb{C}}$.*

Define $\mathrm{PGL}_n$ as the functor which associates to any scheme $T$ the group $\operatorname{Aut}(M_n(\mathscr{O}_T))$.

**Corollary 1.1.13.** *There is a natural bijection between $\mathrm{PGL}_n$-torsors over $S$, Azumaya algebras of rank $n^2$ over $S$ and Brauer-Severi schemes of relative dimension $n-1$ over $S$.*

## 1.2 Non Abelian cohomology

In this section, we introduce some of the basic theory of non-abelian cohomology developed by J. Giraud in [Gir71] (see also [Deb17] for a shorter introduction to the subject). To define $H^2(S, G)$ we need the notion of $G$-gerbes, in particular, we need the notion of stacks. We will discuss stacks (and gerbes) in Chapter 2, so the reader who is not familiar with these concepts can consult it before, as well as the references that appear there. We use the notation $H^i_g$ for non-abelian cohomology as Milne does in [Mil80, Chapter 4, Theorem 2.5].

**Definition 1.2.1.** *Let $S$ be a scheme and $G$ a group sheaf over $S$ (possible non-commutative). We define pointed sets $H^i(S, -)$ for $i = 0, 1, 2$ as follows;*

*(i) $H^0_g(S, G)$ is the group $G(S)$ with marked point the identity section in the group $G(S)$, [Gir71, Section III.2.4.1].*

(ii) $H^1_g(S,G)$ *is the pointed set* $\mathrm{Tors}_S(G)$ *with marked point* $G$*, the trivial* $G$*-torsor, [Gir71, Definition III.2.4.2].*

(iii) $H^2_g(S,G)$ *is the pointed set of* $G$*-gerbes over* $S$ *with marked point* $BG$*, the trivial* $G$*-gerbe (see 2.3.1 for the definition of* $G$*-gerbes when* $G$ *is commutative; for the general case, see [Gir71, IV.2.2]).*

**Definition 1.2.2.** *[Ols16, 12.1.7] Given a morphism of groups* $f : G \to H$ *and* $P$ *a* $G$*-torsor. We define an* $H$*-torsor, denoted by* $f_*P$ *(be aware that this is not the usual pushforward of sheaves) as follows;* $f_*P$ *is the quotient of* $H \times_S P$ *by* $G$ *where* $G$ *acts by* $g\cdot(h,p) = (hf(g)^{-1}, g\cdot p)$*. The action of* $H$ *on* $f_*P$ *is given on classes by* $h\cdot(h',p) = (hh',p)$*, and makes* $f_*P$ *into an* $H$*-torsor.*

**Proposition 1.2.3.** *[Gir71, III.2.4.2 and 2.4.2.1] Let* $S$ *be a scheme and* $f : G \to H$ *a morphism of groups over* $S$*. The induced morphisms of pointed sets* $G(S) \to H(S)$ *and* $\mathrm{Tors}_S(G) \to \mathrm{Tors}_S(H)$ *(see 1.2.2) turn* $H^0_g(S,-)$ *and* $H^1_g(S,-)$ *into functors from the category of group sheaves over* $S$ *to pointed sets.*

*Remark* 1.2.4. Given a morphism of groups $f : G \to H$ over $S$, it is not always possible to define an induced function $H^2_g(S,G) \to H^2_g(S,H)$. In other words, the theory of $H^2_g(S,-)$ of Giraud is not functorial. This was pointed out in [Gir71, Page 248]. However, if we make further assumptions on $f : G \to H$, then it is possible to give a well-defined function $H^2_g(S,G) \to H^2_g(S,H)$ that is functorial. This holds, for instance; if $G$ and $H$ are commutative groups [Ols16, Exercise 12.F], if $f$ identifies $G$ with a subgroup of the center of $H$, or if $f$ is an epimorphism [Gir71, Remarque 4.2.10].

**Proposition 1.2.5.** *[Gir71, Proposition 4.2.8] Let* $S$ *be a scheme and*

$$0 \to H \xrightarrow{\beta} G \xrightarrow{\alpha} K \to 0$$

*an extension of groups over* $S$*, such that* $\beta$ *identifies* $H$ *with a subgroup of the center of* $G$*. Then, There is an exact sequence of groups*

$$0 \to H^0_g(H) \to H^0_g(G) \to H^0_g(K) \to H^1_g(H) \to H^1_g(G) \to H^1_g(K) \to H^2_g(H).$$

**Proposition 1.2.6.** *Let* $S$ *be a scheme and* $G$ *a commutative group over* $S$*. There are natural isomorphisms of pointed sets* $H^i(S,G) \simeq H^i_g(S,G)$*. Moreover, the sequence of*

*Proposition 1.2.5 agrees with the long exact sequence of the derived functors $H^i(S,-)$ under these isomorphisms.*

*Proof.* See [Sta25, Tag 03AG] or [Ols16, Section 12.1] for the case $i=1$ and [Ols16, Section 12.2] for the case $i=2$. □

## 1.3 Brauer Groups

In this section, we review the basic theory of Brauer groups for schemes. Our main reference is [Mil80, Chapter 4], but the reader can also consult Grothendieck's lectures [Gro68a], [Gro95] and [Gro68b]. It may also be helpful to review the basics of the theory of Brauer groups for fields [CTS21, Chapter 1], [GS06, Chapter 1-4] or [Sta25, 073W].

**Definition 1.3.1.** *[Mil80, Page 141] Two Azumaya algebras $A, A'$ over $X$ (see Definition 1.1.9) are said to be Brauer equivalent, if there exist locally free $\mathscr{O}_X$-modules $E$ and $E'$ of finite rank such that*

$$A \otimes_{\mathscr{O}_X} \mathrm{End}_{\mathscr{O}_X}(E) \simeq A' \otimes_{\mathscr{O}_X} \mathrm{End}_{\mathscr{O}_X}(E').$$

This defines an equivalence relation on the set of Azumaya algebras over $X$, since $\mathrm{End}(E)\otimes \mathrm{End}(E') \simeq \mathrm{End}(E\otimes E')$. It is clear that the tensor product of two Azumaya algebras is again an Azumaya algebra, and moreover, this operation is compatible with the equivalence relation. Therefore, we can give the following definition.

**Definition 1.3.2.** *[Mil80, Page 141] Given a scheme $X$, the Azumaya-Brauer group of $X$, denoted $\mathrm{Br}(X)$, is the set of Brauer equivalent classes of Azumaya algebras over $X$ (Definition 1.3.1). This set is a group under tensor product, with identity given by the class of $\mathscr{O}_X$ and inverse $A^{-1}$ given by the opposite algebra of $A$.*

**Definition 1.3.3.** *Let $X$ be a scheme, the cohomological Brauer group of $X$ is the torsion subgroup of $H^2(X_{ét}, \mathbb{G}_m)$, we denote it by $\mathrm{Br}'(X)$.*

**Theorem 1.3.4.** *[Gir71, V. 4.4]. There is a canonical injective morphism*

$$\mathrm{Br}(X) \to H^2(X_{ét}, \mathbb{G}_m)$$

*functorial on $X$. We refer to this morphism as the Brauer map.*

*Remark* 1.3.5. The morphism $\mathrm{Br}(X) \to H^2(X_{\text{ét}}, \mathbb{G}_m)$ of 1.3.4 can be described by using non-abelian cohomology. Indeed, the Noether-Skolem Theorem [Mil80, Corollary 2.4] give us a central extension

$$0 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 0.$$

So given an Azumaya algebra $A$ of rank $n^2$, its image in $H^2(X, \mathbb{G}_m)$ is the same as the image of the associated $\mathrm{PGL}_n$-torsor (see 1.1.13) under the second boundary map in non abelian cohomology

$$\delta_n : H^1(X, \mathrm{PGL}_n) \to H^2(X, \mathbb{G}_m).$$

Thus, many authors define $\mathrm{Br}(X)$ as $\bigcup_n \mathrm{Im}(\delta_n) \subseteq H^2(X, \mathbb{G}_m)$, [Mat22, Definition 7].

**Corollary 1.3.6.** *[Mil80, Corollary 2.6] Let $X$ be a quasi-compact scheme, then there is a natural injective morphism* $\mathrm{Br}(X) \to \mathrm{Br}'(X)$.

In view of the previous Corollary, Grothendieck proposed the following problem.

**Problem 1.** *Let $X$ be a quasi-compact scheme. Is the Brauer map described in 1.3.4 surjective?*

At this point, considerable research has been done in this problem (see the introduction for a summary). A remarkable result in this direction is Gabber's theorem, which is stated below for the sake of completeness. For a detailed proof, see [dJ03]. In Chapter 3 we study this question when $X$ is an $A$-torsor, and $A$ is an abelian scheme.

**Theorem 1.3.7.** *(Gabber). Let $X$ be a scheme with an ample line bundle, then* $\mathrm{Br}(X) \to \mathrm{Br}'(X)$ *is surjective.*

# Chapter 2

# Quotient stacks and Gerbes

The central object of this thesis are gerbes. Gerbes are a particular class of stacks, so in this section, we introduce the language needed to define them. Throughout the chapter, we present several definitions and prove some theorems concerning quotient stacks. To make abstract notions more tangible, we include examples in every section. Stacks are essential tools in Chapter 3, so we hope this section will be especially helpful for readers unfamiliar with the theory, looking for a quick introduction. The chapter is primarily based on Olsson's book [Ols16] and sometimes we refer to [LMB00], we also encourage the reader to consult [EHKV01] and [KV04], Sections $2, 3$ and $4$ essentially offer a more detailed explanation of some of the theorems found in those works.

## 2.1 Algebraic Stacks

Stacks are technical objects and require a significant amount of preliminary material to define them. Although we attempt to introduce most of the necessary background, a full treatment of the theory would go far beyond the scope of this work.

Let $C$ be a category, a category over $C$ is a pair $(F, p)$, where $F$ is a category and $p : F \to C$ is a functor.

**Definition 2.1.1.** *Let* $(F, p)$ *be a a category over* $C$*;*

*(i) A morphism* $\phi : u \to v$ *in* $F$ *is called cartesian if for any other morphism* $\psi : w \to v$

*and a factorization*

$$p(w) \xrightarrow{h} p(u) \xrightarrow{p(\phi)} p(v)$$

*of* $p(\psi)$, *there exists a unique morphism* $\lambda : w \to u$ *such that* $\phi \circ \lambda = \psi$ *and* $p(\lambda) = h$. *If* $\phi : u \to v$ *is cartesian, then the object* $u$ *is called a pullback of* $v$ *along* $p(\phi)$ *[Ols16, Definition 3.1.1].*

(ii) *We say that* $(F, p)$ *is a category fibered in groupoids if; every morphism in* $F$ *is cartesian, and for every morphism* $\varphi : U \to V$ *in* $C$, *there exists at least one morphism* $\phi : x \to y$ *such that* $p(\phi) = \varphi$, *[Ols16, 3.4.1 and Exercise 3.D].*

(iii) *Given a category fibered in groupoids* $p : F \to C$ *and an object* $U \in C$, *we write* $F(U)$ *for the category whose objects are* $u \in F$ *such that* $p(u) = U$, *and their morphisms are* $f : u \to u'$ *in* $F$ *such that* $p(f) = \mathrm{id}_U$. *The category* $F(U)$ *is a groupoid (i.e. every morphism is an isomorphism), [Ols16, 3.4.1].*

(iv) *A morphism between categories fibered in groupoids is a functor* $\varphi : F \to G$ *such that* $p_F = p_G \circ \varphi$, *[Ols16, Definition 3.1.3 (ii)].*

**Example 2.1.2.** *[Ols16, Example 4.4.13] Let* $\underline{\mathrm{Sch}}$ *be the category of schemes and let* $\mathscr{M}_g$, $g \geq 2$, *be the category whose objects are pairs* $(S, X \to S)$ *where* $X, S$ *are schemes, and* $X \to S$ *is a proper, flat, finitely presented morphism such that all its geometric fibers are smooth curves of genus* $g$. *A morphism* $(S, X) \to (S', X')$ *in* $\mathscr{M}_g$ *is the data of a morphism of schemes* $f : S \to S'$, *and a morphism of schemes* $g : X \to X'$ *over* $S'$, *such that the induced morphism* $X \to X' \times_{S'} S$ *is an isomorphism.*

*Let* $p : \mathscr{M}_g \to \underline{\mathrm{Sch}}$ *be the functor given by* $p(S, X \to S) = S$ *and* $p(f, g) = f$, *then* $\mathscr{M}_g$ *is a category fibered in groupoids over* $\underline{\mathrm{Sch}}$. *Indeed, letting* $S \to T$ *be a morphism and* $X \in \mathscr{M}_g(T)$, *then* $X \times_T S \to S$ *is a proper, flat, finitely presented morphism, because these properties are preserved by base change, the geometric fibers of* $X \times_T S$ *are the same as* $X \to T$, *then* $X \times_T S \to X$ *is a morphism above* $S \to T$, *and it is cartesian by the universal property of fibered products.*

**Example 2.1.3.** *[LMB00, Exemple 3.4.1] Let* $F : C^{op} \to \mathrm{Sets}$ *be a functor. Let* $\mathscr{F}$ *be the category whose objects are pairs* $(U, x)$ *such that* $U \in C$ *and* $x \in F(U)$, *a morphism between* $f : (U, x) \to (V, y)$ *is a morphism* $f : U \to V$ *in* $C$ *such that* $F(f)(y) = x$. *Let* $p : \mathscr{F} \to C$ *be the projection, then* $\mathscr{F}$ *is a category fibered in groupoids over* $C$. *Indeed, let* $f : U \to V$ *be a morphism in* $C$ *and* $x \in F(V)$, *then we have a map* $f : (U, y) \to (V, x)$

*in $\mathscr{F}$ if and only if $y = F(f)(x)$ and this morphism is always cartesian. To see this, let $g : (W, z) \to (V, x)$ be a morphism in $\mathscr{F}$ and $h : W \to U$ a morphism in $C$ such that $f \circ h = g$. Observe that $z = F(g)(x)$, and $F(g) = F(h) \circ F(f)$, thus $z = F(f \circ h)(x)$, so we have a morphism $h : (W, z) \to (V, y)$ that is uniquely determined by $F(f \circ h)(x)$. This construction allows us to embed the category of presheaves over $C$ in the (2-)category of categories fibered in groupoids.*

In order to define stacks, we first need to introduce the concept of descent for a category over a site [Ols16, Section 4.2].

Let $C$ be a category with finite fiber products, and let $p : F \to C$ be a fibered category in groupoids. Choose for each morphism $f : X \to Y$ in $C$ compatible pull back functors; roughly speaking for every $\eta \in F(X)$ choose a pull back $f^*\eta \in F(Y)$, then there is a unique way of pulling back morphisms, so this defines functors;

$$f^* : F(Y) \to F(X)$$

(the fact that you can choose the functors $f^*$ in a compatible way, follows from the existence of cleavages, see [Vis07, Definition 3.9] for details). Now if $\{X_i \to Y\}_{i \in I}$ is a set of morphisms in $C$, define $F(\{X_i \to Y\})$ to be the category whose objects are collections of data $(\{E_i\}_{i \in I}, \{\sigma_{ij}\}_{i,j \in I})$, where $E_i \in F(X_i)$ and for each $i, j \in I$,

$$\sigma_{ij} : p_1^* E_i \to p_2^* E_j,$$

is an isomorphism in $F(X_i \times_Y X_j)$, such that for any three indices $i, j, k \in I$ the following diagram in $F(X_i \times_Y X_j \times_Y X_k)$ commutes:

$$\begin{array}{ccccc}
p_{12}^* p_1^* E_i & \xrightarrow{p_{12}^* \sigma_{ij}} & p_{12}^* p_2^* E_j & \xrightarrow{\simeq} & p_{23}^* p_1^* E_j \\
\downarrow \simeq & & & & \downarrow p_{23}^* \sigma_{jk} \\
p_{13}^* p_1^* E_i & \xrightarrow{p_{13}^* \sigma_{ik}} & p_{13}^* p_2^* E_k & \xrightarrow{\simeq} & p_{23}^* p_2^* E_k
\end{array}$$

We refer to the set of isomorphisms $\{\sigma_{ij}\}$ as descent data on the objects $\{E_i\}$. There is also a natural functor

$$\epsilon : F(Y) \to F(\{X_i \to Y\})$$

which sends $E$ to the collection $(\{f_i^*E\}, \{\sigma_{can}^{ij}\}$ where $\sigma_{can}$ is the canonical isomorphism of $p_1^*f_i^*E$ with $p_2^*f_j^*E$.

We say that a collection of morphisms $\{X_i \to Y\}$ satisfies effective descent if $\epsilon$ is an equivalence of categories.

We are primarily interested in the case where $C$ is the category $\underline{\mathrm{Sch}/S}$ of schemes over $S$ equipped with a Grothendieck topology such as the Zariski, étale, fppf or fpqc topology.

There are several well-known examples of effective descent; see [GW20, Appendix C].

**Example 2.1.4.** *[Har77, Exercise 1.22] Let $X$ be a scheme and $Op(X)$ be the category of open sets of $X$, let $F$ be the category of pairs $(U, \mathscr{F})$ where $U$ is an open set of $X$ and $\mathscr{F}$ is a sheaf on $U$, a morphism on $F$ between $(V, \mathscr{G})$ and $(U, \mathscr{F})$ is an inclusion $V \to U$ (if it exists) together with an isomorphism $\sigma : \mathscr{F}_V \to \mathscr{G}$. Then any set of morphisms $\{f_i : V_i \to U\}$ such that the $\{V_i\}$ form a cover of $U$ satisfies effective descent, this is just the property of gluing sheaves.*

Finally, we give the definition of stacks.

**Definition 2.1.5.** *[Ols16, Definition 4.6.1] Let $C$ be a site and $(F, p)$ be a category fibered in groupoids over $C$. We say that $F$ is a stack over $C$, if for every object $X \in C$ and every covering $\{X_i \to X\}$, the functor*

$$F(X) \to F(\{X_i \to X\})$$

*is an equivalence of categories.*

An important property of the category of stacks is the existence of finite fiber products. We present the construction below as an illustrative example. For a more detailed discussion of fiber products of stacks the reader may consult [Ols16, 3.4.9].

**Example 2.1.6.** *Let $p : F \to H, q : G \to H$ be morphisms of stacks over some category $C$. We define the fiber product $F \times_H G$ as follows: the objects of $F \times_H G$ are triples $(f, g, \phi)$ where $f \in F$, $g \in G$, and $\phi : p(f) \to q(g)$ is an isomorphism in $H$. A morphism between two objects $(f, g, \phi)$ and $(f', g', \phi')$ is a pair $(\psi, \theta)$ where $\psi : f \to f', \theta : g \to g'$ are* *morphisms in $F, G$ respectively making the following diagram commute*

$$\begin{array}{ccc} p(f) & \xrightarrow{p(\psi)} & p(f') \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} \\ q(g) & \xrightarrow{q(\theta)} & q(g'). \end{array}$$

*One can check that the groupoids $(F \times_H G)(U)$ are equivalent to the fiber product of groupoids $F(U) \times_{H(U)} G(U)$, therefore, it is a stack. Moreover, $F \times_H G$ fits into a 2-commutative diagram*

$$\begin{array}{ccc} F \times_H G & \longrightarrow & G \\ \downarrow & & \downarrow \\ F & \longrightarrow & H. \end{array}$$

*which satisfies the universal property of fibered products up to 2-commutativity.*

*Remark* 2.1.7. When we say 2-commutative we mean that two functors may not be equal, however, they are isomorphic as functors. When dealing with diagrams involving stacks, the diagram is often 2-commutative rather than commutative. However, we typically use the terms commutative and 2-commutative interchangeably.

