Pontificia Universidad Católica de Chile

Master's thesis

Deformation cone of polytopes: H-representation and SageMath implementation

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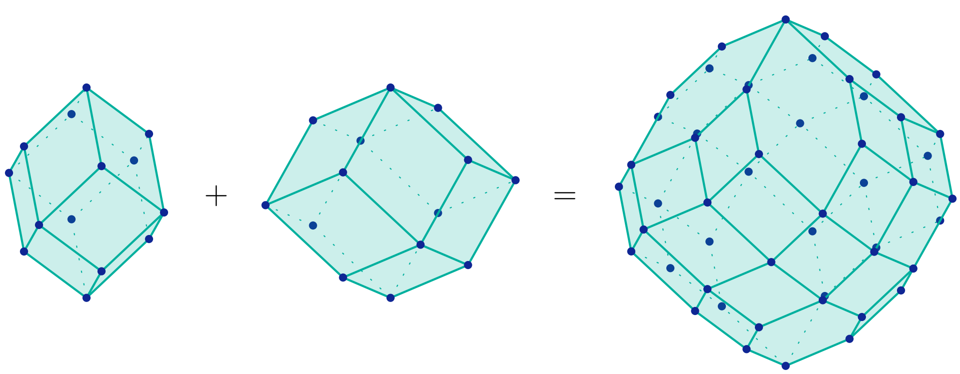

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Contents

1 Introduction

Minkowski sums are a basic polytope operation that has been studied for over a century. They were given that name in honor of mathematician Hermann Minkowski, who did not define them, but provided one of the first results about sums of polytopes, the Brunn-Minkowski Theorem. However, there were not a lot significant results in the subject until the development of computers, when the study of linear programming gained interest. Since then, a lot of applications have been found for this deceivingly simple operation, firstly in algebraic contexts where it has been used in defining mixed volumes and calculating Gröbner bases [10]. And secondly, it has also some practical applications in fields like motion planning [13], collision detection [8] and computational biology [3].

The *Minkowski sum* of two polytopes P and Q is defined as

$$\mathsf{P} + \mathsf{Q} = \{\mathbf{p} + \mathbf{q} \,:\, \mathbf{p} \in \mathsf{P}, \mathbf{q} \in \mathsf{Q}\}$$

We say a polytope Q is a *summand* of P if there is another polytope R such that $\mathsf{P} = \mathsf{Q} + \mathsf{R}$ and it is a *weak summand* if $\lambda\mathsf{Q}$ is a summand of P for some $\lambda > 0$. A polytope is *indecomposable* if its only summands are dilations of itself. A natural question that arises from these definitions is, when is a polytope indecomposable? and how can we write a given polytope as a sum of indecomposable ones? This was first explored more that 50 years ago by Shephard [11], when he described a way of detecting when a particular polytope is summand of another. But the question of how to decompose a polytope still remained.

In 1973, McMullen defined the Type cone [15]. This was used to describe combinatorically isomorphic polytopes; nevertheless, its closure provides a parametrization of the set of weak summands of the polytope. Later, the concept of Nef cone was defined in algebraic geometry (see Chapter 6 of [5]) for projective varieties, but this can also be applied to the normal fan of polytopes to characterize its weak summands. This means there are at least two different ways of representing the cone of summands of a polytope, in general this polyhedron is called *deformation cone* of P.

Both of these cones are defined by equations and inequalities, the Type cone by 1-Minkowski weights and the Nef cone by wall-crossing inequalities. The rays of these cones correspond to indecomposable polytopes, so in order to compute a decomposition of a polytope in indecomposable summands we need to obtain the representation of the cone by ray generators. To reduce the amount of inequalities, and thus making the computation easier, in 1991 Batyrev proposed a representation of the Nef cone based in primitive collections of rays in the normal fan of the polytope [2]. Originally, this was done only for smooth fans, and the proof of this characterization for general fans was not done until 2008, when Cox and Renesse [6] published two proofs, both of them using tools from algebraic geometry. With this result, it is possible to consider only the wall-crossing inequalities that have positive coefficients in a primitive collection of rays.

On the other hand, techniques have been developed to detect indecomposable polytopes by studying polytope graphs, as it was presented by Kallay in 1982 [12]. But this result only detects when a polytope is indecomposable, and it does not provide a decomposition in the case it is not.

The main goals of this thesis are to provide proofs of the aforementioned results using only elemental properties of polytopes and implementing these definitions in SageMath to construct the deformation cone of a polytope and build its indecomposable summands.

The first chapter consists of preliminaries following mostly the books "Lectures on polytopes" by Günter Ziegler [20] and "Convex polytopes" by Branko Grünbaum [11]. Firstly, polytopes are defined as the convex hull of a finite set of points and as the intersection of a finite amount of halfspaces along with some of their basic properties, like support function, face lattice and a couple of classic examples like simple and simplicial polytopes. Then, we introduce the normal fan of a polytope and study when a function over the normal fan is the support function of a polytope along with some characterizations of piecewise linear convex functions. After this, we define the Minkowski sum of polytopes along with some properties, we also define Minkowski difference as implemented in SageMath along with some equivalent definitions.

The second chapter contains different ways of constructing the deformation cone of a polytope. In first place, we present Shephard's criterion, which is proved by analyzing the convexity of the difference of support functions, as opposed to Shephard's proof which constructed the summand explicitly. Then, we introduce 1-Minkowski weights as defined by Federico Castillo et al. [4] and use them to construct the Type cone of a polytope, and we define wall-crossing inequalities as defined by Federico Ardila et al. [1] and use them to construct the Nef cone. After this, we reduce the amount of inequalities by introducing the concept of primitive collections of rays in the normal fan and applying Batyrev's criterion. Finally, we show some examples of indecomposable polytopes and we use Batyrev's criterion to count the number of indecomposable summands of a $d-$cube.