**Example 2.1.8.** *[LMB00, Exemple 3.4.1] Recall from Example 2.1.3, that a presheaf $F$ defines a category fibered in groupoids $\mathscr{F}$. If $F$ is a sheaf, then $\mathscr{F}$ is a stack (actually if and only if, see Lemma 2.1.9). This follows because the gluing condition on the sheaf ensures that $\mathscr{F}(T) \to \mathscr{F}(\{T_i \to T\})$ is essentially surjective, and fully faithfulness follows from the definition of $\mathscr{F}$, where the only morphisms are the identity. As we can see, stacks compared to sheaves, carry the data of the automorphisms of their objects. However, there are stacks where the only automorphism are the identities, such as the stack associated to a sheaf. The next lemma shows that in fact, this is the only example.*

**Lemma 2.1.9.** *[LMB00, Proposition 2.4.1.1] Let $F$ be a stack over a scheme $S$, then the following conditions are equivalent.*

*(i)* *$F$ is isomorphic to a sheaf.*

*(ii)* *For all schemes $T \to S$, the fiber $F(T)$ is equivalent to a set.*

*(iii)* *$\Delta_{F/S} : F \to F \times_S F$ is a monomorphism.*

*Proof.* $(i) \Leftrightarrow (ii)$. From left to right follows from the definition of the stack associated to a sheaf. Conversely, let $G : (\underline{\mathrm{Sch}/S})^{op} \to \underline{\mathrm{Sets}}$ be the functor that associates $T \to S$, the set $F(T)$, and for $f : T \to T'$ choose a pullback $f^* : F(T') \to F(T)$. Since $F(T)$ is equivalent to a set, the pullbacks are unique. Hence, $F$ defines a presheaf, that is in fact a sheaf due to the stack condition. The sheaf $G$ is clearly isomorphic to $F$.

$(ii) \Leftrightarrow (iii)$. It suffices to prove that for all schemes $T$, the functor $\Delta : F(T) \to F(T) \times F(T)$ is fully faithful. It is clearly faithful. Let $a, b \in F(T)$, a morphism in $F(T) \times F(T)$ between $(a, a)$ and $(b, b)$ is the data of two isomorphisms $f_1 : a \to b$ and $f_2 : a \to b$, we need to show that $f_1 = f_2$. Observe that $f_1 \circ f_2^{-1} = id_b$ because $F(T)$ is discrete. Thus, we conclude that $f_1 = f_2$. Conversely, if $\Delta_{F/S}$ is a monomorphism, take $x \in F(T)$, we have a cartesian diagram (see Lemma 2.2.4)

$$\begin{array}{ccc} \underline{\mathrm{Aut}}_{F_T}(x) & \longrightarrow & T \\ \downarrow & & \downarrow{\scriptstyle (x,x)} \\ F & \longrightarrow & F \times_S F. \end{array}$$

Hence $\underline{\mathrm{Aut}}_{F_T}(x) \to T$ is a monomorphism, thus $\underline{\mathrm{Aut}}_{F_T}(x)$ is trivial. □

Now we define what it means for a morphism of stacks to be representable. It is convenient to deal with algebraic spaces, though we do not define them here. The reader may consult [Ols16, Chapter 5] or [LMB00, Chapitre 1].

**Definition 2.1.10.** *[LMB00, Definition 3.9] A morphism of stacks $F' \to F$ is representable if for any morphism from a scheme $B \to F$ the fiber product $B \times_F F'$ is represented by an algebraic space. The morphism is schematic if $B \times_F F'$ is represented by a scheme.*

**Definition 2.1.11.** *[LMB00, 3.10] Let $S$ be a scheme and $P(f)$ be a property of morphisms of algebraic spaces $f : X \to Y$ over $S$ that is stable under base change, and local for the étale topology in $Y$. A representable morphism $\varphi : F' \to F$ has property $P(\varphi)$ if; for all morphisms from a scheme $B \to F$, the induced morphism $\varphi_{F'} : B_{F'} \to B$ has property $P(\varphi_{F'})$.*

See [LMB00, 3.10] or [Sta25, Tag02KN] for examples of properties local in the étale topology.

The following definitions of algebraic stacks and Deligne-Mumford stacks are [Ols16, Definitions 8.1.4 and 8.3.1]. See [LMB00, Definition 4.1] and [Alp25, Definitions 3.1.4, 3.1.6] for a different approach.

**Definition 2.1.12.** *A stack $F$ over $\underline{\mathrm{Sch}/S}$ is an algebraic stack if;*

*(i) The diagonal morphism $F \to F \times_S F$ sending $a \mapsto (a, a, \mathrm{id}_{p(a)})$, is representable.*

*(ii) There exists a smooth surjective morphism $X \to F$ from a scheme.*

**Definition 2.1.13.** *A stack $F$ over $\mathrm{Sch}/S$ is Deligne-Mumford if;*

*(i) The diagonal morphism $F \to F \times_S F$ sending $a \mapsto (a, a, \mathrm{id}_{p(a)})$, is representable.*

*(ii) There exists an étale surjective morphism $X \to F$ from a scheme.*

**Example 2.1.14.** *The most basic example of algebraic stacks are algebraic spaces, such as schemes. In fact, they are Deligne-Mumford stacks. If we consider $\mathscr{M}_g$ as in Example 2.1.2, then $\mathscr{M}_g$ is a Deligne-Mumford stack. This was first proved by Deligne and Mumford in [DM69, Proposition 5.1], and it is because of this that stacks with étale covers receive that name.*

**Definition 2.1.15.** *[LMB00, 4.7] Let $S$ be a scheme and $P(X)$ a property of algebraic spaces over $S$ that is local for the smooth topology on $X$ (i.e. for any smooth surjective morphism $X' \to X$, $X$ has property $P$ if and only if $X'$ has it). An algebraic stack $F$ over $S$ has property $P(F)$ if there exists a smooth surjective morphism $X \to F$ such that $X$ satisfies $P(X)$.*

See [LMB00, 4.7] or [Sta25, Tag04YH] for properties local in the smooth topology.

*Remark* 2.1.16. Although we do not discuss the 2-Yoneda lemma in this thesis, it is helpful to keep it in mind; the reader may consult [Ols16, Section 3.2]. Two important consequences are; first, a morphism from a scheme to a stack, $X \to F$, corresponds to an object in the category $F(X)$. Second, given morphisms $f, g : X \to F$ corresponding to $x \in F(X)$ and $x' \in F(X)$ respectively, then, an isomorphism of the functors $f \to g$ corresponds to an isomorphism $x' \to x$ in $F(X)$. More generally, given a pair of morphisms $x : X \to F$ and $y : Y \to F$ corresponding to objects (that we denote by the same letter) $x \in F(X)$ and $y \in F(Y)$. Then a pair $(f, g)$, where $f : X \to Y$ is a morphism of schemes and $g : x \to y \circ f$ is an isomorphism of functors, corresponds to a choice of pullback $f^*y \in F(X)$ and an isomorphism $g : f^*y \to x$ in $F(X)$.

## 2.2 Quotient Stacks

In this section, we present the definition of quotient stacks, prove some of their properties, and provide some examples. Quotient stacks are among the main objects of study in this thesis. Among stacks, quotient stacks are particularly well-behaved, because their geometry corresponds to the equivariant geometry of a scheme or algebraic space.

**Definition 2.2.1.** *[Ols16, Example 8.1.12] Let $S$ be a scheme, $G$ a group scheme over $S$, and $X$ a scheme (or an algebraic space) over $S$ with an action of $G$. Let $[X/G]$ be the stack over $S$ whose objects are triples $(T, P, \pi)$ such that*

*(i) $T$ is a scheme over $S$.*

*(ii) $P$ is a $G_T$-torsor.*

*(iii) $\pi : P \to X_T$ is a $G_T$-equivariant morphism.*

*A morphism $(T, P, \pi) \to (T', P', \pi')$ is a pair $(f, g)$ where $f : T \to T'$ is a morphism over $S$ and $g : P \to P'_T$ is an isomorphism of $G_T$-torsors such that the diagram*

$$\begin{array}{ccc} P & \xrightarrow{\quad g \quad} & P'_T \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle f^*\pi'} & \\ & X_T. & \end{array}$$

*commutes. When $X = S$ and the action of $G$ on $S$ is trivial, the stack $[S/G]$ is called the classifying stack of $G$, and denoted $BG$, or $B_S G$ if we want to emphasize that we are working over $S$.*

*Remark* 2.2.2. Clearly $[X/G]$ is a category fibered in groupoids, since over $T$, the data of a morphism is $(\mathrm{id}, g)$, where $g$ is an isomorphism. Descent for sheaves and morphism of sheaves (see [Ols16, 4.2.11]) implies that $[X/G]$ is a stack. We will show below that if $G$ is smooth over $S$, then $[X/G]$ is an algebraic stack.

**Lemma 2.2.3.** *[Alp25, Exercise 2.4.38] Let $F$ be a stack over $S$. Let $X, Y$ be two schemes over $S$ and $f : X \to F$, $g : Y \to F$ be two morphism of stacks. There is a cartesian* *diagram*

$$\begin{array}{ccc} X \times_F Y & \longrightarrow & X \times_S Y \\ \downarrow & & \downarrow{\scriptstyle f\times g} \\ F & \xrightarrow{\Delta} & F \times_S F. \end{array}$$

*Proof.* Given $U$ a scheme over $S$, we have that $(X \times_F Y)(U)$ is the category whose objects are triples $(x : U \to X, y : U \to Y, \phi : f(x) \to g(y))$. A morphism between $(x, y, \phi)$ and $(x', y', \phi')$ is a pair $(p : U \to U, q : U \to U)$, such that $x' \circ p = x$, $y' \circ q = y$ and $g(q) \circ \phi = \phi' \circ f(p)$.

On the other hand, objects of $((X \times_S Y) \times_{F\times_S F} F)(U)$ are triples $((x : U \to X, y : U \to Y), u, \phi : (u, u) \to (f(x), g(y)))$. A morphism between $((x, y), u, \phi) \to ((x', y'), u', \phi')$ is a pair $((p : U \to U, q : U \to U), h)$, such that $x' \circ p = x$, $y' \circ q = y$, $h : u \to u'$ and $(f(p) \times g(q)) \circ \phi' = \phi \circ (h \times h)$.

It follows that the morphism given by sending $(x, y, \phi) \mapsto ((x, y), f(x), \mathrm{id} \times \phi)$ and $(p, q) \mapsto ((p, q), f(p))$ is an equivalence of categories. □

**Lemma 2.2.4.** *[Alp25, 2.4.39] Let $F$ be a stack over $S$, $T$ a scheme over $S$, and $x, y \in F(T)$. Consider the induced morphism $(x, y) : T \to F \times_S F$ given by Yoneda's lemma. Then, the following diagram*

$$\begin{array}{ccc} \underline{\mathrm{Isom}}_{F_T}(x, y) & \longrightarrow & T \\ \downarrow & & \downarrow{\scriptstyle (x,y)} \\ F & \xrightarrow{\Delta} & F \times_S F. \end{array}$$

*is cartesian.*

*Proof.* The proof is similar to Lemma 2.2.3. □

**Theorem 2.2.5.** *[Ols16, Example 8.1.12] Let $G$ be a smooth group over $S$, then $[X/G]$ is an algebraic stack.*

*Proof.* First, we show that the morphism $X \to [X/G]$ (induced by the trivial torsor $G_X$ and the equivariant morphism $\sigma : G_X \to X$ given by the action) is smooth and surjective.

From the definition of fiber products and the properties of torsors, one can check that for all schemes $T$ over $S$ and every object $(P, \phi) \in [X/G](T)$; there is a cartesian diagram

$$\begin{array}{ccc} P & \xrightarrow{\pi_X \circ \phi} & X \\ \downarrow & & \downarrow \\ T & \xrightarrow{(P,\phi)} & [X/G], \end{array}$$

where $\pi_X$ is the projection $X \times_S T \to X$. Since $G$-torsors are locally of the form $G \to S$, and this is a smooth surjective morphism, we see that $X \to [X/G]$ is also smooth and surjective.

To see that the diagonal is representable, we need to check that given two objects $p_1 = (P, \pi), p_2 = (P', \pi')$ of $[X/G](T)$ the sheaf $I = \underline{\mathrm{Isom}}_{F_T}(p_1, p_2)$ is representable by an algebraic space. It suffices to do this locally so we can assume that $P = P' = G_T$. In this case, for any $T' \to T$ the sheaf $I$ associates the set of elements $g \in G(T')$ such that the diagram

$$\begin{array}{ccc} G_{T'} & \xrightarrow{g} & G_{T'} \\ & {}_{\pi}\searrow \quad \swarrow_{\pi'} & \\ & X_T. & \end{array}$$

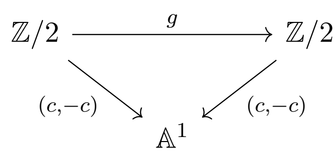
commutes. Since this is a diagram of $G$-equivariant morphisms, its commutativity is equivalent to the condition $\pi(e) = \pi'(g)$. Thus, $I$ is identified with the fiber product

$$\begin{array}{ccc} I & \longrightarrow & G_T \\ \downarrow & & \downarrow{\scriptstyle \pi(e)\times\pi'} \\ X_T & \xrightarrow{\Delta} & X_T \times_T X_T. \end{array}$$

It follows that $I$ is representable. In conclusion $[X/G]$ is an algebraic stack. □

*Remark* 2.2.6. In the previous proof, we use the smoothness of $G$ to show that the morphism $X \to [X/G]$ is a smooth cover. However, by [LMB00, Théorème 10.1], it suffices to require $G \to S$ faithfully flat and locally of finite presentation.

**Example 2.2.7.** *Consider* $X = \mathbb{A}^1_{\mathbb{C}}$ *and the group scheme* $G = \mathbb{Z}/2$ *over* $\operatorname{Spec}\mathbb{C}$*,* $G$ *acts on* $\mathbb{A}^1$ *via composition with the automorphism that sends* $x \mapsto -x$*. We study the* $\mathbb{C}$*-points of* $[X/G]$*. A* $\mathbb{C}$*-point is a* $G$*-torsor* $T \to \operatorname{Spec}\mathbb{C}$ *and a* $G$*-equivariant map* $T \to \mathbb{A}^1$*, since* *the only $G$-torsor over* $\operatorname{Spec}\mathbb{C}$ *up to isomorphism is $G$, a $\mathbb{C}$-point of $[X/G]$ is just a $G$-equivariant map $G \to \mathbb{A}^1$, this is the same as the orbits of the points $X(\mathbb{C})$ under the action, so $G \to \mathbb{A}^1$ is given by choosing the points (and the order) $(c, -c)$ if $c \neq 0$, or $\{0\}$ otherwise. The automorphism group of a $\mathbb{C}$-point $x$ consists of the morphisms of $G$-torsors $g : G \to G$ making the following diagram*




*commute. Then, if $c \neq 0$, we see that $\underline{\mathrm{Aut}}((c, -c))$ is trivial. If $c = 0$, then $\underline{\mathrm{Aut}}(x) = G$, this corresponds to the fact that $0$ is the only $\mathbb{C}$-point of $\mathbb{A}^1$ which has no non-trivial stabilizer. Therefore, $[\mathbb{A}^1_{\mathbb{C}}/\mathbb{Z}/2](\mathbb{C})$ has the same objects as $\mathbb{A}^1_{\mathbb{C}}(\mathbb{C})$, where we identify $c \in \mathbb{A}^1_{\mathbb{C}}(\mathbb{C})$ with $(c, -c)$. Observe that even when $(c, -c)$ and $(-c, c)$ are different $\mathbb{C}$-points ($c \neq 0$), there is an isomorphism between them in $[\mathbb{A}^1_{\mathbb{C}}/\mathbb{Z}/2](\mathbb{C})$ (in fact, if and only if). Therefore, the category $[\mathbb{A}^1_{\mathbb{C}}/\mathbb{Z}/2](\mathbb{C})$ recovers the points $\mathbb{A}^1_{\mathbb{C}}(\mathbb{C})$, but also the orbits and stabilizers of the action.*

**Example 2.2.8.** *The stack $\mathscr{M}_g$ (Example 2.1.14) of curves of genus $g \geq 2$ is of the form $[H_g/\mathrm{GL}_{5g-5}]$ where $H_g$ is a scheme (see the references below for a construction of $H_g$); this is how Deligne and Mumford prove the algebraicity of $\mathscr{M}_g$ [DM69, Proposition 5.1] (see also [Ols16, 8.4.3]). Moreover, the stack $\mathscr{M}_0$ of smooth curves of genus $0$ is the stack $B\mathrm{PGL}_2$, [Ols16, Remark 8.4.15], and the stack $\mathscr{M}_1$ can be identified with the stack $B\mathscr{E}$, where $\mathscr{E}$ is the universal elliptic curve over $\mathscr{M}_{1,1}$, though we do not go into the details of this construction.*

**Definition 2.2.9.** *[EHKV01, Definition 2.9] Let $S$ be a scheme and $F$ a stack over $S$. We say that $F$ is a quotient stack if $F \simeq [X/G]$ where $X$ is an algebraic space over $S$ and $G$ is a subgroup scheme of $\mathrm{GL}_{n,S}$ for some $n$.*

*Remark* 2.2.10. Given a scheme $S$, a group $G$ over $S$ and an $S$-scheme $X$ with an action of $G$. The morphism $X \to S$ being a $G$-torsor is stable under base change and a local property for the étale topology on $S$ [Alp25, Exercise 6.2.16]. Thus, we can speak of a morphism of stacks $F \to H$ with a $G_H$ action on $F$ being a $G$-torsor (see 2.1.11).

The following lemma tells us that stacks admitting a representable morphism to a quotient stack are also quotient stacks.

**Lemma 2.2.11.** *[Alp25, Definition 6.1.15] Let $F$ be an algebraic stack over a scheme $S$ and $G$ a fppf group scheme over $S$, the following are equivalent:*

*(i)* $F \simeq [X/G]$ *where* $X$ *is an algebraic space over* $S$.

*(ii) There is a* $G$*-torsor* $U \to F$ *where* $U$ *is an algebraic space over* $S$.

*(iii) There is a representable morphism* $F \to BG$.

*Proof.* $(i) \Rightarrow (iii)$ . Since the structure morphism $X \to S$ is $G$-equivariant, by taking quotients we get a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & S \\ \downarrow & & \downarrow \\ [X/G] & \longrightarrow & BG. \end{array}$$

We claim that the morphism $[X/G] \to BG$ is representable. For this we prove that the above diagram is cartesian. Indeed, if $R$ is the fiber product of the diagram, then we get a morphism $X \to R$. Since $S \to BG$ is a $G$-torsor, the morphism $R \to [X/G]$ is also a $G$-torsor. One can check that the morphism $X \to R$ is a morphism of $G$-torsors, in particular it is an isomorphism. Now, if $T \to BG$ is any morphism, by base changing the diagram along $T \to S$, we see that $[X/G] \times_{BG} T \simeq X_T$, this is an algebraic space so $[X/G] \to BG$ is representable. $(iii) \Rightarrow (ii)$. The morphism $S \to BG$ given by the trivial torsor $G$ over $S$ is a $G$-torsor (see Theorem 2.2.5). Therefore, $S \times_{BG} F \to F$ is also a $G$-torsor, moreover $S \times_{BG} F$ is an algebraic space since $F \to BG$ is representable.

$(ii) \Rightarrow (i)$. If $U \to F$ is a $G$-torsor, then $F \simeq [U/G]$ since $F$ is a categorical quotient for the action of $G$ on $U$. □

**Lemma 2.2.12.** *[Alp25, Exercise 3.4.19] Let $S$ be a scheme, $X$ an algebraic space over $S$, and $H \to G$ an immersion of groups over $S$ such that $H$ acts on $X$, then;*

*(i)* $H$ *acts freely on* $G \times_S X$ *via* $h \cdot (g, x) = (gh^{-1}, h \cdot x)$, *and we denote* $G \times^H X$ *the algebraic space* $(G \times_S X)/H$ *(be careful with the notation used in 1.2.2, we reserve the notation* $f_*$ *when* $X$ *is a torsor).*

(ii) *$G$ acts on $G \times^H X$ via $g \cdot (g', x) = (gg', x)$.*

(iii) *There is an isomorphism between $[X/H]$ and $[(G \times^H X)/G]$.*

*Proof.* (i) $H$ acts freely on $G$ because it is a subgroup, since the action on $G \times_S X$ is the diagonal action, $H$ also acts freely on $G \times_S X$.

(ii) $G$ acts on $G \times_S X$ by left multiplication on the left factor, this action respects the action of $H$. Therefore, it descends to an action on $G \times^H X$.

(iii) It is enough to show that the following diagram

$$\begin{array}{ccc} G \times_S X & \longrightarrow & X \\ \downarrow & & \downarrow \\ G \times^H X & \longrightarrow & [X/H]. \end{array}$$

is cartesian. Since if this were true, then the morphism $G \times^H X \to [X/H]$ is a $G$-torsor. Hence, $[(G \times^H X)/G] \simeq [X/H]$.

Now, let $X' := (G \times^H X) \times_{[X/H]} X$, the diagram above is commutative so there is a morphism $G \times_S X \to X'$ of $G$-torsors, therefore $X'$ and $G \times_S X$ are isomorphic.

□

By the above lemma, given a quotient stack $F = [X/G]$, we can always find an algebraic space $Z$ such that $F \simeq [Z/\mathrm{GL}_n]$.

*Remark* 2.2.13. Before continuing, we need to define the notions of relative Spec and relative Proj associated to a quasi-coherent sheaf on a stack. We prefer to skip the definition of quasi-coherent sheaves on a stack, the reader may consult [Ols16, Chapter 9], [LMB00, Chapitre 13] and [Sta25, Tag06TF]. What is important to us is that given a quasi-coherent sheaf $A$ on a stack $F$ and a morphism from a scheme $t : T \to F$, there is a well defined pull-back $t^*A$ that is a quasi-coherent sheaf over $T$. Furthermore, given two morphisms $t_1, t_2 : T \to F$, and an isomorphism of functors $f : t_1 \to t_2$, there is a well-defined morphism $t_1^*A \to t_2^*A$.

**Definition 2.2.14.** *[Ols16, 10.2.1] Let $F$ be an algebraic stack over $S$, and let $A$ be a quasi-coherent sheaf of algebras on $F$. Define the stack $\underline{\mathrm{Spec}}_F(A)$ as follows. The objects* *of $\underline{\mathrm{Spec}}_F(A)$ are triples $(T, x, \pi)$ where $T$ is a scheme over $S$, $x \in F(T)$ is an object (corresponding to a morphism $x : T \to F$, see 2.1.16), and $\pi : x^*A \to \mathscr{O}_T$ is a morphism of $\mathscr{O}_T$-algebras. A morphism $(T, x, \pi) \to (T', x', \pi')$ is a pair $(f, g)$ where $f : T \to T'$ is a morphism of schemes over $S$ and $g : x \to f^*x'$ is a morphism in $F(T)$ (corresponding to a natural isomorphism of functors between $f^*x'$ and $x$, see 2.1.16) such that the following diagram commutes*

$$\begin{array}{ccc} x^*A & \xrightarrow{g} & (f^*x')^*A \\ & \searrow^{\pi} \quad \swarrow_{f^*\pi'} & \\ & \mathscr{O}_T. & \end{array}$$

*If $A$ is a locally free sheaf of finite rank in $F$, we denote by $\mathbb{V}(A)$ the stack $\underline{\mathrm{Spec}}_F(A)$ and call it the vector bundle associated with $A$.*

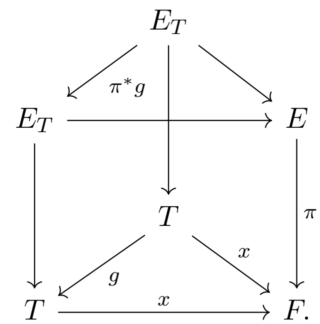
**Definition 2.2.15.** *[Ols16, 10.2.5] Let $F$ be an algebraic stack over $S$, let $A = \bigoplus_{d\geq 0} A_d$ be a quasi-coherent sheaf of graded algebras on $F$. Define the stack $\underline{\mathrm{Proj}}_F(A)$ as follows. The objects of $\underline{\mathrm{Proj}}_F(A)$ are triples $(T, x, \pi)$, where $T$ is a scheme over $S$, $x \in F(T)$ is an object (corresponding to a morphism $x : T \to F$, see 2.1.16) and $\pi : T \to \mathrm{Proj}_T(x^*A)$ is a section of the scheme $\mathrm{Proj}_T(x^*A) \to T$. A morphism $(T, x, \pi) \to (T', x', \pi')$ is a pair $(f, g)$, where $f : T \to T'$ is a morphism of schemes over $S$ and $g : x \to f^*x'$ is a morphism in $F(T)$ (corresponding to a natural isomorphism of functors between $x$ and $f^*x'$, see 2.1.16) such that the following diagram commutes*

$$\begin{array}{ccc} T & \xrightarrow{\mathrm{id}} & T \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle f^*\pi'} \\ \mathrm{Proj}_T(x^*A) & \xrightarrow{g} & \mathrm{Proj}_{T'}((f^*x')^*A). \end{array}$$

*Given a quasi-coherent sheaf $E$ on $F$ we define $\mathbb{P}_F(E)$ as the stack $\underline{\mathrm{Proj}}_F(\mathrm{Sym}E^\vee)$.*

Both constructions define algebraic stacks over $F$. For proofs of $\underline{\mathrm{Spec}}$ and $\underline{\mathrm{Proj}}$ being algebraic stacks and further properties, see [Ols16, Section 10.2], or [LMB00, Chapter 14].

*Remark* 2.2.16. Given a morphism of stacks $\pi : E \to F$ and a $T$-point $x \in F(T)$ (corresponding to a morphism $T \to F$, see 2.1.16) the stabilizer group $G = \underline{\mathrm{Aut}}_F(x)$ of $x$ acts on the fiber $E_T \to T$ by pulling back the action on $x$, we illustrate this in the following diagram




In the next two theorems we make use of these actions to deduce properties of quotient stacks.

**Theorem 2.2.17.** *[EHKV01, Lemma 2.12] Let $F$ be an algebraic stack of finite type over a Noetherian scheme $S$. The following are equivalent:*

*(i) $F$ is a quotient stack.*

*(ii) There exists a vector bundle $V \to F$ such that at every geometric point, the stabilizer action on the fiber is faithful.*

*(iii) There exists a vector bundle $V \to F$ and a locally closed substack $V^0 \subseteq V$ such that $V^0$ is representable and $V^0$ surjects onto $F$.*

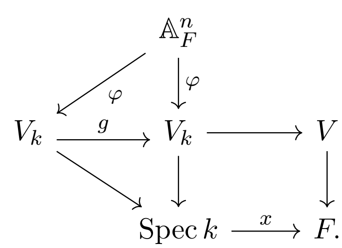
*Proof.* (*i*) $\Rightarrow$ (*iii*). Suppose that $F \simeq [X/\mathrm{GL}_n]$. Consider the representation of $\mathrm{GL}_n$ given by the inclusion $\mathrm{GL}_n \subseteq \mathbb{A}^{n^2} = M_n$. Then $\mathrm{GL}_n$ acts linearly on $\mathbb{A}^{n^2}$ and freely on the open subset $\mathrm{GL}_n$. Consider the diagonal action of $\mathrm{GL}_n$ on $X \times_S \mathrm{GL}_n$ and $X \times_S \mathbb{A}^{n^2}$. Then, the maps $X \times_S \mathrm{GL}_n \to X \times_S \mathbb{A}^{n^2} \to X$ are $\mathrm{GL}_n$-equivariant, and we can consider the following diagram where the squares are cartesian

$$\begin{array}{ccccc}
X \times_S \mathrm{GL}_n & \longrightarrow & X \times_S \mathbb{A}^{n^2} & & \\
\downarrow & & \downarrow & & \\
[X \times_S \mathrm{GL}_n/\mathrm{GL}_n] & \longrightarrow & [X \times_S \mathbb{A}^{n^2}/\mathrm{GL}_n] & \longleftarrow & X \times_S \mathbb{A}^{n^2} \\
 & & \downarrow & & \downarrow \\
 & & [X/\mathrm{GL}_n] & \longleftarrow & X.
\end{array}$$

Thus we see that $[X \times_S \mathbb{A}^{n^2}/\mathrm{GL}_n] \to [X/\mathrm{GL}_n]$ is a vector bundle and the morphism $[X \times_S \mathrm{GL}_n/\mathrm{GL}_n] \to [X \times_S \mathbb{A}^{n^2}/\mathrm{GL}_n]$ is an immersion. Moreover, $[X \times \mathrm{GL}_n/\mathrm{GL}_n]$ is an algebraic space because the action is free, and it surjects onto $F$. Hence $[X \times_S \mathbb{A}^{n^2}/\mathrm{GL}_n]$ is the desired vector bundle.

*(iii)* $\Rightarrow$ *(ii)*. Let $x : \operatorname{Spec} k \to F$ be a geometric point of $F$. We are going to prove that $V$ has the desired property. Observe that $V^0 \cap V_x \neq \emptyset$ because $V^0 \to F$ is surjective. We want to prove that the kernel of $\underline{\mathrm{Aut}}(x) \to \underline{\mathrm{Aut}}(V_x)$ is trivial. If this were not the case, and there is some $g \in \ker(\underline{\mathrm{Aut}}(x) \to \underline{\mathrm{Aut}}(V_x))$, then $g$ must be in $\underline{\mathrm{Aut}}(v)$, for all $v \in V_x$, in particular for $v \in V^0 \cap V_x$. However, $V^0$ is representable, therefore it has trivial stabilizers, so the action must be faithful.

*(ii)* $\Rightarrow$ *(i)*. Let $P = \underline{\mathrm{Isom}}_F(\mathbb{A}^n_F, V)$ be the stack over $F$ classifying isomorphisms between $\mathbb{A}^n_F$ and $V$. Since $V$ is a vector bundle, $P$ is an algebraic stack. Let $x : \operatorname{Spec} k \to F$ be a geometric point of $F$, $\varphi : \mathbb{A}^n_x \to V_x$ an isomorphism, and let $g$ be an automorphism of $\varphi$. Then $g$ is the data of an automorphism $G_x := \underline{\mathrm{Aut}}_F(x)$ which preserves the isomorphism $\varphi$. We can summarize this information in the following commutative diagram:




Since $\varphi$ is an isomorphism , we can see that $g$ must fix all fibers $V_x$. As $G_x$ acts faithfully on fibers $g$ must be the identity. Therefore, $P$ has trivial stabilizers, hence $P$ is an algebraic space (Lemma 2.1.9). Moreover, $P$ is a $\mathrm{GL}_n$-torsor, as a consequence $F \simeq [P/\mathrm{GL}_n]$. □

**Definition 2.2.18.** *[EHKV01, Page 6] A morphism of algebraic stacks $f : E \to F$ is said to be projective if it factors, up to isomorphism, as a closed immersion followed by the projection $E \to \mathbb{P}_F(\mathscr{E}) \to F$, where $\mathscr{E}$ is a finite-type, quasi-coherent sheaf on $F$.*

We saw in Lemma 2.2.11 that a stack admitting a representable morphism to a quotient stack, is itself a quotient stack. The following lemma may be viewed as a converse.

**Theorem 2.2.19.** *[EHKV01, Lemma 2.13] Let $\pi : E \to F$ be a flat projective map of* *stacks (of finite type over a Noetherian base scheme $S$) which is surjective. If $E$ is a quotient stack, then so is $F$.*

*Proof.* Since $E$ is a quotient stack, by Theorem 2.2.17, there is a locally free sheaf $\mathscr{E}$ on $E$ such that for every geometric point $x$ of $E$, the stabilizer action of $G_x := \underline{\mathrm{Aut}}_E(x)$ on the fibers is faithful. Because $\pi$ is projective, we can find a relatively very ample line bundle $\mathscr{O}(1)$ on $E$. Denote $\mathscr{L}(k) := \mathscr{L} \otimes \mathscr{O}(k)$. By Serre's theorem on projective morphisms ([GD61, Theorem 2.2.1]), we can find some $k$ such that $R^i\pi_*\mathscr{E}(k) = 0$ for $i > 0$ and $\pi^*\pi_*\mathscr{E} \to \mathscr{E}$ is surjective (it suffices to check this conditions after a smooth cover of $F$, so we are reduced to the case of schemes). Moreover, we can assume that these conditions hold after arbitrary base change. Therefore, by cohomology and base change (see, [Har77, Theorem 12.11]) $\pi_*\mathscr{E}(k)$ is a locally free sheaf on $F$.

Now we would like to prove that $\pi_*\mathscr{E}(k)$ has the property that the geometric stabilizers of $F$ act faithfully on the fibers. In order to do this, we will need that the geometric stabilizers of $E$ act faithfully on $\mathscr{E}(k)$. However, this may not be the case, so to solve this we may replace $\mathscr{E}$ by $\mathscr{E} \oplus \mathscr{O}_E$. Indeed, if the actions of $G_x$ on $\mathscr{E}$ and $\mathscr{O}(1)$ cancel each other in $\mathscr{E}(k)$, then we can just add an $\mathscr{O}(k)$ so that the action is still faithful. We explain this in detail below.

Let $BG_x \to E$ be the residual gerbe at a geometric point $x$ (see 2.3), and call $V, W$ the $G_x$-vector spaces associated with the restriction of $\mathscr{E}$ and $\mathscr{O}(k)$ on the fiber respectively (here we are using that the category of quasi-coherent sheaves on $BG$ is equivalent to the category of $G$-vector spaces, [Ols16, Exercise 9.H]). The $G_x$-vector space $W$ is one dimensional, so let $w$ be a generator, and suppose that the action of $G_x$ on $V \otimes W$ is not faithful. Then, there is a $g$ such that;

$$g \cdot (v \otimes w) = (\psi_g(v) \otimes g \cdot w) = (\psi_g(v) \otimes \lambda_g w) = v \otimes w$$

for all $v \in V,\ w \in W$, and $\lambda_g \in k(x)$. Rearranging, we get

$$(\lambda_g \psi_g(v) - v) \otimes w = 0$$

so $\lambda_g \circ \psi_g = \mathrm{id}_V$. If this is the case, let $L$ be the $G_x$-vector space associated to the fiber of

$\mathscr{O}_E$. Now, the action on $(\mathscr{E} \oplus \mathscr{O}_E)(k)$ looks like

$$g \cdot (v \otimes w, t \otimes w) = (v \otimes w, g \cdot t \otimes g \cdot w) = (v \otimes w, t \otimes \lambda_g w).$$

If every element is fixed, then $\lambda_g = 1$, but $\psi_g = \lambda_g^{-1}$ and $\psi_g$ acts faithful on $\mathscr{E}$, so this is a contradiction. As a result, either the action on $\mathscr{E}(k)_x$ or on $(\mathscr{E} \bigoplus \mathscr{O}_E)(k)_x$ must be faithful for all geometric points $\operatorname{Spec} L \to E$.

So far what we have is the following; we can suppose that there is a vector bundle $\mathscr{E}$ on $E$ such that $\pi_*\mathscr{E}$ is locally free on $F$, the geometric stabilizer action of $E$ on $\mathscr{E}$ is faithful and the canonical morphism $\pi^*\pi_*\mathscr{E} \to \mathscr{E}$ is surjective. Furthermore, this properties hold over all the fibers. Now we show that the action of geometric stabilizers on $\pi_*\mathscr{E}$ is faithful. Let $\operatorname{Spec} k \xrightarrow{x} F$ be a geometric point, let $H_x$ be the sheaf of automorphism of $x$ and consider the following diagram

$$\begin{array}{ccc}
\mathbb{V}(\mathscr{E}_x) & \longrightarrow & \mathbb{V}(\mathscr{E}) \\
\downarrow & & \downarrow \\
E_x & \longrightarrow & E \\
\downarrow & & \downarrow \\
\operatorname{Spec} k & \xrightarrow{x} & F
\end{array}$$

The action of $H_x$ on $\mathbb{V}(\mathscr{E}_x)$ is faithful since it factors through the action of $H_x$ on $E_x$ but we already know that the action of any geometric point of $\operatorname{Spec} L \to E_x$ is faithful. Since we have a surjective morphism $\pi^*\pi_*\mathscr{E}_x \to \mathscr{E}_x$ and $H_x$ acts faithful on $\mathscr{E}_x$, the same holds for $\pi^*\pi_*\mathscr{E}_x$. However, the action of $H_x$ on $\pi^*\pi_*\mathscr{E}_x$ is induced by the action of $H_x$ on $\pi_*\mathscr{E}_x$. Therefore, the action of $H_x$ on $\pi_*\mathscr{E}_x$ is also faithful, hence Theorem 2.2.17 implies that $F$ is a quotient stack. □

## 2.3 Gerbes

Throughout this section, we present a second class of stacks called *gerbes*. More precisely, we focus on $G$-gerbes, where $G$ is a commutative group scheme. There are several approaches to the theory of gerbes. Gerbes are ubiquitous objects in algebraic geometry and appear naturally in the theory of moduli. For instance, they measure the obstructions of certain natural geometric constructions (e.g., see [JLM24, Cor. 3.24, Prop. 3.26] for examples relating to cyclic cover, and also [BBGN07] relating to universal families). In this section, we follow the treatment as in [Ols16, Chapter 12]. This approach has the advantage of offering a simple and workable definition, although it requires us to restrict to commutative group schemes $G$. For other approaches, see [EHKV01, Section 3], or [Gir71, Section III.3].