In the third chapter, we explain how some of these concepts were implemented in the mathematical software SageMath and we analyze the time it takes to construct the cone of summands using the different methods previously presented.

The contributions of this thesis are as follows. First, we compile results that have appeared separately in the literature, but never been compiled with a consistent notation. Second, we provide proofs of the aforementioned results using only elemental discrete geometry tools, so this text can be useful as documentation for its implementation in SageMath or as an introduction to polytope decomposition without the need of algebraic geometry tools. Third, we provide an implementation of these results in SageMath, so it can be used in the future for further experimentation. Finally, we provide an alternative and shorter proof for one of the results in [4] about the deformation cone of $d-$cubes.

2 Preliminaries

2.1 Polytopes

Definition 2.1.1. An H-polytope is the bounded intersection of finitely many halfspaces. That is to say, there exists $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^d \setminus \{0\}$ and $b_1, \ldots, b_n \in \mathbb{R}$ such that

$$\mathsf{P} = \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}_1^t\mathbf{x} \leq b_1, \ldots, \mathbf{a}_n^t\mathbf{x} \leq b_n\right\}.$$

The inequalities defining an H-polytope may not be minimal. If an inequality can be omitted without altering P, we will say the inequality is *redundant*, if not, it is *irredundant*. In general, we would like to work with inequalities such that none of them are redundant, this minimal set exists and is unique (up to scaling).

Definition 2.1.2. The *convex hull* of a set $X \subset \mathbb{R}^d$ is defined by

$$\operatorname{conv}(X) := \left\{\lambda_1\mathbf{a}_1 + \cdots + \lambda_n\mathbf{a}_n : \mathbf{a}_i \in X, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1\right\}.$$

A *V-polytope* P is defined as the convex hull of a finite set of points in $\mathbb{R}^d$.

This definition also allows redundancy. A point in V will be redundant if its elimination from V yields the same polytope P. We will define the *vertex set* of a polytope as the minimal set of points V such that $\mathsf{P} = \operatorname{conv}(V)$. This set always exists and is unique, it will be denoted as $\operatorname{vert}(\mathsf{P})$.

Theorem 2.1.3 (Minkowski-Weyl)**.** *A set $\mathsf{P} \subset \mathbb{R}^d$ is an H-polytope if and only if it is a V-polytope.*

This theorem can be generalized for polyhedra, in that case, the foward direction can be proved by writing the polyhedron as the intersection between a polyhedral cone and an affine subspace and showing by the double description method that the intersection between a polyhedral cone and an affine subspace is a polyhedron. The other direction can be proved by writing a V-polyhedron as the projection of an H-polyhedron of higher dimension and then using Fourier-Motzkin elimination to show that the projection of an H-polyhedron is also an H-polyhedron. Finally, the theorem for polytopes can be concluded from the fact that polytopes are just compact polyhedra. For more details, see Chapter 1 of [20].

Given the previous equivalence, we will simply call H-polytopes and V-polytopes, polytopes. It is important to note that polytopes are always compact and convex sets, properties that are a direct consequence of both their definitions.

Definition 2.1.4. Let $\mathsf{P} \subset \mathbb{R}^d$ be a polytope and $\mathcal{L}$ the set of affine subspaces of $\mathbb{R}^d$ that contain P. The *affine hull* of P is defined as

$$\operatorname{aff}(\mathsf{P}) := \bigcap_{L \in \mathcal{L}} L.$$

A set of n points is *affinely independent* if $\dim(\operatorname{aff}(\mathsf{P})) = n$.

The *relative interior* of P is defined as the interior of P relative to $\operatorname{aff}(\mathsf{P})$, and it is denoted by $\operatorname{relint}(\mathsf{P})$.

The *dimension* of P is defined as the dimension of its affine hull.

From now on, a d-polytope will refer to a polytope of dimension d.

Example 2.1.5. An important example of d-polytope is the d-simplex. This is the convex hull of $d+1$ affinely independent points in $\mathbb{R}^d$.

Definition 2.1.6. Given a polytope P we will define its *support function* as $h_{\mathsf{P}}(\mathbf{c}) : \mathbb{R}^d \to \mathbb{R}$ such that

$$h_{\mathsf{P}}(\mathbf{c}) := \max_{\mathbf{x}\in\mathsf{P}}\{\mathbf{c}^t\mathbf{x}\}.$$

Given a $\mathbf{c} \in \mathbb{R}^d$ we can define its corresponding *face* in P by

$$\mathsf{P}^{\mathbf{c}} := \left\{\mathbf{x} \in \mathsf{P} : \mathbf{c}^t\mathbf{x} = h_{\mathsf{P}}(\mathbf{c})\right\}. \tag{1}$$

By convention, $\emptyset$ is also a face of P.

The support function is well defined because polytopes are compact, so there is always an element in P that maximizes any given linear map.

Proposition 2.1.7. *Given a d-polytope P with vertex set $V = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ and $\mathbf{c} \in \mathbb{R}^d$,*

$$\mathsf{P}^{\mathbf{c}} = \operatorname{conv}(\{\mathbf{v}_i : \mathbf{c}^t(\mathbf{v}_i) = h_{\mathsf{P}}(\mathbf{c})\}). \tag{2}$$

Moreover, the faces of a polytope are also polytopes.