**Definition 2.3.1.** *[Ols16, Definition 12.2.2] Let $S$ be a scheme and $G$ a commutative group over $S$. A $G$-gerbe over $S$ is a stack $F$ together with an isomorphism of sheaves of groups*

$$i_x : G_T \to \underline{\mathrm{Aut}}_F(x)$$

*for every object $x \in F(T)$ and every scheme $T \to S$, such that the following conditions hold:*

*(i) For any scheme $T \to S$ there exists a covering $\{T_i \to T\}$ such that $F(T_i)$ is nonempty for every $i$.*

*(ii) For any two objects $x, x' \in F(T)$ there exists a cover $\{\pi_i : T_i \to T\}$ such that $\pi_i^* x$ and $\pi_i^* x'$ are isomorphic in $F(T_i)$ for every $i$.*

*(iii) For every isomorphism $\sigma : x \to x'$ in $F(T)$ the following diagram commutes*

$$\begin{array}{ccc} & G_T & \\ {}^{i_x}\swarrow & & \searrow^{i_{x'}} \\ \underline{\mathrm{Aut}}(x) & \xrightarrow{\quad\sigma\quad} & \underline{\mathrm{Aut}}(x') \end{array}$$

*A morphism of $G$-gerbes $(F, i) \to (H, j)$ is a morphism of stacks $f : F \to H$ such that for every object $x \in F$ the diagram*

$$\begin{array}{ccc} & G & \\ {}^{i_x}\swarrow & & \searrow^{j_{f(x)}} \\ \underline{\mathrm{Aut}}(x) & \xrightarrow{\quad f_*\quad} & \underline{\mathrm{Aut}}(f(x)) \end{array}$$

*commutes.*

**Example 2.3.2.** *The classifying stack $BG$ is a $G$-gerbe, called the trivial gerbe (below we* *are going to see that every gerbe is locally of this form). Let us see that $BG$ satisfies all the required conditions for being a $G$-gerbe. Given a $G$-torsor $P \to S \in BG(S)$ the group $\underline{\mathrm{Aut}}(P/S)$ is also a $G$-torsor under the action of $G$ over $P$ (here we are using that $G$ is commutative), thus $\underline{\mathrm{Aut}}(P/S) \simeq G$. Given any scheme $T \to S$ there exists the trivial torsor $G_T \to T$, thus $BG(T) \neq \emptyset$, this shows that $BG$ satisfies the first condition. The second condition follows from the fact that every $G$-torsor is locally isomorphic to $G$. Finally given two torsors $P \to T$ and $Q \to T$ and a $G$-equivariant isomorphism $f : P \to Q$ we need to check the commutativity of the diagram*

$$\begin{array}{ccc} & G_T & \\ {\scriptstyle i_P}\swarrow & & \searrow{\scriptstyle i_Q} \\ \underline{\mathrm{Aut}}(P) & \xrightarrow{\quad f \quad} & \underline{\mathrm{Aut}}(Q). \end{array}$$

*From left to right the composition is $f(g \cdot (f^{-1}(-))) : Q \to Q$, since $f$ is $G$-equivariant, we have $f(g \cdot (f^{-1}(-))) = g \cdot f(f^{-1}(-)) = g \cdot (-)$, so the last condition is verified.*

*Remark* 2.3.3. The first two conditions in the definition of gerbes, as well as the commutativity of $G$, imply that for every two objects $x, x' \in F(T)$, the sheaf $\underline{\mathrm{Isom}}_F(x, x')$ is a $G$-torsor, under the action given by the isomorphism $i_x$ and the natural action of $\underline{\mathrm{Aut}}_F(x)$. Indeed, $\underline{\mathrm{Aut}}(x)$ acts simply transitively on $\underline{\mathrm{Isom}}_F(x, x')$ via composition, and for some cover $\{T_i \to T\}$, the sets $\underline{\mathrm{Isom}}_{F(T_i)}(x, x')$ are non-empty for all $i$.

**Lemma 2.3.4.** *[Ols16, Lemma 12.2.4] Any morphism of $G$-gerbes is an isomorphism.*

*Proof.* Let $\varphi : F \to F'$ be a morphism of $G$-gerbes, we need to show that for every scheme $T \to S$ the morphism $\varphi(T) : F(T) \to F'(T)$ is an equivalence of categories. Let us see that $\varphi(T)$ is essentially surjective. Given $y \in F'(T)$, due to the first two conditions of a gerbe, there is a cover of $T' \to T$ such that the essential image of $F'(T')$ is non-empty, say $y' = F(x')$ for some $x' \in F(T')$ and $y'$ is isomorphic to $y_{T'}$. Hence, locally on $T$, $y$ is in the essential image of $\varphi$. This is sufficient to check essential surjectivity since the stack conditions allow us to descend objects. To check that $\varphi$ is fully faithful, given $x, x' \in F(T)$, observe that the induced morphism;

$$\underline{\mathrm{Isom}}_F(x, x') \to \underline{\mathrm{Isom}}_{F'}(\varphi(x), \varphi(x'))$$

is a morphism of $G$-torsors, therefore an isomorphism, so the lemma follows. $\square$

**Proposition 2.3.5.** *[Sta25, Tag 05QH] Let $S$ be a scheme, $G$ a commutative group scheme over $S$, and $\mathscr{G}$ a $G$-gerbe over $S$. There exists a cover $T \to S$, such that $\mathscr{G}_T \simeq BG_T$. Moreover, $\mathscr{G} \simeq BG$ if and only if $\mathscr{G}(S) \neq \emptyset$.*

*Proof.* Let $T \to S$ be a cover such that $\mathscr{G}(T) \neq \emptyset$ and fix $x \in \mathscr{G}(T)$. Define $\varphi : \mathscr{G}_T \to BG$ by $\varphi(U \to T, u \in \mathscr{G}(U)) = (\underline{\mathrm{Isom}}_{\mathscr{G}}(u, x_U) \to U)$, and observe it is a morphism of stacks. Moreover, the diagram

$$\begin{array}{ccc} & G & \\ {\scriptstyle i_u}\swarrow & & \searrow{\scriptstyle i_{\varphi(u)}} \\ \underline{\mathrm{Aut}}(u) & \xrightarrow{\varphi_*} & \underline{\mathrm{Aut}}(\varphi(u)) \end{array}$$

is commutative by definition of $i_{\varphi(u)}$, thus $\varphi$ is a morphism of $G$-gerbes. By the above lemma we conclude that $BG_T \simeq \mathscr{G}_T$. This proof is basically the same as for $G$-torsors. $\square$

**Example 2.3.6.** *[Alp25, Exercise 6.4.21] Since gerbes are essentially twisted forms of $BG$, one natural way to construct examples is by gluing them over a cover to obtain a new one.*

*Suppose that we have an open cover $S_1, S_2 \subseteq S$, and an isomorphism of $G$-gerbes*

$$\varphi_{12} : B_{S_{12}}G \to B_{S_{21}}G$$

*($S_{12} = S_1 \cap S_2 = S_{21}$) compatible with the canonical isomorphism $S_{12} \to S_{21}$. We can define a fibered category $\mathscr{G} \to S$ in the following way: an element $a \in \mathscr{G}$ is a tuple $(T \to S, x_1, x_2)$ where $T \to S$ is a morphism of schemes, and $x_i \in BG(T_{S_i})$ are such that $\varphi_{12}(x_1) = x_2$. In the same way, a morphism $a \to a'$ is a pair of morphisms $x_1 \to x_1'$, $x_2 \to x_2'$ compatible with $\varphi$. The fibered category $\mathscr{G}$ is a stack, and in fact is a $G$-gerbe. See [Gir71, II Section 2.1.5] and [dGF25] for a more general construction.*

*Let $k$ be an algebraically closed field, $S = \mathbb{P}^1_k$ and $U, V \simeq \mathbb{A}^1_k$ be the standard cover glued along $W = \mathbb{A}^1_k - \{0\}$. Let $G = \mathbb{Z}/2$, we are going to glue two copies of $BG \times \mathbb{A}^1_k$ along $BG \times W$. The morphism $f : W \to W$ defined by $x \mapsto x^2$, gives $W$ the structure of a $\mathbb{Z}/2$-torsor over $W$, where $\mathbb{Z}/2$ acts by $x \mapsto -x$. Consider the involution $i : BG \times W \to BG \times W$*

*given by*

$$(g : T \to W, T \times G \to T) \mapsto (g : T \to W, g^*(f : W \to W)),$$

*it suffices to specify the morphism on the trivial torsor, since* $BG$ *is the stackification of the prestack whose only object is the trivial torsor, see [Alp25, Exercise 2.5.21, Theorem 2.5.18]). Let* $\mathscr{G} \to \mathbb{P}^1_k$ *be the* $\mathbb{Z}/2$*-gerbe obtained by the gluing data* $i$*. For* $\mathscr{G}$ *to be trivial is equivalent to* $\mathscr{G}(\mathbb{P}^1) \neq \emptyset$*, however, an element in* $\mathscr{G}(\mathbb{P}^1)$ *is a pair* $(P \to \mathbb{A}^1, Q \to \mathbb{A}^1)$ *of* $\mathbb{Z}/2$*-torsors, which are isomorphic over* $W$ *after applying* $i$*. However, a* $\mathbb{Z}/2$*-torsor over* $\mathbb{A}^1_k$ *is just a line bundle of order 2 (with a choice of trivialization). Since* $\mathrm{Pic}\,(\mathbb{A}^1) = 0$*, we see that* $P$ *and* $Q$ *are trivial. However,* $i(\mathbb{Z}/2)$ *is not trivial, thus* $\mathscr{G}(\mathbb{P}^1) = \emptyset$*, so* $\mathscr{G}$ *is non-trivial.*

*The gerbe described above is also isomorphic to the root stack of* $\mathscr{O}(1)$ *over* $\mathbb{P}^1$*, see [Ols16, Section 10.3] for a different approach.*

Next, we present one of the motivating problems of this thesis.

**Problem 2.** *Let* $S$ *be a scheme and* $G$ *a commutative, finite, locally free group scheme over* $S$*. Is any* $G$*-gerbe a quotient stack?*

This is a very general question, so you can try to impose certain conditions on $S$, such as affine or projective, as well as restrict to certain finite, locally free groups. In Chapter 3, using the work of Gabber [dJ03], Kresch-Vistoli [KV04], Edidin-Hassett-Kresch-Vistoli [EHKV01], and Bragg-Hall-Mathur [BHM25], we give some answers to this question when $S$ is a quasi-projective scheme over a field.

The next theorem plays a key role in relating Problems 1 and 2.

**Theorem 2.3.7.** *[EHKV01, Theorem 3.6]. Let* $X$ *be a Noetherian scheme. Let* $\beta$ *be an element of* $H^2(X, \mathbb{G}_m)$*. The following are equivalent:*

*(i)* $\beta$ *lies in the image of the Brauer map.*

*(ii) There exists a surjective, flat, projective morphism of schemes* $\pi : Y \to X$*, such that* $\pi^*\beta = 0$ *in* $H^2(Y, \mathbb{G}_m)$*.*

*(iii) The* $\mathbb{G}_m$*-gerbe associated with* $\beta$ *is a quotient stack.*

*Furthermore, if* $n\beta = 0$ *and* $\alpha \in H^2(X, \mu_n)$ *is a preimage of* $\beta$ *under the morphism* $H^2(X, \mu_n) \to H^2(X, \mathbb{G}_m)[n]$ *(given by the Kummer sequence), then conditions* $(i), (ii)$

*and (iii) are equivalent to*

*(iv) The $\mu_n$-gerbe associated with $\alpha$ is a quotient stack.*

*Proof.* $(i) \Rightarrow (ii)$. Let $\gamma \in H^1(X, \mathrm{PGL}_n)$ be a preimage of $\beta$ and $\pi : P \to X$ be the Brauer-Severi scheme associated with $\gamma$. By descent, $\pi$ is a surjective, flat, projective morphism, the base change of $P \to X$ along itself has a section given by the diagonal morphism. It is well known that a Brauer-Severi scheme with a section must be Brauer trivial [Poo17, Proposition 4.5.10], that is, $\pi^*\beta = 0$.

$(ii) \Rightarrow (iii)$. If we let $\mathscr{G}$ be a $\mathbb{G}_m$-gerbe represented by $\beta$, then $(ii)$, implies that the diagram

$$\begin{array}{ccc} B\mathbb{G}_m & \xrightarrow{\varphi} & \mathscr{G} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\pi} & X \end{array}$$

is cartesian, so $\varphi$ is also a surjective, flat, projective morphism. Therefore, by Theorem 2.2.19, $\mathscr{G}$ is also a quotient stack.

$(iii) \Rightarrow (i)$. Again, let $f : \mathscr{G} \to X$, be the $\mathbb{G}_m$-gerbe associated with $\beta$. Due to Theorem 2.2.17, there exists a vector bundle $V$ on $\mathscr{G}$, with faithful stabilizer action on fibers. Moreover, this vector bundle comes with an action of $\mathbb{G}_m$, which induces a decomposition of $V$ into eigenbundles indexed by characters in $\mathbb{G}_m^\vee(X) := \mathrm{Hom}_{gp}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$ (see [Lie08, Proposition 3.1.1.4]). Since the action of $\mathbb{G}_m$ on the stabilizers is faithful, the elements of $\mathbb{G}_m^\vee(X)$ whose eigenbundles are nonzero, must generate $\mathbb{G}_m^\vee(X)$. Therefore, there are integers $r, s$, such that the decomposition of $(V)^{\otimes r} \times (V^\vee)^{\otimes s}$ into eigenbundles has non-zero $V_1$. Then, $A = f_*\mathrm{End}(V_1)$ is an Azumaya algebra on $X$, such that $\mathscr{G}_A \simeq \mathscr{G}$, [Ols16, Proposition 12.3.11].

$(iii) \Leftrightarrow (iv)$. Let $\mathscr{G}$ be the $\mu_n$-gerbe associated with $\alpha$, and $\mathscr{G}'$ the $\mathbb{G}_m$-gerbe associated with $\beta$. Then, there is a representable morphism $\varphi : \mathscr{G} \to \mathscr{G}'$, which is $\mu_n \to \mathbb{G}_m$ invariant [Ols16, Exercise 12.F]. Theorem 2.2.11 implies that if $\mathscr{G}'$ is a quotient stack, then $\mathscr{G}$ is also a quotient stack. In the other direction, let $V$ be a locally free coherent sheaf on $\mathscr{G}$ such that the stabilizer action on fibers is faithful. We will show that $\varphi_*V$ contains a finite rank vector bundle, with faithful action of $\mathbb{G}_m$. It suffices to do this étale locally, so we can assume $X$ is the spectrum of a strictly local ring, and $\mathscr{G} \simeq B\mu_n$, $\mathscr{G}' \simeq B\mathbb{G}_m$. Decompose $\varphi_*V$ as $\bigoplus_{l\in\mathbb{Z}}(\varphi_*V)_l$, we claim that all $(\varphi_*V)_l$ are locally free, here we follow the argument of [Mat22, Proposition 10] .Consider the line bundle $\mathscr{L}' \in \mathrm{Pic}\,(B\mu_n)$ such that $W = \mathscr{L}'^{\oplus m}$, there is a line bundle $\mathscr{L} \in \mathrm{Pic}\,(B\mathbb{G}_m)$ such that $\varphi^*\mathscr{L} \simeq \mathscr{L}'$. It suffices to show that $\mathscr{L}^{\otimes l} \otimes (\varphi_* V)_l^\vee$ is locally free. If $\pi : B\mathbb{G}_m \to X$ is the structure morphism, then

$$\mathscr{L}^{\otimes l} \otimes (\varphi_* V)_l^\vee \simeq \pi_*(\mathscr{L}^{\otimes l} \otimes (\varphi_* V)^\vee) = \mathrm{Hom}(\mathscr{L}^{\otimes l}, \varphi_* V)$$

by adjunction we see that $\mathscr{L}^{\otimes l} \otimes (\varphi_* V)_l^\vee \simeq \mathrm{Hom}(\mathscr{L}'^{\otimes l}, \mathscr{L}^{\oplus m})$, which is a free module over $X$. Therefore, $(\varphi_* V)_l$ is locally free for all $l$. Now, observe that the $\mu_n$ action on $V$ is the adjoint action of $\mathbb{G}_m$ on $\varphi_* V$ (see [Lie08, Lemma 3.1.1.5]), so if $V_l \neq 0$ then $(\varphi_* V)_l \neq 0$ . Since the action of $\mu_n$ is faithful, the characters $l \in \mathbb{Z}/n \simeq \mathrm{Hom}(\mu_n, \mathbb{G}_m)$ for which $V_l \neq 0$ generate $\mathbb{Z}/n$. Hence, the same holds for the $l \in \mathbb{Z}$ such that $(\varphi_* V)_l \neq 0$. Let $S$ be a finite set of integers such that $(\varphi_* V)_l \neq 0$ and $S$ generates $\mathbb{Z}$, then $\bigoplus_{s\in S}(\varphi_* V)_s$ is a vector bundle on $\mathscr{G}'$ with faithful stabilizer action. □

The next example serves as a counterexample for both Problems 1 and 2. See [Ber05] for a different approach using cohomological tools.

**Example 2.3.8.** *[EHKV01, Example 2.21] Consider the scheme $X$ obtained as two copies of $Y = \mathbb{C}[x, y, z]/(xy-z^2)$ glued along the open subscheme $Y^{reg} = Y - \{0\}$, so $X$ is the cone with double vertex. First, we describe some properties of $X$. Due to [MP73, Proposition 3.2] the scheme $Y$ has no non-trivial vector bundles, since $Y$ is normal, and the glueing is over a locus whose complement has codimension 2, the scheme $X$ also has no nontrivial vector bundles [Sta25, Tag 0EBJ]. We will construct a $\mu_2$-gerbe on $X$, with no non-trivial vector bundles; thus by Theorems 2.2.17 and 2.3.7 we conclude that $\mathrm{Br}(X) \neq \mathrm{Br}'(X)$. To construct this gerbe we follow a similar process as in Example 2.3.6. Let $L \in \mathrm{Pic}\,(Y^{reg})$ be the unique two torsion line bundle and $\mathscr{L} \in \mathrm{Pic}\,(B\mu_2 \times Y^{reg})$ the pull back of $L$. Then $\mathscr{L}$ with a choice of trivialization defines an involution*

$$i : B\mu_2 \times Y^{reg} \to B\mu_2 \times Y^{reg}.$$

*Let $\mathscr{G} \to X$ be the $\mu_2$-gerbe obtained by glueing two copies of $B\mu_2 \times Y$ along $i$. We show that $\mathscr{G}$ can not have a vector bundle with faithful stabilizer action. Let $V$ be a vector bundle on $\mathscr{G}$, then $V$ has a decomposition given by the characters of $\mu_2$ (see [Lie08, Proposition 3.1.1.4]), i.e. $V = V(1) \bigoplus V(-1)$ for some vector bundles, where the action on $V(1)$ is trivial, and the action on $V(-1)$ is given by multiplication by $-1$. Thus, it suffices to show* *that if $V$ is a vector bundle on $\mathscr{G}$, such that the action of $\mu_2$ on $V$ is multiplication by $-1$, then $V=0$. Let $V_i$ denote the restriction of $V$ to the $i$-th copy of $B\mu_2\times Y$. By the construction of $\mathscr{G}$ we have an isomorphism*

$$V_1|_{B\mu_2\times Y^{reg}}\simeq i^*(V_2|_{B\mu_2\times Y^{reg}}).$$

*From the definition of $i$ we deduce that $i^*(V_2|_{B\mu_2\times Y^{reg}})\simeq \mathscr{L}\otimes V_2|_{B\mu_2\times Y^{reg}}$. Therefore, we have an isomorphism*

$$V_1|_{B\mu_2\times Y^{reg}}\simeq \mathscr{L}\otimes V_2|_{B\mu_2\times Y^{reg}}.$$

*If $Y^{reg}\to B\mu_2\times Y^{reg}$ is the section given by $L$, then we get an isomorphism*

$$V_1|_{Y^{reg}}\simeq L\otimes V_2|_{Y^{reg}}.$$

*However, $V_Y$ is free of some rank, therefore $L^{\oplus m}\simeq \mathscr{O}^{\oplus m}$, but this is impossible. Indeed, we have a cartesian diagram given by the action of $\mu_2$ on $\mathbb{A}^2$ $((x,y)\mapsto(-x,-y))$;*

$$\begin{array}{ccc}\mathbb{A}^2-\{0\} & \longrightarrow & \mathbb{A}^2\\ \downarrow & & \downarrow\\ Y^{reg} & \longrightarrow & [\mathbb{A}^2/\mu_2].\end{array}$$

*Hence $Y^{reg}$ is an open substack inside $[\mathbb{A}^2/\mu_2]$ with complement of codimension 2, thus, the isomorphism $L^{\oplus m}\simeq\mathscr{O}^{\oplus m}$ extends (see [Sta25, 0EBJ]) to an isomorphism over $[\mathbb{A}^2/\mu_2]$ between a free sheaf and a non-trivial locally free sheaf.*

*In conclusion, the scheme $X$ has $\mu_2$-gerbe that is not a quotient stack, so $\mathrm{Br}(X)\neq\mathrm{Br}'(X)$.*

We conclude this section with a brief discussion of residual gerbes providing their properties. Our exposition follows [BL24, Appendix A], see also [LMB00, Chapter 11], and [Sta25, Tag06ML].