Proof. Let $\mathsf{F} := \operatorname{conv}(\{\mathbf{v}_i : \mathbf{c}^t(\mathbf{v}_i) = h_{\mathsf{P}}(\mathbf{c})\})$. Clearly, $\mathsf{F} \subset \mathsf{P}^{\mathbf{c}}$, so it is only necessary to prove the other containment. Let $\mathbf{x}$ be a point in $\mathsf{P}^{\mathbf{c}}$, then $\mathbf{x} \in \mathsf{P}$, which means there are $\lambda_1, \dots, \lambda_n \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$ and $\mathbf{x} = \sum_{i=1}^n \lambda_i\mathbf{v}_i$. Then,

$$\begin{aligned} h_{\mathsf{P}}(\mathbf{c}) &= \mathbf{c}^t\mathbf{x} \\ &= \mathbf{c}^t\sum_{i=1}^n \lambda_i\mathbf{v}_i \\ &= \sum_{i=1}^n \lambda_i\mathbf{c}^t\mathbf{v}_i. \end{aligned}$$

Let $\mathbf{v}_j$ be a vertex such that $\mathbf{c}^t\mathbf{v}_j = \max_{j\in[m]}\{\mathbf{c}^t\mathbf{v}_j\}$. Then,

$$\begin{aligned} h_{\mathsf{P}}(\mathbf{c}) &\leq \sum_{i=1}^n \lambda_i\mathbf{c}^t\mathbf{v}_j \\ &= \mathbf{c}^t\mathbf{v}_j \end{aligned}$$

so we know that $\mathsf{F} \neq \emptyset$. If $\mathbf{x} \notin \operatorname{conv}(\{\mathbf{v}_i : \mathbf{c}^t(\mathbf{v}_i) = h_{\mathsf{P}}(\mathbf{c})\})$ then there is a $\lambda_k \neq 0$ such that $\mathbf{v}_k \notin \mathsf{F}$ and

$$\sum_{i=1}^n \lambda_i\mathbf{c}^t\mathbf{v}_i < \sum_{i\neq k} \lambda_i\mathbf{c}^t\mathbf{v}_i + \lambda_k\mathbf{c}^t\mathbf{v}_j \tag{3}$$

which means that $\sum_{i=1}^n \lambda_i\mathbf{c}^t\mathbf{v}_i < h_{\mathsf{P}}(\mathbf{c})$. $\rightarrow\!\leftarrow$

Finally, the fact that the faces of polytopes are polytopes is a consequence of it being the convex hull of a finite set of points. □

Proposition 2.1.8. *Every polytope has a finite number of faces.*

Proof. Let $\mathsf{P} \subset \mathbb{R}^d$ be a polytope, then every face is defined as the convex hull of a subset of the vertices of P. The number of vertices is finite, so the number of faces is also finite. □

We will call $\mathcal{F}_d(\mathsf{P})$ the set of faces of P of dimension d. If P is a polytope of dimension d, then we will call *edges* its faces of dimension 2 and *facets* its faces of dimension $d-1$. If we have an irredundant set of inequalities that define a polytope, we will say the inequalities are *facet defining*, because for full dimensional polytope, each irredundant inequality intersects with the polytope in a facet.

Example 2.1.9. When all the faces of a polytope are simplices, then we will say the polytope is *simplicial*. Notice that for a polytope to be simplicial it is enough for its facets to be simplicial, because all subsets of affinely independent sets are affinely independent, so faces of simplices are also simplices.

On the other hand, we say a d-polytope is *simple* if every vertex is in exactly d edges (also in d facets).

The support function provides another way of characterizing polytopes, this follows from the fact that given h_{P} then there is a finite set $\{\mathbf{a}_i\}_{i=1}^n \subset \mathbb{R}^d \setminus \{0\}$ such that

$$\mathsf{P} = \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}_1^t\mathbf{x} \leq h_{\mathsf{P}}(\mathbf{a}_1), \ldots, \mathbf{a}_n^t\mathbf{x} \leq h_{\mathsf{P}}(\mathbf{a}_n)\right\}. \tag{4}$$

Naturally, that suggests the question of when is a function the support function of a polytope. This will be studied in the following section.

Definition 2.1.10. The *face lattice* of a polytopes P is the partially ordered set given by the faces of P ordered by inclusion. Two polytopes are *combinatorially equivalent* if their face lattices are isomorphic.

Example 2.1.11. The face lattice of a d-simplex is always the boolean lattice so all d-simplices are combinatorially equivalent.

Example 2.1.12. The standard $d-$cube is the polytope generated by taking the convex hull of all the points in $\mathbb{R}^d$ with 0-1 coordinates. Equivalently, it is the polytope such that for every vector in the canonical basis $\mathbf{e}_i \in \mathbb{R}^d$ it satisfies the facet-defining inequalities

$$\mathbf{e}_i^t\mathbf{x} \leq 1 \quad \text{and} \quad -\mathbf{e}_i^t\mathbf{x} \leq 0$$

Figure 1 shows a standard $2-$cube and its face lattice.

We will denote the facets defined by $\mathbf{e}_i^t\mathbf{x} \leq 1$ as F_i and the facets defined by $-\mathbf{e}_i^t\mathbf{x} \leq 0$ as F_i'. Then, edges of the polytope are given by vertices $\mathbf{v} = (v_1, \ldots, v_d)$ and $\mathbf{w} = (w_1, \ldots, w_d)$ for which there is an $i \in \{1, \ldots, d\}$ such that $w_j = v_j$ for all $j \neq i$ and $w_i \neq v_i$. Then, the edges are the intersection of the facets F_j when $v_j = 1$ and F_j' when $v_j = 0$ for every $j \neq i$. This means, the only pairs of facets that have empty intersection are F_i and F_i'.

In general, we will call any $d-$polytope that has d pairs of facets F_i and F_i' such that $\mathsf{F}_i \cap \mathsf{F}_i' = \emptyset$ and all faces are the intersection of at most one facet in each pair, a $d-$cube. All $d-$cubes have the same face lattice and are combinatorically equivalent to the standard $d-$cube.