Let $F$ be a stack, we can define the set of topological points of $F$ denoted by $|F|$ as follows: a point $x\in|F|$ is an equivalence class of pairs $(k,\operatorname{Spec}k\to F)$, where $k$ is a field, $\operatorname{Spec}k\to F$ is a morphism, and two pairs $(k,\operatorname{Spec}k\to F)$, $(k',\operatorname{Spec}k'\to F)$ are declared to be equal, if there is a field $L$ containing $k, k'$, such that the following diagram

$$\begin{array}{ccccc} & & \operatorname{Spec} k & & \\ & \nearrow & & \searrow & \\ \operatorname{Spec} L & & & & F \\ & \searrow & & \nearrow & \\ & & \operatorname{Spec} k' & & \end{array}$$

commutes.

Associated to a point $x \in |F|$, there exists a residue field $k(x)$, and a residual gerbe $\mathscr{G}(x)$, which is a gerbe over $k(x)$. These two objects fit into the following diagram:

$$\begin{array}{ccc} \mathscr{G}(x) & \xrightarrow{\ i\ } & F \\ \Big\downarrow{\scriptstyle \rho} & & \\ \operatorname{Spec} k(x). & & \end{array}$$

Instead of giving a construction of $k(x)$ and $\mathscr{G}(x)$, we just state their good properties. For a proof of the existence of these objects and their properties, the reader may consult [BL24, Appendix A.2].

**Proposition 2.3.9.** *Let $F$ be an algebraic stack with quasi-compact diagonal, $x \in |F|$, and $\mathscr{G}(x)$, $k(x)$ as in 2.3. The following properties hold.*

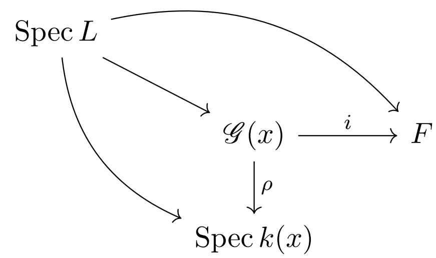
*(i) $\mathscr{G}(x)$ is an algebraic stack, and is a gerbe over $k(x)$. In particular $\mathscr{G}(x)$ is reduced and $|\mathscr{G}|$ is a singleton.*

*(ii) The morphism $\mathscr{G}(x) \to F$ is a monomorphism, and hence it induces an isomorphism on the automorphism groups.*

*(iii) The morphism $|i| : |\mathscr{G}(x)| \to |F|$, sends the unique point of $|\mathscr{G}(x)|$ to $x$.*

*(iv) Given $\operatorname{Spec} L \to F$ in the equivalence class of $x$, there is a unique factorization,*

*yielding a diagram:*




*in particular* $k(x) \subseteq L$.

## 2.4 Coverings of Stacks

In this section, we use the work of [KV04, Theorem 2.1], and [DHM22, Theorem 4.4], to show that if $G$ is a finite flat commutative group scheme over a quasi-projective scheme $S$ over a field $k$, and $\mathscr{G}$ is a $G$-gerbe over $S$, then $\mathscr{G}$ admits a finite flat cover. We emphasize that this result is already implied by [DHM22, Theorem 4.4]; thus we are not presenting a new theorem.

We begin by recalling the concept of coarse moduli space.

**Definition 2.4.1.** *[Ols16, Definition 11.1.1] Let $F$ be an algebraic stack and $X$ an algebraic space, a morphism $\pi : F \to X$ is said to be a coarse moduli space if;*

*(i) For every algebraically closed field $k$, the induced map $F(k)/\!\sim \to X(k)$, from the set of isomorphism classes of objects of $F$ over* $\operatorname{Spec} k$ *is bijective.*

*(ii) The morphism $\pi$ is universal for morphisms to algebraic spaces. In other words, for any morphism $\varphi : F \to Y$ such that $Y$ is an algebraic space, $\varphi$ factors uniquely through $\pi$.*

The main theorem concerning the existence of a coarse moduli space is due to Sean Keel and Shigefumi Mori in [KM97, Corollary 1.3]. We state a version of it below just for the sake of completeness.

**Theorem 2.4.2.** *[Con05, Theorem 1.1] Let $S$ be a scheme, and let $F$ be an algebraic stack locally of finite presentation over $S$, with finite diagonal. There exists a coarse moduli space $\pi : F \to X$, and it satisfies the following properties:*

(i) *The structure map $X \to S$ is separated if $F \to S$ is separated, and it is locally of finite type if $S$ is locally Noetherian.*

(ii) *The morphism $\pi$ is proper and quasi-finite.*

(iii) *If $X' \to X$ is a flat morphism between algebraic spaces, then $\pi' : F' := F \times_X X' \to X'$ is a coarse moduli space.*

(iv) *The induced morphism $\mathscr{O}_X \to \pi_* \mathscr{O}_F$ is an isomorphism.*

**Example 2.4.3.** *Let $G$ be a finite group acting on an affine scheme* $\operatorname{Spec} A$*, then* $[\operatorname{Spec} A/G]$ *has a coarse moduli space, and it is given by the canonical morphism* $\pi : [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ *[Alp25, Theorem 4.3.6]. Thus, the stack* $[\mathbb{A}^1_{\mathbb{C}}/\mathbb{Z}/2]$ *of Example 2.2.7 admits a coarse moduli space, it is given by* $[\mathbb{A}^1_{\mathbb{C}}/\mathbb{Z}/2] \to \mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x^2]$*. In particular, if we take $BG$, the structure morphism $BG \to S$ is a coarse moduli space. Therefore, if $\mathscr{G}$ is a $G$-gerbe, the structure morphism $\mathscr{G} \to S$ is also a coarse moduli space.*

The following definition is [Har77, Page 103], see [Sta25, Tag 0B42] for other equivalent definitions.

**Definition 2.4.4.** *A morphism of schemes $f : X \to S$ is quasi-projective, if it factors through a quasi-compact immersion $X \to \mathbb{P}^n_S$ for some $n$. If $S$ has an ample invertible sheaf, then this definition is equivalent to $f$ being of finite type and the existence of a $f$-relatively ample invertible sheaf on $X$.*

In what follows, we prove some lemmas in order to establish the main theorem of this section.

**Lemma 2.4.5.** *[KV04, Lemma 3.1][DHM22, Proposition 4.1] Let $f : X \to Y$ be a proper morphism of quasi-projective schemes over an infinite base field $k$, with constant fiber dimension $r > 0$. Choose a projective embedding $X \to \mathbb{P}^N_k$ for some $N$ and a proper closed subscheme $C \subseteq \mathbb{P}^N_k$. Then for sufficiently large $d$ there is an open set $U_d \subseteq H^0(\mathbb{P}^N_k, \mathscr{O}(d))$, such that for any $H \in U_d$ the following properties are satisfied.*

1. *$X \cap H \to X$ is a Cartier divisor.*
2. *$H$ misses all the associated points of $C$.*
3. *$X \cap H \to Y$ has constant fiber dimension $r - 1$.*
4. *If $Z \to X$ is a locally closed subscheme that is regular, then $Z \cap H$ is also regular.*

*Proof.* We are going to show that for $i = 1, 2, 3, 4$ there exists $d_i$ with the following property; for all $d \geq d_i$, there is a non-empty open subset $U_d \subseteq H^0(\mathbb{P}^N_k, \mathscr{O}(d))$ such that for all closed points $H \in H^0(\mathbb{P}^N_k, \mathscr{O}(d))$ condition $i$) is satisfied. Hence, by taking a sufficiently large $d$, and intersecting the corresponding open sets, the lemma will follow.

For 1) and 2), consider the finite set $S$ of associated points of $C$ and $X$. Take $p \in S$, and let $\varphi_p : H^0(\mathbb{P}^N_k, \mathscr{O}(d)) \to k(p)$ be the evaluation morphism at $p$. The morphism $\varphi_p$ is not the zero map, therefore, $\ker \varphi_p$ is a proper closed subset, let $U_p = H^0(\mathbb{P}^N_k, \mathscr{O}(d)) - \ker \varphi_p \neq \emptyset$. Then, it suffices to take $U = \bigcap_{p \in S} U_p$, this intersection is non empty, since the $U_p$ are dense and there are only finitely many associated points, any hypersurface in $U$ satisfies conditions 1) and 2). If we assume $Z$ regular, since $\mathscr{O}(d)$ is base point free, Bertini's theorem (see [Sta25, Tag 0FD6]) implies the existence of a dense open $U'_d \subseteq H^0(\mathbb{P}^N_k, \mathscr{O}(d))$ for all $d > 0$, such that $X \cap H$ is regular for all $H \in U_d$.

Condition 3) is more subtle because we need to avoid every component of every fiber of $f : X \to Y$. First, we reduce to the case when $f : X \to Y$ is proper, flat, with geometrically irreducible fibers. Let $P_f$ be the statement that condition 3) holds for $f$. Observe that if $Y' \to Y$ is a morphism between quasi-projective schemes over $k$, and $P_f$ holds, then $P_{f'}$ holds, where $f' : X' \to Y'$ is the base change of $f$ along $Y' \to Y$. Moreover, if $Y' \to Y$ is surjective, and $P_{f'}$ holds, then $P_f$ also holds.

Observe that if we have two disjoint subschemes $f_1 : Y_1 \to Y$, $f_2 : Y_2 \to Y$, such that $P_{f_1}, P_{f_2}$ hold, then $P_{f_1 \coprod f_2}$ also holds (it suffices to take $d$ large enough and intersect the corresponding opens). Thus, if $f_1 \coprod f_2 : Y_1 \coprod Y_2 \to Y$ is surjective, $P_f$ also holds. Therefore, we can use Noetherian induction, and prove the result for a non-empty open subscheme $Y' \subseteq Y$. Indeed, by Noetherian induction $P_g$ holds where $g$ is the base change of $f$ along $(Y - Y')_{red} \to Y$. Since $(Y - Y')_{red} \coprod Y' \to Y$ is surjective, we have $P_f$ holds.

Now we successively shrink $Y$. First, the morphism $Y_{red} \to Y$ is surjective, so we can assume $Y$ reduced. Since $Y$ has a finite number of components, we can suppose that $Y$ is irreducible, therefore $Y$ is integral. Since the regular locus of $Y$ is open, we can suppose that $Y$ is regular and affine. After base change to a connected étale cover we may assume that the irreducible components $\{X_i\}$ of the generic fiber $X \to Y$ are geometrically irreducible (see [Sta25, Tag 020J]), and their scheme theoretic closure $\{\overline{X}_i\}$ are the irreducible components of $X$. If $f_i : X_i \to Y$ is the restriction of $f$ to a component of $X$, then $P_f$ holds if and only if $P_{f_i}$ holds for every $i$, thus we can suppose that $X$ is irreducible, and $f : X \to Y$

has geometrically irreducible generic fiber. Finally, by generic flatness [Sta25, Tag 052A], we can suppose that $f : X \to Y$ is flat, and by [Sta25, Tag 0559] we can further suppose that $f : X \to Y$ is a proper flat morphism, with geometrically irreducible fibers.

The associated map $X \to \mathbb{P}^N_Y$ over $Y$ is a closed immersion, therefore, we have a sequence;

$$0 \to I_X \to \mathscr{O}_{\mathbb{P}^N_Y} \to \mathscr{O}_X \to 0.$$

Let $\pi : \mathbb{P}^N_Y \to Y$ be the projection. By a theorem of Serre on projective morphisms [GD61, Theorem 2.2.1], there exists $d_0$ such that for all $d \geq d_0$ and $i > 0$

$$R^i\pi_*(I_X(d)) = R^i\pi_*(\mathscr{O}_{\mathbb{P}^N_Y}(d)) = R^i\pi_*(\mathscr{O}_X(d)) = 0$$

therefore, we have an exact sequence;

$$0 \to \pi_* I_X(d) \to \pi_*\mathscr{O}_{\mathbb{P}^N_Y}(d) \to \pi_*\mathscr{O}_X(d) \to 0. \tag{2.4.1}$$

Furthermore, because the sheaves $I_X(d), \mathscr{O}_{\mathbb{P}^N}(d), \mathscr{O}_X(d)$ are flat over $Y$, by [Har77, III Theorem 12.11] the sequence 2.4.1 is a sequence of locally free sheaves whose formation is compatible with arbitrary base change on $Y$. Passing to the associated vector bundles over $Y$, we have a morphism

$$i : \mathbb{V}(\pi_* I_X(d)) \to H^0(\mathbb{P}^N_k, \mathscr{O}(d)) \times Y$$

such that at any point $y \in Y$, $\mathbb{V}(\pi_* I_X(d))_y = \mathbb{V}(\pi_* I_{X_y}(d))$ can be described as the space of degree $d$ forms with coefficients in $k(y)$ vanishing on $X_y \subseteq \mathbb{P}^N_{k(y)}$. Hence, it suffices to show that the image of the composition $p_1 \circ i : \mathbb{V}(\pi_* I_X(d)) \to H^0(\mathbb{P}^N_k, \mathscr{O}(d)) \times Y \to H^0(\mathbb{P}^N_k, \mathscr{O}(d))$ is contained in a proper closed subset of $H^0(\mathbb{P}^N_k, \mathscr{O}(d))$.

Since $X_y$ is of positive dimension, for $d$ large enough, $p_{X_y}(d) > \dim Y$ (the Hilbert polynomial of $X_y$). Let $V = \mathbb{V}(\pi_* I_X(d))$, we have the following inequalities

$$\dim V \leq \dim Y + \dim V_y = \dim Y + p_{I_{X_y}}(d) = \dim Y + p_{\mathbb{P}^N_{k(y)}}(d) - p_{X_y}(d) < p_{\mathbb{P}^N_{k(y)}}(d).$$

This shows that $\dim p_1(i(V)) < \dim_k H^0(\mathbb{P}^n_{k(y)}, \mathscr{O}(d))$, thus it must be contained in a proper closed subscheme, the complement satisfies condition 3), so the lemma holds. □

The following theorem is a mixture of [DHM22, Theorem 4.4], [KV04, Theorem 2.1] and [Kre09, Theorem 2.1]

**Theorem 2.4.6.** *Let $k$ be a field, and $\mathscr{X}$ a quotient stack, of finite type over $k$, with finite diagonal, such that its coarse moduli space is a quasi-projective scheme. Then, there exists a scheme $Z$ over $k$ and a finite flat cover $Z \to \mathscr{X}$. Moreover, if $\mathscr{X}$ is Deligne-Mumford, then the cover $Z \to \mathscr{X}$ can be made to be étale over a finite set of points of $\mathscr{X}$.*

*Proof.* The idea is that we can find a projective bundle $\mathscr{P} \to \mathscr{X}$ with good properties, such that the associated morphism on coarse moduli space $P \to X$ is quasi-projective. Then, by a repeated use of Lemma 2.4.5, we can slice $P$ inside a projective space to get a finite flat cover of $X$ which factors through $P$.

We can assume $k = \overline{k}$. Since $\mathscr{X}$ is a quotient stack, follows from the proof of Theorem 2.2.17, that there exists a vector bundle $V$ on $\mathscr{X}$ with faithful action on the fibers and a representable dense open substack. The same conditions hold for $\mathscr{P} := \mathbb{P}(V \oplus \mathscr{O}_{\mathscr{X}})$ (c.f. proof of Theorem 2.2.19). The structure morphism $\mathscr{P} \to \mathscr{X}$ is a smooth projective morphism with constant fiber dimension $r > 0$. Denote by $U \subseteq \mathscr{P}$ the representable dense open substack, so that $C = \mathscr{P} - U$ (with the reduced closed substack structure) has fiber dimension at most $r - 1$. Let $\mathscr{P}^t = \mathscr{P} \times_{\mathscr{X}} ... \times_{\mathscr{X}} \mathscr{P}$, the projections $\mathscr{P}^t \to \mathscr{P}$ are representable, therefore, the complement of $C^t = C \times_{\mathscr{X}} ... \times_{\mathscr{X}} C$ is representable by an algebraic space. The dimension of $C^t$ is at most $t(r-1) + \dim \mathscr{X}$, while the fiber dimension of $\mathscr{P}^t \to \mathscr{X}$ is $rt$. Thus, if we take $t$ large enough, after replacing $\mathscr{P}$ with $\mathscr{P}^t$ we may assume that $\mathscr{P} \to \mathscr{X}$ is a smooth projective morphism with a representable dense open substack $U \subseteq \mathscr{P}$, such that $C = \mathscr{P} - U$ has dimension strictly less than the fiber dimension of $\mathscr{P} \to \mathscr{X}$.

Since $\mathscr{P} \to \mathscr{X}$ is representable, $\mathscr{P}$ also has finite diagonal, by Theorem 2.4.2 there is a coarse moduli space $\mathscr{P} \to P$. Denote by $\mathscr{X} \to X$ the coarse moduli space of $\mathscr{X}$, and consider the induced morphism $P \to X$. Since $\mathscr{P} \to P$ and $\mathscr{P} \to X$ are proper morphisms, the same holds for $P \to X$, [Sta25, Tag 04NX]. Furthermore, $P \to X$ also has constant fiber dimension $r > 0$.