2.2 Normal fan

Definition 2.2.1. A *cone* is the intersection of finitely many hyperplanes such that it has only one vertex. That is to say, there exists $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^d \setminus \{0\}$ and $b_1, \ldots, b_n \in \mathbb{R}$ such that

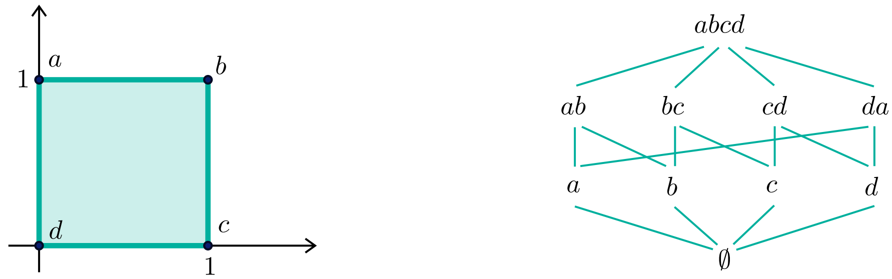
$$\mathsf{C} = \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}_1^t\mathbf{x} \leq b_1, \ldots, \mathbf{a}_n^t\mathbf{x} \leq b_n\right\}.$$

and, if the representation is irredundant, the intersection of all the hyperplanes of the form $\mathbf{a}_i^t\mathbf{x} = b_i$ is a single point. The *rays* of the cone are its 1-dimensional faces.

Definition 2.2.2. A *fan* in $\mathbb{R}^d$ is a non empty family of cones

$$\mathcal{F} = \{\mathsf{C}_1, \ldots, \mathsf{C}_N\}$$

such that,



(a) 2-cube (b) Face lattice

Figure 1

1. Every non empty face of a cone in $\mathcal{F}$ is in $\mathcal{F}$.

2. The intersection of two cones in $\mathcal{F}$ is a face of both cones.

The *support* of a fan is the space created by taking the union of all the cones in the fan. We say a fan is *complete* if its support is $\mathbb{R}^d$, it is *pointed* if it contains the cone $\{\mathbf{0}\}$ and it is *lineless* if for every $\mathbf{c} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ such that $\mathbf{c}$ spans a ray in $\mathcal{F}$, then $-\mathbf{c}$ does not span a ray in $\mathcal{F}$. The cones of dimension k in $\mathcal{F}$ are denoted by $\mathcal{F}_k$.

Definition 2.2.3. If two fans $\mathcal{F}$ and $\mathcal{G}$ have the same support, we say $\mathcal{F}$ *coarsens* $\mathcal{G}$ if every cone of $\mathcal{F}$ is the union of cones in $\mathcal{G}$. This relationship will be denoted by $\mathcal{G} \preceq \mathcal{F}$. Conversely, $\mathcal{F}$ *refines* $\mathcal{G}$ if each cone in $\mathcal{F}$ is a subset of a cone in $\mathcal{G}$.

Definition 2.2.4. If $\mathcal{F}$ and $\mathcal{G}$ are both fans in the same space $\mathbb{R}^d$, then we define their *common refinement* as

$$\mathcal{F} \wedge \mathcal{G} := \{\mathsf{C} \cap \mathsf{C}' \,:\, \mathsf{C} \in \mathcal{F},\, \mathsf{C}' \in \mathcal{G}\}.$$

Fans are a useful way of understanding support functions and compare polytopes. To do this we need to define the following.

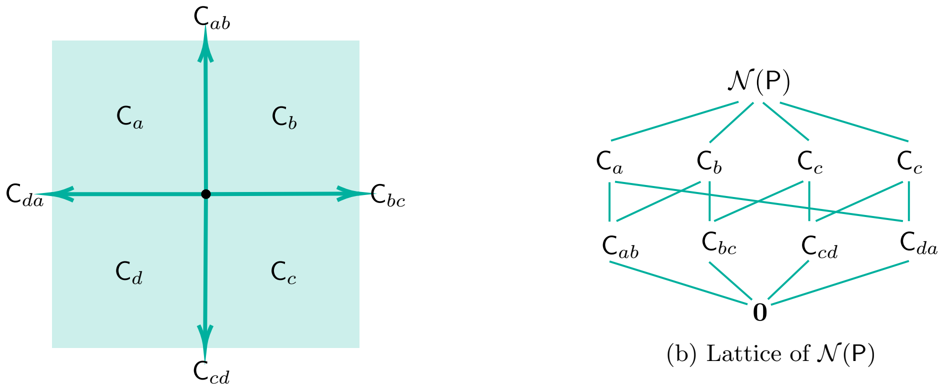
Definition 2.2.5. Let P be a polytope in $\mathbb{R}^d$. Its *(outer) normal fan* is the fan comprised by vectors in $\mathbb{R}^d$ (interpreted as linear maps) that are maximized in a face of P. In other words, for every non-empty face F of P there is a cone

$$\mathsf{C}_\mathsf{F} := \left\{\mathbf{c} \in \mathbb{R}^d : \mathsf{F} \subseteq \mathsf{P}^\mathbf{c}\right\}.$$

We denote the normal fan of P by $\mathcal{N}(\mathsf{P})$.

This fan will always be complete because the polytope is bounded, so every lineal map must be maximized in some face. If P is a polytope of dimension at least 2, then $\mathsf{P}^\mathbf{0} = \mathsf{P}$ and there is no other $\mathbf{c} \in \mathbb{R}^d$ such that $\mathsf{P}^\mathbf{c} = \mathsf{P}$, so $\mathcal{N}(\mathsf{P})$ is pointed. If $\mathcal{N}(\mathsf{P})$ contained a line, then there is a $\mathbf{c} \in \mathbb{R} \setminus \{\mathbf{0}\}$ such that $h_\mathsf{P}(\mathbf{c}) = h_\mathsf{P}(-\mathbf{c})$, so P is contained in an affine linear space orthogonal to $\mathbf{c}$, meaning it is not full dimensional. Therefore, the normal fan of a full-dimensional polytope is lineless.

Remark 2.2.6. It is useful to notice that every face on a polytope contains a vertex, so the normal fan will always be the collection of the normal cones of the vertices and all of their faces. This means that in order to know the normal fan of a polytope, it is enough to know the normal cones of the vertices.