We will show that $P \to X$ is a quasi-projective morphism. Indeed, since $\mathscr{P} \to \mathscr{X}$ is projective, there is a relatively ample line bundle $\mathscr{L}$ on $\mathscr{P}$. By [Ryd15], there is a line bundle $\mathscr{M}$ on $P$ such that $\mathscr{M}_{\mathscr{P}}$ is relatively ample. We claim that $\mathscr{M}$ is relatively ample to $P \to X$. Let $T \to \mathscr{X}$ be a finite cover by a scheme (see [EHKV01, Theorem 2.7] for the existence of finite covers), then $\mathscr{M}_{\mathscr{P}_T}$ is relatively ample, so the same holds for $\mathscr{M}$ (see [Sta25, 0GFB]). Thus, $P$ is quasi-projective because $X$ is.

Now we apply Lemma 2.4.5 to $P \to X$ to obtain a hypersurface $H \subseteq \mathbb{P}^n_k$ such that $P \cap H \to X$ has constant fiber dimension $r-1$, $H$ does not contain any component of $C$ and $P \cap H \to P$ is a Cartier divisor. We claim that the morphism $P \cap H \to X$ is flat. The idea is that locally on all the fibers of $P \to X$, the hypersurface $H$ is given by a single element that is not a zero divisor. Let $p \in P \cap H$ and $x$ its image in $X$. By the local criterion for flatness [Sta25, Tag 00MK] it suffices to show that $\operatorname{Tor}_1^{\mathscr{O}_x}(k(x), \mathscr{O}_{P \cap H, p}) = 0$. Observe that $H$ intersects $P$ properly, therefore $P_x \cap H$ is also a Cartier divisor on $P_x$. Hence, by [Sta25, Tag 01WS], if locally $H = \mathbb{V}(h)$, then we have an exact sequence

$$0 \to (h) \to \mathscr{O}_{P,p} \to \mathscr{O}_{P \cap H, p} \to 0.$$

By tensoring with $k(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x$ we get

$$(h) \to \mathscr{O}_{P_x,p} \to \mathscr{O}_{P_x \cap H, p} \to 0.$$

The sequence is exact on the left since $P_x \cap H \to P_x$ is a Cartier divisor, so $(h)$ is not a zero divisor. Therefore $\operatorname{Tor}_1^{\mathscr{O}_x}(k(x), \mathscr{O}_{P_x \cap H, p}) = 0$. Repeating this process $r-1$ times, we end up with a finite, flat morphism $Z \to X$ such that $Z \subseteq P$ is disjoint from $C$. Thus, we can lift $Z \to U$ and since $\mathscr{P} \to \mathscr{X}$ is smooth, the morphism $Z \to \mathscr{X}$ is finite flat.

Now suppose that $\mathscr{X}$ is a Deligne-Mumford stack, we will prove that if $x \in X$ is a closed point, then there is an open set of $W_x \subseteq H^0(\mathbb{P}^N_k, \mathscr{O}(d))$, such that for all $H \in W_x$, the intersection $H \cap P \to X$ is smooth over $x$ (hence, it suffices to prove the result for one point). It is enough to do this locally, so we may assume that $X = \operatorname{Spec} A$ is the spectrum of a complete local ring and $\mathscr{X}$ is of the form $[\operatorname{Spec} B/G]$ where $B$ is a complete local ring and $G$ is a finite group (here we are using that $\mathscr{X}$ is Deligne-Mumford, see [LMB00, Théorème 6.2]). Furthermore, since in the end $P \cap H$ will miss $C$, we will just work with $U$.

Let $\operatorname{Spec} k \xrightarrow{x} \operatorname{Spec} A$ be the inclusion of the closed point. Consider the following diagram;

$$\begin{array}{ccccc}
U_x \times_{\mathscr{X}} \mathscr{X}_{x,\mathrm{red}} & \longrightarrow & \mathscr{X}_{x,\mathrm{red}} & & \\
\downarrow & & \downarrow & & \\
U_x & \longrightarrow & \mathscr{X}_x & \longrightarrow & \operatorname{Spec} k \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle x} \\
U & \longrightarrow & \mathscr{X} & \longrightarrow & X.
\end{array}$$

The stack $\mathscr{X}_{x,\mathrm{red}}$ is a reduced stack with $|\mathscr{X}_{x,\mathrm{red}}|$ a singleton, so we can identify it with the residual gerbe $BN$ at the point $x \in |\mathscr{X}|$, where $N = \underline{\mathrm{Aut}}_{\mathscr{X}}(x)$ . Moreover, since $BN$ is reduced and the morphism $U_x \times_{\mathscr{X}} BN \to BN$ is smooth we can also identify $U_x \times_{\mathscr{X}} BN$ with $U_{x,\mathrm{red}}$. Now $BN$ is also regular, therefore $U_{x,\mathrm{red}}$ is also regular. Therefore, by Lemma 2.4.5, there exists a dense open set of hypersurfaces $H$ such that $H \cap U_{x,\mathrm{red}}$ is also regular over $k$. In other words, $H \cap U_{x,\mathrm{red}} \to \operatorname{Spec} k$ is smooth.

Now, let $\operatorname{Spec} k \to \mathscr{X}$ be the inclusion of the point $x \in |\mathscr{X}|$. We have the following diagram where all the squares are cartesian;

$$\begin{array}{ccccc}
V \cap H & \longrightarrow & V & \longrightarrow & \operatorname{Spec} k \\
\downarrow & & \downarrow & & \downarrow \\
U_{x,\mathrm{red}} \cap H & \longrightarrow & U_{x.\mathrm{red}} & \longrightarrow & BN \\
\downarrow & & \downarrow & & \downarrow \\
U \cap H & \longrightarrow & U & \longrightarrow & \mathscr{X}.
\end{array}$$

Since $\mathscr{X}$ is Deligne-Mumford, the morphism $\operatorname{Spec} k \to BN$ is étale, hence $V \cap H \to H \cap U_{x,\mathrm{red}}$ is also an étale morphism, so $V \cap H$ is smooth over $\operatorname{Spec} k$. Therefore, the morphism $U \cap H \to \mathscr{X}$ is smooth in an open neighborhood of $x \in \mathscr{X}$ [Sta25, Tag 02GU]. Repetition of this process leaves us with a finite flat cover $Z \to \mathscr{X}$ that is étale in a neighborhood of $x$. □

**Corollary 2.4.7.** *Let $S$ be a quasi-projective scheme over a field $k$, $G$ a finite flat group scheme over $S$, and $\mathscr{G}$ a $G$-gerbe over $S$. Suppose that $\mathscr{G}$ is a quotient stack, then, there is a finite flat cover $Z \to S$ such that $\mathscr{G}_Z \simeq BG$.*

# Chapter 3

# Gerbes for Finite Flat Group Scheme

This final chapter focuses on Problems 1 and 2. In the first two sections, we present some foundational theory of finite flat group schemes, and provide answers to problem 1. More precisely; we prove that for isotrivial finite flat $G$ over a reduced quasi-projective scheme, $G$-gerbes are quotient stacks. It is worth mentioning that the case $G = \mu_n$ implies $\mathrm{Br} = \mathrm{Br}'$ for quasi-projective schemes (Gabber's Theorem 1.3.7), the case of $\alpha_{p^r}$ and $\mathbb{Z}/p^r$ can be derived from the work of Bragg-Hall-Mathur, [BHM25]. Our proof relies heavily on these results. In the remaining sections, we introduce abelian schemes and consequently abelian torsors in order to address Problem 1. As a consequence of our result on $G$-gerbes, we prove that for abelian schemes $A$ with isotrivial $p$-torsion, the equality $\mathrm{Br} = \mathrm{Br}'$ holds for $A$-torsors over reduced quasi-projective schemes of characteristic $p$.

## 3.1 Finite Flat Group Schemes

Through the section $S$ is always assumed to be Noetherian, groups over $S$ are always finite, flat and commutative unless other thing is stated. References for this section are [Sti12] [Dem86], [ABD$^+$66] and [Mes72].

We begin by recalling the concept of Cartier duality.

**Definition 3.1.1.** *[Sti12, 3.2] Let $G$ be a finite flat commutative group scheme over $S$, we* *define the Cartier dual of $G$ as the group $G^D := \mathscr{H}om_{S-gp}(G, \mathbb{G}_m)$.*

It is a well known fact that Cartier duality defines a contravariant involutory autoequivalence of the category of finite flat commutative group schemes over $S$ [Sti12, Proposition 4].

**Example 3.1.2.** *[Sti12, 3.2.3] If $G = \mathbb{Z}/n$, then $G^D = \mathscr{H}om(G, \mathbb{G}_m) = \mathscr{H}om(\mathbb{Z}/n, \mathbb{G}_m) \simeq \mu_n$, where the last isomorphism is given by sending a morphism $\varphi : \mathbb{Z}/n \to \mathbb{G}_m$, to $\varphi(1)$. This shows that $(\mathbb{Z}/n)^D \simeq \mu_n$ and $\mu_n^D \simeq \mathbb{Z}/n$.*

*If $S$ is of characteristic $p$, then $\alpha_p^D \simeq \alpha_p$, the morphism $\alpha_p \mapsto \alpha_p^D$ given by sending a section $t \in \alpha_p(T)$ to the morphism $\varphi_t : \alpha_{p,T} \to \mathbb{G}_{m,T}$, where $\varphi_t(s) = \sum_{a=0}^{p-1} (st)^a/a!$ defines an isomorphism $\alpha_p \to \alpha_p^D$.*

*This shows that if $S$ is of characteristic $p$, then $\mathbb{Z}/p$, $\mu_p$ and $\alpha_p$ are not mutually isomorphic.*

Cartier duality allows us to make the following definitions.

**Definition 3.1.3.** *Let $S$ be a scheme and $G$ a finite flat group scheme over $S$.*

*(i) $G$ is a constant group scheme if it is the constant sheaf associated to an abstract group [Dem86, Section 2.2].*

*(ii) $G$ is étale if the structure morphism $G \to S$ is étale [Dem86, Section 2.2].*

*(iii) $G$ is radiciel if the structure morphism $f : G \to S$ is injective and for every $g \in G$ the extension $k(g)/k(f(g))$ is purely inseparable [Sta25, Tag 01S2].*

*(iv) $G$ is of multiplicative type if étale locally on $S$, $G^D$ is a constant group scheme [Dem86, Section 2.8].*

*(v) $G$ is unipotent if it is a subgroup of $\underline{\mathrm{Aut}}_{Flag}(V_\bullet)$, where*

$$V_\bullet : 0 \subseteq V_1 \subseteq ... \subseteq V_n$$

*is a flag on $S$ such that $V_{i+1}/V_i$ is a line bundle for all $i$ [BHM25, Remark 5.11].*

*Remark* 3.1.4. When $V_\bullet$ is the flag given by $V_i = \mathscr{O}_S^{\oplus i}$ and quotients $\mathscr{O}_S$, then the group $\underline{\mathrm{Aut}}_{\mathrm{Flag}}(V_\bullet)$ is isomorphic to $\mathbb{U}_{n,S}$ the group of upper triangular matrices with 1's on the diagonal. Moreover, for any flag $V_\bullet$ with $V_{i+1}/V_i$ a line bundle, the group $\underline{\mathrm{Aut}}_{\mathrm{Flag}}(V_\bullet)$ is a Zariski inner form of $\mathbb{U}_{n,S}$. When $S$ is a field, the definition of unipotent groups given in 3.1.3 agrees with the classical definition. See [ABD+66, XVII Théorème 3.5] for other equivalent definitions.

The following lemma will suffice for our purposes in working with unipotent groups over more general bases.

**Lemma 3.1.5.** *[BHM25, Corollary 5.13] Let $S$ be a quasi-projective scheme over a field $k$, $G$ a group scheme over $S$, and $S' \to S$ a morphism of schemes. If $G$ is unipotent over $S$, then $G_{S'}$ is unipotent over $S'$. Furthermore, if $S' \to S$ is an fppf-cover and $G_{S'}$ is unipotent over $S'$ then $G$ is unipotent over $S$.*

The next theorem implies the existence of a decomposition of $G$ into an étale group and a radiciel group. Over a Henselian ring, this decomposition agrees with the connected-étale sequence [Sti12, Proposition 37]; which over a Henselian ring always exists, but is not compatible with base change.

Recall that given a local $k$-algebra $A$, its separable rank is defined to be the separable rank of the extension $k \to A/\mathfrak{m}_A$). If $X = \operatorname{Spec} A$ is a finite scheme over a field $k$, then the separable rank of $X$ is the sum over all $\mathfrak{p} \in X$, of the separable rank of $A_{\mathfrak{p}}$.

**Theorem 3.1.6.** *[Mes72, Lemma 4.8] Let $S$ be a Noetherian scheme over $\mathbb{F}_p$ and $G$ a finite flat group scheme over $S$ such that the function $s \mapsto$ separable rank$(G_s)$ is locally constant. Then there is an exact sequence*

$$0 \to G^0 \to G \to G^{ét} \to 0$$

*where $G^0$ is a radiciel group and $G^{ét}$ is an étale group. Furthermore, this sequence is unique up to unique isomorphism, and is functorial on $S$.*

## 3.2 Gerbes are Quotient Stacks

This section is devoted to provide some positive answers to Problem 2 and represents the core of this thesis. We use the theory and tools developed in the previous chapters to prove that given an isotrivial, finite, flat, commutative group scheme $G$, then every $G$-gerbe is a quotient stack.

**Lemma 3.2.1.** *[EHKV01, Corollary 2.16] Let $S$ be a Noetherian scheme and $G$ a finite flat group scheme over $S$. Then the trivial gerbe $BG$ over $S$ is a quotient stack.*

*Proof.* Let $\pi : S \to BG$ be the morphism defined by the trivial $G$-torsor over $S$. Since $G$ is finite flat the morphism $\pi : S \to BG$ is as well, moreover, $S$ is a quotient stack, thus by Theorem 2.2.19 $BG$ is a quotient stack. □

**Lemma 3.2.2.** *Let $S$ be a quasi-projective scheme over a field. Consider an exact sequence of commutative, finite flat groups over $S$*

$$0 \to H \to G \to K \to 0.$$

*If $H_{S'}$-gerbes and $K_{S'}$-gerbes are quotient stacks over $S'$, for any finite flat morphism $S' \to S$, then the same holds for $G$-gerbes.*

*Proof.* Consider the induced sequence in cohomology

$$H^2(S, H) \to H^2(S, G) \to H^2(S, K).$$

Let $\alpha \in H^2(S, G)$ be the class of a $G$-gerbe, and $\beta$ its image in $H^2(S, K)$. The class $\beta$ is represented by a quotient stack, so by Corollary 2.4.7, there is a finite flat cover $S' \to S$ such that $\beta_{S'} = 0$. Hence, after pulling back to $S'$ we can assume that $\alpha$ is in the image of some $\gamma \in H^2(S, H)$. By the same reasoning, we can find a finite flat cover $S'' \to S$, such that $\alpha_{S''} = 0$. In other words, after a finite flat cover the $G$-gerbe $\alpha$ becomes isomorphic to $BG$. The lemma follows from 2.2.19. □

**Corollary 3.2.3.** *Suppose that $G$ and $H$ are finite flat commutative groups over a Noetherain scheme $S$, with the property that $G$-gerbes and $H$-gerbes are quotient stacks, and this property holds after finite flat covers. Then $G \times_S H$-gerbes are also quotient stacks.*

The following theorem is a consequence of Gabber's Theorem 1.3.7.

**Theorem 3.2.4.** *Let $S$ be a quasi-projective scheme over $k$, and $G$ a finite flat group scheme of multiplicative type over $S$. Then, every $G$-gerbe over $S$ is a quotient stack.*

*Proof.* Since $G$ is of multiplicative type, étale locally $G^D$ is a constant group. Suppose that $S' \to S$ is an étale cover such that $G^D_{S'} = \underline{M}$, where $M$ is an abstract finite group. Consider the $\underline{\mathrm{Aut}}(\underline{M})$-torsor $\underline{\mathrm{Isom}}_S(\underline{M}, G^D)$, since $\underline{\mathrm{Aut}}(\underline{M})$ is a finite étale group scheme, we conclude that $\underline{\mathrm{Isom}}_S(\underline{M}, G^D)$ must be trivial after a finite flat cover. Therefore, by

Theorem 2.2.19 we can suppose that $G \simeq \mathscr{H}om_{S-gp}(\underline{M}, \mathbb{G}_m) \simeq \mu_{n_1} \times ... \times \mu_{n_2}$, (see Example 3.1.2). By Lemma 3.2.3, we are reduced to the case $G = \mu_n$. However, this last case follows from Gabber's Theorem 1.3.7 and Theorem 2.3.7. □

**Theorem 3.2.5.** *Let $S$ be a quasi-projective scheme over a field $k$ and $G$ a commutative, finite étale group scheme over $S$. Then, every $G$-gerbe over $S$ is a quotient stack.*

*Proof.* We follow the same idea as in Lemma 3.2.4. After an étale cover of $S$, the group $G$ becomes a constant group $\underline{M}$ where $M$ is an abstract finite group. Consider the $\underline{\mathrm{Aut}}(M)$-torsor $\underline{\mathrm{Isom}}_S(\underline{M}, G)$, since $\underline{\mathrm{Aut}}(\underline{M})$ is a finite étale group scheme, after a finite flat cover we are reduced to the case $G = \underline{M}$. Write $M$ as a product of groups of the form $\mathbb{Z}/n$, Corollary 3.2.3 reduce us to the case; $M \simeq \mathbb{Z}/n$ and $n$ a power of a prime number. Suppose that $\mathrm{char}\, k \nmid n$, then there exists a finite flat cover of $S$ such that $\mu_n \simeq \mathbb{Z}/n$ as group schemes (add the roots of unity of $\Gamma(S, \mathscr{O}_S)$). So by Theorem 2.2.19 we can assume $\mathbb{Z}/n \simeq \mu_n$ and the result follows from 3.2.4. The case $\mathrm{char}\, k \mid n$ will follow from Theorem 3.2.6, since $\mathbb{Z}/p^r$ is unipotent for all $r$. □

The following theorem is an immediate consequence of [BHM25, Theorem 5.5].

**Theorem 3.2.6.** *Let $S$ be a quasi-projective scheme over a field $k$, and $G$ a fppf unipotent group over $S$. Then, every $G$-gerbe over $S$ is a quotient stack.*

*Proof.* Let $\mathscr{G}$ be a $G$-gerbe over $S$, then there is an étale cover $S' \to S$ such that $\mathscr{G}_{S'} \simeq BG_{S'}$. Since $G$ is unipotent $G_{S'}$ is unipotent as well (Lemma 3.1.5), therefore the morphism $BG_{S'} \to S'$ is unipotent (see [BHM25, Definitions 5.1 and 5.9]). In conclusion, $\mathscr{G} \to S$ is locally unipotent (see [BHM25, Definition 5.1]), therefore by [BHM25, Theorem 5.5] $\mathscr{G}$ is a quotient stack. Observe that we are using that $S$ is quasi-projective and so it has the resolution property. □

Let $\mathscr{X}_n$ be the moduli stack of finite locally free commutative group schemes of order $p^n$. See [TO70, pages 12-15] for a description in the case of $n = 1$, and [Zha00, Lemma 1] for the general case. Recall that a group $G$ over $S$ is said to be isotrivial if all geometric fibers of $G$ are isomorphic, we have the following theorem.