(a) $\mathcal{N}(\mathsf{P})$

(b) Lattice of $\mathcal{N}(\mathsf{P})$

Figure 2

We can define the poset of a fan just as we did for polytopes. This means its elements are the cones and the order is given by inclusion. Clearly, if a face F is contained in another face F', then F' maximizes at least the same linear maps as F, so inclusions are reversed. In order for it to have a maximal element, we also consider the complete fan as an element in the poset, so the lattice of $\mathcal{N}(\mathsf{P})$ is the dual of the lattice for P. This gives rise to the following property.

Proposition 2.2.7. *Let* P *be a full dimensional polytope in* $\mathbb{R}^d$*. The cone in* $\mathcal{N}(\mathsf{P})$ *corresponding to the* m *dimensional face* $\mathsf{P}^{\mathbf{c}}$ *has dimension* $d - m$*.*

Example 2.2.8. Let P be the 2-cube in Figure 1. Then, its normal fan and its corresponding lattice will be the ones depicted in Figure 2.

Another property that is a consequence of this duality is that if a polytope P is simple, all the full dimensional cones in $\mathcal{N}(\mathsf{P})$ have d rays, corresponding to the d facets that contain each vertex.

Definition 2.2.9. Let $\mathcal{F} = \{\mathsf{C}_1, \ldots, \mathsf{C}_m\}$ be a complete fan in $\mathbb{R}^d$. A function $g : \mathbb{R}^d \to \mathbb{R}$ is a *piecewise linear function* over $\mathcal{F}$ if there are linear functions $\ell_1, \ldots, \ell_m : \mathbb{R}^d \to \mathbb{R}$ such that $g|_{\mathsf{C}_i} = \ell_i$ for every $i \in \{1, \ldots, m\}$.

Piecewise linear functions are continuous when restricted to any cone in the fan and these cones are closed and cover $\mathbb{R}^d$, so the whole function is continuous.

Definition 2.2.10. A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be *convex* if for all $0 \leq \lambda \leq 1$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$,

$$f(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1 - \lambda) f(\mathbf{y}). \tag{5}$$

Definition 2.2.11. A function $f : \mathbb{R}^d \to \mathbb{R}$ is called *positive homogeneous* if, for any $\lambda \geq 0$ and $\mathbf{x} \in \mathbb{R}^d$,

$$f(\lambda\mathbf{x}) = \lambda f(\mathbf{x}). \tag{6}$$

Remark 2.2.12. Every piecewise linear function over a complete fan such that all its cones contain $\mathbf{0}$ is positive homogeneous. This happens because in that case, if $\mathbf{x} \in \mathbb{R}^d$ and $\lambda \geq 0$, then $\mathbf{x}$ and $\lambda\mathbf{x}$ are in the same cone, so f is the same linear function in both points and $f(\lambda\mathbf{x}) = \lambda f(\mathbf{x})$.

Lemma 2.2.13. *The support function of a polytope* P *is piecewise linear, convex over* $\mathcal{N}(\mathsf{P})$*, and positive homogeneous.*

Proof. Let P be a polytope in $\mathbb{R}^d$ with support function $h_{\mathsf{P}} : \mathbb{R}^d \to \mathbb{R}$, and let F be a face of P with a corresponding cone C_{F} in $\mathcal{N}(\mathsf{P})$, then for every $\mathbf{c} \in \mathsf{C}_{\mathsf{F}}$ and any $\mathbf{x} \in \mathsf{F}$,

$$\mathbf{c}^t\mathbf{x} = \max_{\mathbf{y}\in\mathsf{P}}\{\mathbf{c}^t\mathbf{y}\} = h_{\mathsf{P}}(\mathbf{c}).$$

So h_{P} is linear in C_{F} and in general it is piecewise linear in $\mathcal{N}(\mathsf{P})$.

Let P be a polytope with support function h_{P}, then for every $\mathbf{c}, \mathbf{d} \in \mathbb{R}^d$ and $\lambda \in [0, 1]$,

$$\begin{aligned} h_{\mathsf{P}}(\lambda\mathbf{c} + (1-\lambda)\mathbf{d}) &= \max_{\mathbf{x}\in\mathsf{P}}\{(\lambda\mathbf{c} + (1-\lambda)\mathbf{d})^t\mathbf{x}\} \\ &= \max_{\mathbf{x}\in\mathsf{P}}\{\lambda(\mathbf{c}^t\mathbf{x}) + (1-\lambda)(\mathbf{d}^t\mathbf{x})\} \\ &\leq \lambda\max_{\mathbf{x}\in\mathsf{P}}\{\mathbf{c}^t\mathbf{x}\} + (1-\lambda)\max_{\mathbf{x}\in\mathsf{P}}\{\mathbf{d}^t\mathbf{x}\} \\ &\leq \lambda h_{\mathsf{P}}(\mathbf{c}) + (1-\lambda)h_{\mathsf{P}}(\mathbf{d}). \end{aligned}$$

To show the function is positive homogeneous, by Remark 2.2.12, it is enough to show that every cone in $\mathcal{N}(\mathsf{P})$ contains 0, but this is clear from the fact than $\mathbf{0}^t\mathbf{x} = 0$ for every $\mathbf{x} \in \mathbb{R}^d$, so $\mathbf{0}$ is maximized in every face. □

This characterization is actually an equivalence, to prove this we have to recall the following version of the Farkas lemma and a useful equivalence for convexity.

Lemma 2.2.14. *[20, Proposition 1.3] Let $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, $b_1, \dots, b_n \in \mathbb{R}$, $\mathbf{a}_0 \in \mathbb{R}^d$, $b_0 \in \mathbb{R}$ and*

$$\mathsf{P} = \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}_1^t\mathbf{x} \leq b_1, \dots, \mathbf{a}_n^t\mathbf{x} \leq b_n\right\}.$$

Then $\mathbf{a}_0\mathbf{x} \leq b_0$ is valid for P if and only if one or both of the following apply:

(i) there exists $\lambda_1, \dots, \lambda_n \in \mathbb{R}_{\geq 0}$ such that

$$\sum_{i=1}^{n}\lambda_i\mathbf{a}_i = \mathbf{a}_0 \qquad \text{and} \qquad \sum_{i=1}^{n}\lambda_i b_i \leq z_0, \tag{7}$$

(ii) there exists $\lambda_1, \dots, \lambda_n \in \mathbb{R}_{\geq 0}$ such that

$$\sum_{i=1}^{n}\lambda_i\mathbf{a}_i = \mathbf{0} \qquad \text{and} \qquad \sum_{i=1}^{n}\lambda_i b_i < 0. \tag{8}$$

Lemma 2.2.15. *[9, Lemma 5.6] A positive homogeneous function $f : \mathbb{R}^d \to \mathbb{R}$ is convex if and only if for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$,*

$$f(\mathbf{x} + \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y}). \tag{9}$$

In general, if a function $f : \mathbb{R}^d \to \mathbb{R}$ satisfies (9) for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, we will say it is subadditive.

Lemma 2.2.16. *A piecewise linear function $f : \mathbb{R}^d \to \mathbb{R}$ over a fan $\mathcal{F}$ is convex if and only if*

$$f(\mathbf{c}) = \max\{f|_{\mathsf{C}}(\mathbf{c}) : \mathsf{C} \in \mathcal{F}_d\}. \tag{10}$$

Proof. For the forward direction, let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function, and $\mathbf{c} \in \mathbb{R}^d$ in the cone $\mathsf{C} \in \mathcal{F}_d$. Let $\mathsf{C}' \in \mathcal{F}_d$ be a different cone and $\mathbf{c}' \in \mathsf{C}'$ a point in its interior, then there is a $\lambda \in (0, 1)$ such that $\mathbf{d} := \lambda\mathbf{c} + (1-\lambda)\mathbf{c}' \in \mathsf{C}'$. So,

$$\begin{aligned} \lambda f|_{\mathsf{C}'}(\mathbf{c}) + (1-\lambda)f|_{\mathsf{C}'}(\mathbf{c}') &= f(\mathbf{d}) \\ &\leq \lambda f(\mathbf{c}) + (1-\lambda)f(\mathbf{c}') \\ &= \lambda f|_{\mathsf{C}}(\mathbf{c}) + (1-\lambda)f|_{\mathsf{C}'}(\mathbf{c}'). \end{aligned}$$

Therefore, $f|_{\mathsf{C}'}(\mathbf{c}) \leq f|_{\mathsf{C}}(\mathbf{c})$. Finally applying the maximum for all cones, we get (10).

For the backward direction, let (10) hold and $\mathbf{c}, \mathbf{c}' \in \mathbb{R}^d$ be points in the full-dimensional cones C and C' respectively. Then for every $\lambda \in [0,1]$, $\mathbf{d} := \lambda\mathbf{c} + (1-\lambda)\mathbf{c}'$ lies in some full-dimensional cone D, and

$$\begin{aligned} f(\mathbf{d}) &= \lambda f|_{\mathsf{D}}(\mathbf{c}) + (1-\lambda) f|_{\mathsf{D}}(\mathbf{c}') \\ &\leq \lambda f|_{\mathsf{C}}(\mathbf{c}) + (1-\lambda) f|_{\mathsf{C}'}(\mathbf{c}') \\ &= \lambda f(\mathbf{c}) + (1-\lambda) f(\mathbf{c}') \end{aligned}$$

so f is convex.

□

Theorem 2.2.17. *[9, Theorem 6.8] A function $f : \mathbb{R}^d \to \mathbb{R}$ is the support function of a polytope if and only if it is positive homogeneous, convex and piecewise linear over a complete pointed fan.*

Proof. The forward direction is given by Lemma 2.2.13. For the backward direction, let $f : \mathbb{R}^d \to \mathbb{R}$ be a piecewise linear, positive, homogeneous and convex function over a complete pointed fan $\mathcal{F}$, then we will construct a polytope P such that $h_{\mathsf{P}} = f$. If $\mathbf{a}_{\mathsf{R}}$ denotes a generator for the ray $\mathsf{R} \in \mathcal{F}_1$, we define the polytope

$$\mathsf{P} := \left\{ \mathbf{x} \in \mathbb{R}^d : \mathbf{a}_{\mathsf{R}}^t \mathbf{x} \leq f(\mathbf{a}_{\mathsf{R}})\, \forall \mathsf{R} \in \mathcal{F}_2 \right\}.$$

The number of rays in a fan is finite, so we only have to show that P is not empty to prove it is a polytope. Let $\mathsf{C} \in \mathcal{F}_d$, then there is a $\mathbf{x}_{\mathsf{C}}$ such that $f|_{\mathsf{C}}(\mathbf{c}) = \mathbf{c}^t \mathbf{x}_{\mathsf{C}}$ for every $\mathbf{c} \in \mathbb{R}^d$. By Lemma 2.2.16 we know that for every $\mathsf{R} \in \mathcal{F}_2$, $f|_{\mathsf{C}}(\mathbf{a}_{\mathsf{R}}) \leq f(\mathbf{a}_{\mathsf{R}})$, therefore

$$\mathbf{a}_{\mathsf{R}}^t \mathbf{x}_{\mathsf{C}} \leq f(\mathbf{a}_{\mathsf{R}})$$

so $\mathbf{x}_{\mathsf{C}} \in \mathsf{P}$.