**Theorem 3.2.7.** *Let $S$ be a reduced quasi-projective scheme over a field $k$ of characteristic $p > 0$, $G$ an isotrivial, commutative, finite flat group scheme of order $p^n$ over $S$. Then, every $G$-gerbe over $S$ is a quotient stack.*

*Proof.* Our strategy is to decompose $G$ into simpler groups such that we can use the previous theorems of the section. To do this, we show that after a fppf cover $G$ is obtained by pull back from a group over a field.

Consider the induced morphism $S \to \mathscr{X}_n$ given by $G$. Let $x \in |\mathscr{X}_n|$ be the point corresponding to the type of the geometric fibers $G_{\bar{s}} \in |\mathscr{X}_n|$. Let $\mathscr{G}$ be the residual gerbe of $x$ (see 2.3), we have a diagram;

$$\begin{array}{ccc} \operatorname{Spec} \overline{k(s)} & \xrightarrow{\bar{s}} & S \\ \downarrow & & \downarrow G \\ \mathscr{G} & \longrightarrow & \mathscr{X}_n. \end{array}$$

Proposition 2.3.9 implies the existence of the arrow $\operatorname{Spec} \overline{k(s)} \to \mathscr{G}$ for all $s \in S$. Now the morphism $S_{\mathscr{G}} \to S$ (obtained by base change of the above diagram) is a pro-immersion (see [HR18, Lemma 2.1]), therefore an isomorphism, because $S$ is reduced and $S_{\mathscr{G}}$ contains all the points in $S$. If $G_{\text{univ}} \to \mathscr{X}_n$ is the universal group scheme over $\mathscr{X}_n$, then we have a cartesian diagram;

$$\begin{array}{ccccc} G & \longrightarrow & G' & \longrightarrow & G_{\text{univ}} \\ \downarrow & & \downarrow & & \downarrow \\ S & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{X}_n. \end{array}$$

The stack $\mathscr{G}$ is a gerbe over a field $k(x)$. Since our goal is to prove that $G$-gerbes are quotient stacks, by Theorem 2.2.19 we can replace $k(x)$ by a finite separable extension $L$, and therefore assume that $\mathscr{G} \simeq BH$ where $H$ is an affine group scheme over a field.

Let $\operatorname{Spec} L \to BH$ be the fppf-cover given by the trivial $H$-torsor on $L$. We claim that $G'_L$ has a filtration of the form

$$G'_\bullet : \ 0 \subseteq G'_{L,\mu} \subseteq G'^0_L \subseteq G'_L$$

where $G'_L/G'^0_L$ is an étale group, and $G'^0_L/G'_{L,\mu}$ is unipotent. Indeed, apply Theorem 3.1.6 to $G'_L$ to get a connected subgroup $G'^0_L \to G'_L$ with étale quotient. Now apply theorem

3.1.6 to $(G'^0_L)^D$ to get a sequence

$$0 \to A \to (G'^0_L)^D \to B \to 0$$

where $A$ is connected, and $B$ is étale. Applying Cartier duality to the last sequence we get a subgroup $G'_{L,\mu} := B^D$ of $G'^0_L$ such that $G'^0_L/G'_{L,\mu}$ is $A^D$. In fact, $(G'^0_L/G'_{L,\mu})$ is unipotent. Its Cartier dual is connected and this is one equivalent definition for being unipotent over a field, see [Sti12, Proposition 5.3] and [ABD+66, XVII Théorème 3.5]. Now, we would like to descend this filtration to $BH$. Let $\varphi : G'_H \to G'_H$ be the descent data associated to $G'$ over $H = \operatorname{Spec} L \times_{BH} \operatorname{Spec} L$. Then $\varphi$ preserves the filtration $G'_\bullet$. This is true because Theorem 3.1.6 implies that the decomposition is functorial, thus for every group $G''$ in the filtration we have $\varphi_H(G'') \subseteq G''$, by comparing the rank of $G''$ and $\varphi(G'')$ we see that $\varphi$ induces an automorphism of the filtration. Therefore, we can descend it to $BH$, and by pulling back to $S$, we obtain a filtration

$$0 \subseteq G_\mu \subseteq G^0 \subseteq G$$

defined over $S$.

Observe that $G/G^0$ is obtained by pull back from an étale group, thus the same property holds for $G/G^0$, in the same way we conclude that $G_\mu$ is of multiplicative type. The group $(G^0/G_\mu)_{S_L}$ is the pull back along $S_L \to \operatorname{Spec} L$ of $G'^0_L/G'_{L,\mu}$ which is unipotent. Since $S_L \to S$ is an fppf cover, Lemma 3.1.5 implies that $G^0/G_\mu$ is also unipotent.

Now consider the partial quotients;

$$0 \to G_\mu \to G^0 \to G^0/G_\mu \to 0$$

$$0 \to G^0 \to G \to G^{\text{ét}} \to 0.$$

Since $G_\mu$ is of multiplicative type and $G^0/G_\mu$ is unipotent, Theorems 3.2.4, 3.2.6 and Lemma 3.2.2 implies that $G^0$-gerbes are quotient stacks. Then again, since $G^{\text{ét}}$ is an étale group, Theorem 3.2.5 and Lemma 3.2.2 implies that $G$-gerbes are quotient stacks, so the theorem follows. □

## 3.3 Abelian Torsors

In this section, we define abelian torsors. Abelian schemes have been studied in detail, for example in [Mum70]. In this section, we follow [GW23, Chapter 27].

**Definition 3.3.1.** *[GW23, Definition 27.89] A group scheme $A$ over $S$ is called an abelian scheme if it satisfies the following properties:*

*(i) The structure morphism $A \to S$ is proper, flat and locally of finite presentation.*

*(ii) All fibers of $A \to S$ are geometrically reduced and geometrically connected.*

*An abelian torsor is an $A$-torsor for some abelian scheme $A$.*

*Remark* 3.3.2. Let $f\colon X \to S$ be a morphism of schemes which is proper, flat and locally of finite presentation. Then $X$ is an abelian torsor for an abelian scheme over $S$ if and only if all the geometric fibers of $f$ admit the structure of an abelian variety. Moreover, a morphism between two such $S$-schemes is automatically a morphism of torsors, see [JM22, Proposition 2.1] and [LS24, Theorem 5.3]. In particular, any genus one curve is automatically a torsor under an elliptic curve.

**Proposition 3.3.3.** *[GW23, Proposition 27.92] Let $f : A \to S$ be an abelian scheme. The following properties hold for $f$;*

1. *is faithfully flat and quasi-compact,*
2. *is smooth,*
3. *is of finite presentation,*
4. *is universally open,*
5. *has geometrically integral fibers.*
6. *For any morphism $S' \to S$, the canonical morphism $\mathscr{O}_{S'} \to f'_*\mathscr{O}_{X_{S'}}$ is an isomorphism.*
7. *$A$ is a commutative group scheme.*

*Remark* 3.3.4. By fppf descent [GW20, Appendix C], if $A$ is an abelian scheme and $X$ an $A$-torsor, then $X$ also satisfies properties $1-6$ listed in 3.3.3.

**Example 3.3.5.** *One example of abelian schemes in dimension* 1 *are elliptic curves. Let* $k$ *be a field with* $\operatorname{char} k \neq 2, 3$*, then the curves* $E = \mathbb{V}(zy^2 - x(x-1)(x-\lambda)) \subseteq \mathbb{P}^2_k$*, where* $\lambda \neq 0, 1$ *are examples.*

Another important example is the Jacobian of a curve:

**Example 3.3.6.** *[GW23, Proposition 27.25] Let* $S$ *be a scheme, and* $C \to S$ *a proper, flat morphism of relative dimension* 1*, such that all the fibers are geometrically reduced and geometrically connected. Consider the Picard functor* $\operatorname{Pic}_{C/S}$ *defined as the fppf-sheafification of the presheaf*

$$T \mapsto (\operatorname{Pic}(C \times_S T))/\operatorname{Pic}(T).$$

*Another description of* $\operatorname{Pic}_{C/S}$ *is; the fppf sheaf* $R^1 f_* \mathbb{G}_m$*. An element in* $\operatorname{Pic}_{C/S}(T)$ *is given locally by a line bundle* $L \in \operatorname{Pic}(C \times T')$*, where* $T' \to T$ *is an fppf cover. Thus, given a line bundle* $\mathscr{L} \in \operatorname{Pic}_{C/S}(T)$ *and a fppf cover* $T' \to T$ *with an* $L \in \operatorname{Pic}(X \times T')$ *as above, we can define a locally constant function;*

$$\deg_{\mathscr{L}} : T \to \mathbb{Z}$$

$$\deg_{\mathscr{L}}(t) = \deg(L|_{C_t}).$$

*This induces a decomposition*

$$\operatorname{Pic}_{C/S} = \coprod_{d \in \mathbb{Z}} \operatorname{Pic}^d_{C/S}$$

*where*

$$\operatorname{Pic}^d_{C/S}(T) := \{\mathscr{L} \in \operatorname{Pic}_{C/S}(T) : \deg_{\mathscr{L}} = d\}.$$

*Then,* $\operatorname{Pic}^0_{C/S}$ *is an Abelian scheme over* $S$*, usually called the Jacobian of* $C$*. Furthermore, tensor product of line bundles induces morphisms*

$$\operatorname{Pic}^d_{C/S} \times \operatorname{Pic}^{d'}_{C/S} \to \operatorname{Pic}^{d+d'}_{C/S}$$

*therefore,* $\operatorname{Pic}^d_{C/S}$ *has a natural* $\operatorname{Pic}^0_{C/S}$*-action. Furthermore, with this action* $\operatorname{Pic}^d_{C/S}$ *is a* $\operatorname{Pic}^0_{C/S}$*-torsor. Indeed, one can prove that étale locally* $\operatorname{Pic}^d_{C/S}$ *has a point by using the deformation theory of line bundles [Har10, Remark 2.8.1], Grothendieck's existence theorem [Alp25, Corollary C.4.9] and Artin approximation [GW23, Corollary 20.59].*

Given an abelian scheme $A/S$ the group structure defines a multiplication by $n$ map $[n]: A \to A$. We will study the group $A[n] := \ker[n]$.

**Lemma 3.3.7.** *[GW23, Proposition 27.186] Given $S$ a Noetherian scheme, $A/S$ an abelian scheme of relative dimension $g$. The morphism $[n]: A \to A$ is a finite flat morphism of degree $n^{2g}$ for all $n \in A$. In particular, the group $A[n]$ is finite flat over $S$.*

We conclude the section by describing an example of a non-quasi-projective abelian torsor.

**Proposition 3.3.8.** *[Ray70, Proposition XIII 2.3] Suppose that $S$ is a quasi-projective scheme over a field $k$, and let $X$ be an $A$-torsor over $S$. If $X$ is quasi-projective over $S$ then $X \in H^1(S, A)_{\mathrm{Tors}}$.*

*Proof.* We can assume $S$ connected. If $X$ is quasi-projective over $S$ then $X$ admits an embedding to $\mathbb{P}^n_k$ for some $n$. Thus we can successively use Lemma 2.4.5 to get a finite flat cover $Z \to S$ which factors through $X \to S$ (c.f. Theorem 2.4.6). Thus, if $\alpha \in H^1(S, A)$ is the class of $X$, then $\alpha_Z \in H^1(Z, A_Z)$ must be trivial. The existence of norm maps for the functor $H^i(-, A)$ (see [Del73, Proposition 6.3.15 iv)]) give us a sequence

$$H^1(S, A) \to H^1(Z, A_Z) \to H^1(S, A)$$

such that the composition is multiplication by $d = \deg(Z \to S)$. Since $\alpha_Z = 0$ we conclude that $d\alpha = 0$ in $H^1(S, A)$, so $X$ is represented by a torsion class. □

Now we explain how to construct a non quasi-projective abelian torsor.

**Example 3.3.9.** *[Ray70, XIII 3.1] Fix a base field $k$. Let $C \to \operatorname{Spec} k$ be the nodal cubic curve, and let $A \to \operatorname{Spec} k$ be an abelian variety with a $k$-point $p$ of infinite order. Let $C' \simeq \coprod_{n \in \mathbb{Z}} \mathbb{P}^1_n / \sim$ where $0 \in \mathbb{P}^1_n \sim \infty \in \mathbb{P}^1_{n+1}$, and $C' \to C$ is the morphism induced by the normalization $\mathbb{P}^1 \to C$ where $0, \infty$ are sent to the node. Then $C' \to C$ is a Galois cover with $\mathrm{Aut}(C'/C) \simeq \mathbb{Z}$. We are going to use Galois descent to construct an $A$-torsor over $C$.*

*Observe that the normalization morphism $\mathbb{P}^1 \to C$ is birational. Now take $c \in A(C)$, then $c$ lifts to a morphism $\mathbb{P}^1 \to A$. Since there are no non-constant morphism $\mathbb{P}^1 \to A$, we conclude that $c$ is constant. In other words, the induced morphism $A(k) \to A(C)$ is an isomorphism. By the same argument over $C'$, which is birational to $\coprod_{\mathbb{Z}} \mathbb{P}^1$ we conclude* *that* $A(k) = A(C') = A_{C'}(C')$*, thus we have a* $C'$*-point of* $A_{C'}$ *of infinite order, which we also call* $p$*. Therefore,* $\mathbb{Z} = \mathrm{Aut}(C'/C)$ *acts trivially on* $A(C')$*.*

*Next, we define an action* $\mathbb{Z} \times A_{C'} \to A_{C'}$ *compatible with the (trivial)* $A_{C'}$*-torsor structure of* $A_{C'}$ *and the Galois structure of* $C'$*, in other words, we want for all* $n$ *a commutative diagram*

$$\begin{array}{ccccccc} A_{C'} & \xrightarrow{n} & A_{C'} & \longrightarrow & A_C & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C' & \xrightarrow{n} & C' & \longrightarrow & C & \longrightarrow & \mathrm{Spec}\, k. \end{array} \tag{3.3.1}$$

*such that the composition* $A_{C'} \xrightarrow{n} A_{C'} \xrightarrow{m} A_{C'}$ *agrees with the morphism* $m+n : A_{C'} \to A_{C'}$*. Let* $\sigma : \mathbb{Z} \times A \times_k C' \to A \times_k C'$ *be given by* $\sigma(n, a, c') \mapsto (a + np, nc')$*, this defines a Galois action on* $A_{C'}$*, thus by Galois descent for* $A$*-torsors (see [BLR90, Section 6]), we get an* $A$*-torsor* $X$ *over* $C$*.*

*Associated to the Galois cover* $C' \to C$ *we have the Hochschield-Serre spectral sequence (see [Mil80, Theorem 2.20]):*

$$H^p(\mathbb{Z}, H^q(C', A)) \Rightarrow H^{p+q}(C, A)$$

*which gives an injection* $H^1(\mathbb{Z}, A(C')) \to H^1(C, A)$*. Because* $\mathbb{Z}$ *acts trivially on* $A(C')$ *we can identify* $H^1(\mathbb{Z}, A(C')) = \mathrm{Hom}_{gp}(\mathbb{Z}, A(C')) \simeq A(C')$*. Following the identifications, given a point* $p \in A(C')$*, the associated* $A$*-torsor in* $H^1(C, A)$*, is the one given by the Galois descent described above. Since* $p$ *is of infinite order* $X$ *is also of infinite order, thus by Lemma 3.3.8,* $X$ *cannot be quasi-projective.*

*Furthermore, the* $A$*-torsor described above is representable. Indeed, all the local rings of* $C$ *are Noetherian of dimension* $\leq 1$*. Hence, by [Ray70, Theorem. XI 3.1.1] we conclude that* $X$ *is representable.*

## 3.4 Brauer groups for Abelian Torsors

In this section, we prove $\mathrm{Br}(X) = \mathrm{Br}'(X)$ for $X$ an $A$-torsor over $S$, where $S$ is a reduced quasi-projective scheme over a field, and $A$ has the property that $A[p]$ is isotrivial for $p = \mathrm{char}\, k$.

The following theorem will appear in the Master's thesis of Francisco Gallardo, and it is based on a result by Hoobler [Hoo72, Theorem 3.3].

**Theorem 3.4.1.** *Let $S$ be the spectrum of a strictly local Henselian ring which contains a field $k$, and $A$ an abelian scheme over $S$. Then for all $n \in \mathbb{N}$, exists an $m$ only depending on $n$ (and not on $A$ or $S$) such that the morphism $m : A \to A$ induces the zero map $H^2(A, \mu_n) \to H^2(A, \mu_n)$.*

We mention a natural question proposed to us by Cristian Aviles:

**Problem 3.** *Does the analogous result of Theorem 3.4.1 hold for semi-abelian schemes? This would be interesting, as semi-abelian schemes appear naturally as Picard schemes (see, e.g., [KM23, Remark 4.9]).*

The following is the main theorem of this thesis.

**Theorem 3.4.2.** *Let $S$ be a reduced quasi-projective scheme over a field $k$ with $\operatorname{char} k = p$, and $A$ an abelian scheme over $S$ such that $A[p]$ is isotrivial. If $X$ is an $A$-torsor over $S$ then $\operatorname{Br}(X) = \operatorname{Br}'(X)$.*

*Proof.* From the Kummer sequence;

$$0 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0$$

we can derive an exact sequence of groups

$$0 \to \operatorname{Pic}(X)/n\operatorname{Pic}(X) \to H^2(X, \mu_n) \to \operatorname{Br}'(X)[n] \to 0.$$

Let $\alpha \in \operatorname{Br}'[n]$ be a cohomology class and $\beta \in H^2(X, \mu_n)$ a preimage of $\alpha$. Due to Theorem 2.3.7, to prove that $\alpha \in \operatorname{Br}(X)$, it suffices to show the existence of a finite flat cover $S' \to S$, such that $\beta_{S'} = 0$. We proceed to prove the existence of such a cover.

Let $f : X \to S$ be the structure morphism, Proposition 3.3.3 implies that $f_*\mathscr{O}_X \simeq \mathscr{O}_S$, in particular $f_*\mu_n \simeq \mu_n$. From the Leray spectral sequence associated to $f$ and $\mu_n$ (see [Mil80, Theorem 1.18]), we derive the sequence;

$$H^2(S, \mu_n) \to \ker[H^2(X, \mu_n) \to H^0(S, R^2 f_*\mu_n)] \to H^1(S, \operatorname{Pic}_{X/S}[n]).$$

Let us suppose that the class $\beta$ lives inside $\ker[H^2(X,\mu_n) \to H^0(S, R^2 f_*\mu_n)]$. Then, its image in $H^1(S, \mathrm{Pic}_{X/S}[n])$ is represented by a $\mathrm{Pic}_{X/S}[n]$-torsor, since $\mathrm{Pic}_{X/S}[n]$ is a finite flat group scheme, the image of $\beta$ can be killed after a finite flat cover. Thus, we can assume that $\beta$ comes from an element in $H^2(S,\mu_n)$. However, since $S$ is a quasi-projective scheme over a field, it follows from Theorems 3.2.4 and 2.4.7, that classes in $H^2(S,\mu_n)$ are also killed by finite flat covers. Therefore, the same holds for $\beta$.