Now, lets prove that $h_{\mathsf{P}} \leq f$. Let $\mathbf{x} \in \mathsf{P}$ and $\mathbf{c} \in \mathbb{R}^d$, then $\mathbf{c}$ is in a full dimensional cone C of $\mathcal{F}$ with some rays $\mathcal{R} \subseteq \mathcal{F}_2$, so there are $\lambda_{\mathsf{R}} \geq 0$ such that $\mathbf{c} = \sum_{\mathsf{R} \in \mathcal{R}} \lambda_{\mathsf{R}} \mathbf{a}_{\mathsf{R}}$ then

$$\begin{aligned} \mathbf{c}^t \mathbf{x} &= \left(\sum_{\mathsf{R} \in \mathcal{R}} \lambda_{\mathsf{R}} \mathbf{a}_{\mathsf{R}} \right)^t \mathbf{x} \\ &= \sum_{\mathsf{R} \in \mathcal{R}} \lambda_{\mathsf{R}} (\mathbf{a}_{\mathsf{R}}^t \mathbf{x}) \\ &\leq \sum_{\mathsf{R} \in \mathcal{R}} \lambda_{\mathsf{R}} f(\mathbf{a}_{\mathsf{R}}). \end{aligned}$$

We also know that f is linear in C, so

$$\begin{aligned} \sum_{\mathsf{R} \in \mathcal{R}} \lambda_{\mathsf{R}} f(\mathbf{a}_{\mathsf{R}}) &= f\left(\sum_{\mathsf{R} \in \mathcal{R}} \lambda_{\mathsf{R}} \mathbf{a}_{\mathsf{R}} \right) \\ &= f(\mathbf{c}) \end{aligned}$$

thus concluding that $\mathbf{c}^t \mathbf{x} \leq f(\mathbf{c})$. Then taking maximum over $\mathbf{x} \in \mathsf{P}$ we get that $h_{\mathsf{P}}(\mathbf{c}) \leq f(\mathbf{c})$ for every $\mathbf{c} \in \mathbb{R}^d$.

On the other hand, if $\mathbf{c} \in \mathbb{R}^d$, from the definition of support function we know that $\mathbf{c}^t \mathbf{x} \leq h_{\mathsf{P}}(\mathbf{c})$ is valid for all $\mathbf{x} \in \mathsf{P}$, so by Lemma 2.2.14 there are $\lambda_{\mathsf{R}} \geq 0$ for $\mathsf{R} \in \mathcal{F}_2$ such that they satisfy (7) or (8).

Suppose that (8) is satisfied. Then,

$$\sum_{\mathsf{R}\in\mathcal{R}} \lambda_{\mathsf{R}} \mathbf{a}_{\mathsf{R}} = \mathbf{0} \qquad \text{and} \qquad \sum_{\mathsf{R}\in\mathcal{R}} \lambda_{\mathsf{R}} f(\mathbf{a}_{\mathsf{R}}) < 0$$

but f is positive homogeneous and subadditve, so

$$0 > \sum_{\mathsf{R}\in\mathcal{R}} \lambda_{\mathsf{R}} f(\mathbf{a}_{\mathsf{R}}) \geq f\left(\sum_{\mathsf{R}\in\mathcal{R}} \lambda_{\mathsf{R}} \mathbf{a}_{\mathsf{R}}\right) = f(\mathbf{0}).$$

However, $f(\mathbf{0}) = 0$, so the equation satisfied must be (7). Then,

$$\sum_{\mathsf{R}\in\mathcal{R}} \lambda_{\mathsf{R}} \mathbf{a}_{\mathsf{R}} = \mathbf{c} \qquad \text{and} \qquad \sum_{\mathsf{R}\in\mathcal{R}} \lambda_{\mathsf{R}} f(\mathbf{a}_{\mathsf{R}}) \leq h_{\mathsf{P}}(\mathbf{c})$$

By subadditivity of f, we get that

$$\begin{aligned} f(\mathbf{c}) &= f\left(\sum_{\mathsf{R}\in\mathcal{R}} \lambda_{\mathsf{R}} \mathbf{a}_{\mathsf{R}}\right) \\ &\leq \sum_{\mathsf{R}\in\mathcal{R}} \lambda_{\mathsf{R}} f(\mathbf{a}_{\mathsf{R}}) \\ &\leq h_{\mathsf{P}}(\mathbf{c}). \end{aligned}$$

Finally, we can conclude that $f = h_{\mathsf{P}}$, so f is the support function of a polytope. □

What follows is a series of propositions that will be useful when working with normal fans in the following sections.

Firstly, there is a notion of strict convexity. In general, strict convexity is defined by changing the inequality in (5) by a strict one, but in the case of piecewise linear functions this will never be possible. In consequence, we define strict convexity by asking for a strict inequality only when points that are not in the same cone.

Proposition 2.2.18. *Let* P *be a polytope with support function* h_{P} *and* $\mathbf{c}, \mathbf{c}' \in \mathbb{R}^d$*. The equivalence* $h_{\mathsf{P}}(\mathbf{c}+\mathbf{c}') = h_{\mathsf{P}}(\mathbf{c}) + h_{\mathsf{P}}(\mathbf{c}')$ *holds if and only if there is a cone in* $\mathcal{N}(\mathsf{P})$ *that contains both of them.*

Proof. The backward direction is a direct conclusion from the fact that h_{P} is piecewise linear over $\mathcal{N}(\mathsf{P})$, so the function is linear in the cone that contains them.