The above argument shows that it suffices to prove that after a finite flat cover, classes in $H^2(X,\mu_n)$ are zero under the canonical morphism to $H^0(S, R^2 f_*\mu_n)$. One sufficient condition for $\beta$ to be zero in $H^0(S, R^2 f_*\mu_n)$, is that for every point $s \in S$, the morphism $H^2(X,\mu_n) \to H^2(X_{\mathscr{O}^{sh}_{S,s}},\mu_n)$ sends $\beta$ to zero (since $H^2(X_{\mathscr{O}^{sh}_{S,s}},\mu_n)$ is the stalk of $R^2 f_*\mu_n$ at $s$). Since $X$ is an $A$-torsor and $A$ is a smooth group, then

$$X_{\mathscr{O}^{sh}_{S,s}} \simeq A_{\mathscr{O}^{sh}_{S,s}}.$$

Hence, by Theorem 3.4.1, there exists an $m$, such that $[m] : A_{\mathscr{O}^{sh}_{S,s}} \to A_{\mathscr{O}^{sh}_{S,s}}$ kills classes in $H^2(A_{\mathscr{O}^{sh}_{S,s}},\mu_n)$ for all $s \in S$. Therefore, if we can find a morphism of $A$-torsors $X' \to X$, which is locally of the form $[m] : A \to A$, then we would have found a finite flat cover of $X$, such that $\beta_{X'}$ is zero under the morphism $H^2(X'_{\mathscr{O}^{sh}_{S,s}},\mu_n) \simeq H^2(A_{\mathscr{O}^{sh}_{S,s}},\mu_n)$ for all $s \in S$. We claim that a morphism with these properties exists. Consider the sequence

$$0 \to A[m] \to A \xrightarrow{[m]} A \to 0$$

and take cohomology to get;

$$H^1(S,A) \xrightarrow{\gamma_m} H^1(S,A) \to H^2(S,A[m]).$$

A morphism $X' \to X$ as the one described above exists, if and only if, the class of $X$ in the group $H^1(S,A)$ lives in the image of the morphism $\gamma_m : H^1(S,A) \to H^1(S,A)$ (see Proposition 1.2.2). By exactness, it suffices to show that up to a finite flat base change $S' \to S$, $X$ maps to zero in $H^2(S,A[m])$.

If $(p,m) = 1$ then $A[m]$ is an étale group scheme, thus classes in $H^2(S,A[m])$ can be killed after a finite flat cover by Theorem 3.2.5 and Corollary 2.4.7. If $m = p^r$ we have an exact sequence

$$0 \to A[p^{r-1}] \to A[p^r] \to A[p] \to 0,$$

induction on $r$ and Lemma 3.2.2 allows us to reduce to the case $A[p]$. Since $A[p]$ is isotrivial, it follows from Theorem 3.2.7 and Corollary 2.4.7 that after a finite flat cover, classes in $H^2(S, A[m])$ are 0. Finally, the mixed case $m = p^r n$ with $(p, n) = 1$ follows from the decomposition $A[m] \simeq A[p^r] \times_S A[n]$ and applying the same argument as before.

In summary, after replacing $S$ with a finite flat cover, the morphism $X' \to X$ exists, so we can assume $\beta \in \ker[H^2(X, \mu_n) \to H^0(S, R^2 f_* \mu_n)]$, and the theorem follows. □

*Remark* 3.4.3. Example 3.3.9 implies the existence of non-quasi-projective $A$-torsors. Therefore, Theorem 3.4.2 is not a direct consequence of Gabber's Theorem 1.3.7.

**Definition 3.4.4.** *Let $k$ be a field of characteristic $p$ and $S$ a scheme over $k$. An abelian scheme $A$ over $S$ is said to be ordinary if for every geometric point* $\operatorname{Spec} K \xrightarrow{s} S$*, the group $A[p]_s$ is isomorphic to $(\mathbb{Z}/p)^g \times \mu_p^g$.*

**Theorem 3.4.5.** *Let $k$ be a field of characteristic $p$, $S$ a quasi-projective scheme over $k$, and $A$ an ordinary abelian scheme over $S$. If $X$ is an $A$-torsor over $S$ then* $\operatorname{Br}(X) = \operatorname{Br}'(X)$.

*Proof.* Following the same argument as in Theorem 3.4.2, we reduce to showing that $A[p]$ -gerbes are quotient stacks. Since $A$ is ordinary, by definition $A[p]$ has constant separable rank. Thus, by Messing's theorem 3.1.6, there exists a sequence

$$0 \to A[p]^0 \to A[p] \to A[p]^{\text{ét}} \to 0$$

where $A[p]^0$ is radiciel, $A[p]^{\text{ét}}$ is étale, and the sequence is compatible with base change. Hence, when restricting to a geometric fiber $\operatorname{Spec} k \xrightarrow{s} S$, we get $A[p]^0 \simeq \mu_p^g$. Then, [ABD+66, Corollaire 4.8] implies that $A[p]^0$ is of multiplicative type. Therefore, by Lemma 3.2.2 and Theorems 3.2.4, 3.2.5 we conclude that $A[p]$ gerbes are quotient stacks, so the theorem holds. □

*Remark* 3.4.6. It is not difficult to find examples of abelian schemes $A$ with the property that $A[p]$ is isotrivial. In fact, the groups $A[p]$ have been used to construct a stratification of the moduli space of polarized abelian schemes over $k$ (see [Oor01, Definition 3.4] and [PU21, Proposition 3.3.5]). In other words, if $\mathscr{A}$ is the moduli space of abelian schemes over $\mathbb{F}_p$, $A$ is an abelian scheme over a reduced, quasi-projective scheme $S$ over $\mathbb{F}_p$, then $A[p]$ is isotrivial if and only if the image of the morphism $S \to \mathscr{A}$ defined by $A$ is contained in just one Ekedahl-Oort stratum [Oor01, Definition 3.4]. For instance, the ordinary locus is a dense open subset in $\mathscr{A}$, in other words, most abelian schemes are ordinary and have isotrivial $p$-torsion.

# Bibliography


[ABD+66] Michael Artin, Jean-Etienne Bertin, Michel Demazure, Alexander Grothendieck, Pierre Gabriel, Michel Raynaud, and Jean-Pierre Serre. *Schémas en groupes.* Séminaire de Géométrie Algébrique de l'Institut des Hautes Études Scientifiques. Institut des Hautes Études Scientifiques, Paris, 1963/1966.

[AG60] Maurice Auslander and Oscar Goldman. The Brauer group of a commutative ring. *Trans. Amer. Math. Soc.*, 97:367–409, 1960.

[AHR20] Jarod Alper, Jack Hall, and David Rydh. A Luna étale slice theorem for algebraic stacks. *Ann. of Math. (2)*, 191(3):675–738, 2020.

[Alp25] Jarod Alper. Stacks and moduli. https://sites.math.washington.edu/~jarod/moduli.pdf, 2025. Accessed: 2025-05-21.

[Azu51] Gorô Azumaya. On maximally central algebras. *Nagoya Math. J.*, 2:119–150, 1951.

[BBGN07] Vikraman Balaji, Indranil Biswas, Ofer Gabber, and Donihakkalu S. Nagaraj. Brauer obstruction for a universal vector bundle. *Comptes Rendus Mathematique*, 345(5):265–268, 2007.

[Ber72] V. G. Berkovich. The Brauer group of abelian varieties. *Funkcional. Anal. i Priložen.*, 6(3):10–15, 1972.

[Ber05] Inta Bertuccioni. Brauer groups and cohomology. *Arch. Math. (Basel)*, 84(5):406–411, 2005.

[BHM25] Daniel Bragg, Jack Hall, and Siddharth Mathur. Unipotent morphisms. https://arxiv.org/abs/2110.15041, 2025. arXiv:2110.15041.

[BL24] Daniel Bragg and Max Lieblich. Murphy's law for algebraic stacks. https://arxiv.org/abs/2402.00862, 2024. arXiv:2402.00862.

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990.

[Bra32] Richard Brauer. Über die algebraische struktur von schiefkörpern. *Journal für die reine und angewandte Mathematik*, 166:241–248, 1932.

[Con05] Brian Conrad. Keel-mori theorem via stacks. *preprint*, 2005.

[CTS21] Jean-Louis Colliot-Thélène and Alexei N. Skorobogatov. *The Brauer-Grothendieck group*, volume 71 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Cham, [2021] ©2021.

[Deb17] Raymond Debremaeker. Cohomology with values in a sheaf of crossed groups over a site. https://arxiv.org/abs/1702.02128, 2017. arXiv:1702.02128.

[Del73] Pierre Deligne. Cohomologie a supports propres. In *Théorie des Topos et Cohomologie Etale des Schémas*, pages 250–480, Berlin, Heidelberg, 1973. Springer Berlin Heidelberg.

[Dem86] Michel Demazure. *Lectures on p-divisible groups*, volume 302 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. Reprint of the 1972 original.

[dGF25] Olivier de Gaay Fortman. Descent for algebraic stacks. https://arxiv.org/abs/2504.13642, 2025. arXiv:2504.13642.

[DHM22] Neeraj Deshmukh, Amit Hogadi, and Siddharth Mathur. Quasi-affineness and the 1-resolution property. *Int. Math. Res. Not. IMRN*, Vol. 2022(3):2224–2249, 2022.

[dJ03] Aise Johan de Jong. A result of gabber. *preprint*, 25:36–57, 2003.

[dJLS23] Aise Johan de Jong, Max Lieblich, and Minseon Shin. Locally free twisted sheaves of infinite rank. *Doc. Math.*, 28(1):133–171, 2023.

[DM69] Pierre Deligne and David Mumford. The irreducibility of the space of curves of given genus. *Publications mathématiques de l'IHÉS*, 36, 01 1969.

[EHKV01] Dan Edidin, Brendan Hassett, Andrew Kresch, and Angelo Vistoli. Brauer groups and quotient stacks. *Amer. J. Math.*, 123(4):761–777, 2001.

[Gab78] Ofer Gabber. *SOME THEOREMS ON AZUMAYA ALGEBRAS*. ProQuest LLC, Ann Arbor, MI, 1978. Thesis (Ph.D.)–Harvard University.

[GD61] Alexander Grothendieck and Jean Dieudonné. éléments de géométrie algébrique. III. étude cohomologique des faisceaux cohérents. I. *Inst. Hautes Études Sci. Publ. Math.*, 11:167, 1961.

[Gir71] Jean Giraud. *Cohomologie non abélienne*, volume Band 179 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin-New York, 1971.

[Gro68a] Alexander Grothendieck. Le groupe de Brauer. I. Algèbres d'Azumaya et interprétations diverses. In *Dix exposés sur la cohomologie des schémas*, volume 3 of *Adv. Stud. Pure Math.*, pages 46–66. North-Holland, Amsterdam, 1968.

[Gro68b] Alexander Grothendieck. Le groupe de Brauer. III. Exemples et compléments. In *Dix exposés sur la cohomologie des schémas*, volume 3 of *Adv. Stud. Pure Math.*, pages 88–188. North-Holland, Amsterdam, 1968.

[Gro95] Alexander Grothendieck. Le groupe de Brauer. II. Théorie cohomologique [MR0244270 (39 #5586b)]. In *Séminaire Bourbaki, Vol. 9*, pages Exp. No. 297, 287–307. Soc. Math. France, Paris, 1995.

[GS06] Philippe Gille and Tamás Szamuely. *Central simple algebras and Galois cohomology*, volume 101 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.

[GW20] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I. Schemes—with examples and exercises*. Springer Studium Mathematik—Master. Springer Spektrum, Wiesbaden, second edition, [2020] ©2020.

[GW23] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry II: Cohomology of schemes—with examples and exercises.* Springer Studium Mathematik—Master. Springer Spektrum, Wiesbaden, [2023] ©2023.

[Har77] Robin Hartshorne. *Algebraic geometry*, volume No. 52 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Heidelberg, 1977.

[Har10] Robin Hartshorne. *Deformation theory*, volume 257 of *Graduate Texts in Mathematics*. Springer, New York, 2010.

[Hoo72] Raymond T. Hoobler. Brauer groups of abelian schemes. *Annales scientifiques de l'École Normale Supérieure, Série 4*, 5(1):45–70, 1972.

[Hoo80] Raymond T. Hoobler. A cohomological interpretation of Brauer groups of rings. *Pacific J. Math.*, 86(1):89–92, 1980.

[HR18] Jack Hall and David Rydh. Addendum to "Étale dévissage, descent and pushouts of stacks" [j. algebra 331 (1) (2011) 194–223]. *Journal of Algebra*, 498:398–412, March 2018.

[HS03] Daniel Huybrechts and Stefan Schröer. The Brauer group of analytic $K3$ surfaces. *Int. Math. Res. Not.*, pages 2687–2698, 2003.

[HS09] Jochen Heinloth and Stefan Schröer. The bigger Brauer group and twisted sheaves. *J. Algebra*, 322(4):1187–1195, 2009.

[JLM24] Ariyan Javanpeykar, Daniel Loughran, and Siddharth Mathur. Good reduction and cyclic covers. *J. Inst. Math. Jussieu*, 23(1):463–494, 2024.

[JM22] Ariyan Javanpeykar and Siddharth Mathur. Smooth hypersurfaces in abelian varieties over arithmetic rings. *Forum of Mathematics, Sigma*, 10:e97, 2022.

[KM97] Seán Keel and Shigefumi Mori. Quotients by groupoids. *Annals of Mathematics*, 145(1):193–213, 1997.

[KM23] Andrew Kresch and Siddharth Mathur. Formal gaga for gerbes, 2023. arXiv:2305.19114.

[Kre09] Andrew Kresch. On the geometry of Deligne-Mumford stacks. In *Algebraic geometry—Seattle 2005. Part 1*, volume 80 of *Proc. Sympos. Pure Math.*, pages 259–271. Amer. Math. Soc., Providence, RI, 2009.

[KV04] Andrew Kresch and Angelo Vistoli. On coverings of Deligne-Mumford stacks and surjectivity of the Brauer map. *Bull. London Math. Soc.*, 36(2):188–192, 2004.

[Lie08] Max Lieblich. Twisted sheaves and the period-index problem. *Compositio Mathematica*, 144(1):1–31, January 2008.

[LMB00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.

[LS24] Bruno Laurent and Stefan Schröer. Para-abelian varieties and Albanese maps. *Bull. Braz. Math. Soc. (N.S.)*, 55(1):39, 2024. Id/No 4.

[Mat18] Siddharth Mathur. *Some theorems on the Resolution Property and the Brauer map*. Ph.d. thesis, University of Washington, 2018. Available at http://hdl.handle.net/1773/42453.

[Mat21] Siddharth Mathur. The resolution property via Azumaya algebras. *J. Reine Angew. Math.*, 774:93–126, 2021.

[Mat22] Siddharth Mathur. Experiments on the Brauer map in high codimension. *Algebra Number Theory*, 16(3):747–775, 2022.

[Mat23] Siddharth Mathur. Extending vector bundles on curves. *Math. Res. Lett.*, 29(5):1537–1550, 2023.

[Mes72] William Messing. *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*, volume Vol. 264 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1972.

[Mil80] James S. Milne. *Étale cohomology*, volume No. 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1980.

[MP73] M. Pavaman Murthy and Claudio Pedrini. $K_0$ and $K_1$ of polynomial rings. In *Algebraic $K$-theory, II: "Classical" algebraic $K$-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, volume

Vol. 342 of *Lecture Notes in Math.*, pages 109–121. Springer, Berlin-New York, 1973.

[MS23] Siddharth Mathur and Stefan Schröer. The resolution property holds away from codimension three. *Trans. Amer. Math. Soc.*, 376(2):1041–1063, 2023.

[Mum70] David Mumford. *Abelian varieties*, volume 5 of *Tata Institute of Fundamental Research Studies in Mathematics*. Tata Institute of Fundamental Research, Bombay; by Oxford University Press, London, 1970.

[Ols16] Martin Olsson. *Algebraic spaces and stacks*, volume 62 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2016.

[Oor01] Frans Oort. A stratification of a moduli space of abelian varieties. In *Moduli of abelian varieties (Texel Island, 1999)*, volume 195 of *Progr. Math.*, pages 345–416. Birkhäuser, Basel, 2001.

[Poo17] Bjorn Poonen. *Rational points on varieties*, volume 186 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2017.

[PU21] Rachel Pries and Douglas Ulmer. On $BT_1$ group schemes and Fermat curves. *New York J. Math.*, 27:705–739, 2021.

[Ray70] Michel Raynaud. *Faisceaux amples sur les schémas en groupes et les espaces homogènes*, volume Vol. 119 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1970.

[Roq05] Peter Roquette. *The Brauer-Hasse-Noether theorem in historical perspective*, volume 15 of *Schriften der Mathematisch-Naturwissenschaftlichen Klasse der Heidelberger Akademie der Wissenschaften [Publications of the Mathematics and Natural Sciences Section of Heidelberg Academy of Sciences]*. Springer-Verlag, Berlin, 2005.

[Ryd15] David Rydh. Do line bundles descend to coarse moduli spaces of artin stacks with finite inertia?, 2015. Accessed: 2025-05-23.

[Sch01] Stefan Schröer. There are enough Azumaya algebras on surfaces. *Math. Ann.*, 321(2):439–454, 2001.

[Sch05] Stefan Schröer. Topological methods for complex-analytic Brauer groups. *Topology*, 44(5):875–894, 2005.

[Shi21] Minseon Shin. The cohomological Brauer group of a torsion $\mathbb{G}_m$-gerbe. *Int. Math. Res. Not. IMRN*, pages 14480–14507, 2021.

[Shi23] Minseon Shin. The cohomological Brauer group of weighted projective spaces and stacks. *Pacific J. Math.*, 324(2):353–370, 2023.

[Shi24] Minseon Shin. The $\mathrm{Br} = \mathrm{Br}'$ question for some classifying stacks. *Comm. Algebra*, 52(2):747–762, 2024.

[Sta25] The Stacks project authors. Stacks project. https://stacks.math.columbia.edu, 2025.

[Sti12] Jakob Stix. A course on finite flat group schemes and p-divisible groups. https://www.math.uni-frankfurt.de/~stix/skripte/STIXfinflatGrpschemes20120918.pdf, 2012. Lecture notes.

[TO70] John Tate and Frans Oort. Group schemes of prime order. *Ann. Sci. École Norm. Sup. (4)*, 3:1–21, 1970.

[Toe12] Bertrand Toen. Derived Azumaya algebras and generators for twisted derived categories. *Invent. Math.*, 189(3):581–652, 2012.

[Tot04] Burt Totaro. The resolution property for schemes and stacks. *J. Reine Angew. Math.*, 577:1–22, 2004.

[Vis07] Angelo Vistoli. Notes on grothendieck topologies, fibered categories and descent theory. https://arxiv.org/abs/math/0412512, 2007. arXiv:math/0412512.

[Zha00] Zhaohui Zhang. *On moduli stacks of finite group schemes*. ProQuest LLC, Ann Arbor, MI, 2000. Thesis (Ph.D.)–Massachusetts Institute of Technology.