For the forward direction, let $\mathbf{c}, \mathbf{c}' \in \mathbb{R}^d$ be such that $h_{\mathsf{P}}(\mathbf{c}) + h_{\mathsf{P}}(\mathbf{c}') = h_{\mathsf{P}}(\mathbf{c}+\mathbf{c}')$ and let $\mathbf{x} \in \mathsf{P}^{\mathbf{c}+\mathbf{c}'}$. Then $\mathbf{c}^t\mathbf{x} + \mathbf{c}'^t\mathbf{x} = (\mathbf{c}+\mathbf{c}')^t\mathbf{x} = h_{\mathsf{P}}(\mathbf{c}) + h_{\mathsf{P}}(\mathbf{c}')$, additionally, $\mathbf{c}^t\mathbf{x} \leq h_{\mathsf{P}}(\mathbf{c})$ and $\mathbf{c}'^t\mathbf{x} \leq h_{\mathsf{P}}(\mathbf{c}')$, so $\mathbf{c}^t\mathbf{x} = h_{\mathsf{P}}(\mathbf{c})$ and $\mathbf{c}'^t\mathbf{x} = h_{\mathsf{P}}(\mathbf{c}')$. In conclusion, $\mathbf{x} \in \mathsf{P}^{\mathbf{c}}$ and $\mathbf{x} \in \mathsf{P}^{\mathbf{c}'}$, so $\mathsf{P}^{\mathbf{c}+\mathbf{c}'} \subset \mathsf{P}^{\mathbf{c}} \cap \mathsf{P}^{\mathbf{c}'}$. Therefore, the cone that is defined by $\mathsf{P}^{\mathbf{c}+\mathbf{c}'}$ must contain $\mathbf{c}$ and $\mathbf{c}'$. □

Secondly, it will be useful to know when polytopes have the same normal fan. An example of when this happens is when a polytope is a positive scaling of another polytope.

Proposition 2.2.19. *Let* $\lambda > 0$ *and* $\lambda\mathsf{P}$ *be the polytope defined by* $\lambda\mathsf{P} := \{\lambda\mathbf{p} : \mathbf{p} \in \mathsf{P}\}$*, then* $h_{\lambda\mathsf{P}} = \lambda h_{\mathsf{P}}$ *and* $\mathcal{N}(\mathsf{P}) = \mathcal{N}(\lambda\mathsf{P})$*.*

Proof. Notice that from the definition of $\lambda\mathsf{P}$ we get that

$$\begin{aligned} h_{\lambda\mathsf{P}}(\mathbf{c}) &= \max\{\mathbf{c}^t(\lambda\mathbf{p}) : \mathbf{p} \in \mathsf{P}\} \\ &= \max\{(\lambda\mathbf{c})^t\mathbf{p} : \mathbf{p} \in \mathsf{P}\} \\ &= h_{\mathsf{P}}(\lambda\mathbf{c}). \end{aligned}$$

And since support functions are positive homogeneous and $\lambda > 0$, we can conclude that $h_{\lambda \mathsf{P}} = \lambda h_{\mathsf{P}}$, so $h_{\lambda \mathsf{P}}$ is piecewise linear over the same fan as h_{P}. This means that both polytopes have the same normal fan. $\square$

Thirdly, the next proposition gives a way to detect whether a function is convex just by checking convexity in adjacent full-dimensional cones.

Proposition 2.2.20. *[14, Theorem 5.4] A piecewise linear function $h : \mathbb{R}^d \to \mathbb{R}$ over $\mathcal{F}$ is convex if and only if for every pair of cones $\mathsf{C}, \mathsf{C}' \in \mathcal{F}_d$ that intersect in a facet,*

$$h|_{\mathsf{C}}(\mathbf{c}) \geq h|_{\mathsf{C}'}(\mathbf{c}) \tag{11}$$

for every $\mathbf{c} \in \mathsf{C}$.

Proof. The forward implication is clear from Lemma 2.2.16. For the backward direction, we'll show that if (11) holds, then for every $\mathbf{c} \in \mathbb{R}^d$ with $\mathbf{c} \in \mathsf{C}_0$ and $\mathsf{C}_0 \in \mathcal{F}_d$, we have that $h|_{\mathsf{C}_0}(\mathbf{c}) \geq h|_{\mathsf{C}}(\mathbf{c})$ for every other $\mathsf{C} \in \mathcal{F}_d$.

Let $\mathsf{C} \in \mathcal{F}_d$, then there is a point $\mathbf{c}' \in \mathsf{C}$ such that the segment between $\mathbf{c}$ and $\mathbf{c}'$ only crosses cones of dimension at least $d-1$. Let $\mathsf{C}_0, \ldots, \mathsf{C}_n = \mathsf{C}$ be the full dimensional cones crossed by the path between $\mathbf{c}$ and $\mathbf{c}'$ in order and suppose there is an $i \in \{0, \ldots, n-1\}$ such that $h|_{\mathsf{C}_i}(\mathbf{c}) < h|_{\mathsf{C}_{i+1}}(\mathbf{c})$. Notice that there is an $\mathbf{a} \in \mathbb{R}^d$ such that for every $\mathbf{x} \in \mathbb{R}^d$

$$h|_{\mathsf{C}_i} - h|_{\mathsf{C}_{i+1}}(\mathbf{x}) = \mathbf{a}^t \mathbf{x}.$$

So $\mathbf{a}^t \mathbf{c} < 0$, and since $h|_{\mathsf{C}_i} - h|_{\mathsf{C}_{i+1}}$ is 0 in the hyperplane spanned by $\mathsf{C}_i \cap \mathsf{C}_{i+1}$ and $\mathbf{c}$ is in the same side of the hyperplane as C_i, $\mathbf{a}$ must be an outward normal vector of C_i.

On the other hand, since C_i and C_{i+1} are adjacent, then $h|_{\mathsf{C}_i}(\mathbf{c}') \geq h|_{\mathsf{C}_{i+1}}(\mathbf{c}')$ for every $\mathbf{c}' \in \mathsf{C}_i$, so $h|_{\mathsf{C}_i} - h|_{\mathsf{C}_{i+1}}(\mathbf{c}') = \mathbf{a}^t \mathbf{c}' \geq 0$, but since $\mathbf{a}$ is an outer normal vector of C_i, $\mathbf{a}^t \mathbf{c}' < 0$, arriving at a contradiction.

In conclusion, $h|_{\mathsf{C}_0}(\mathbf{c}) \geq h|_{\mathsf{C}_1}(\mathbf{c}) \geq \cdots \geq h|_{\mathsf{C}_n}(\mathbf{c})$, so $h|_{\mathsf{C}_0}(\mathbf{c}) \geq h_{\mathsf{C}}(\mathbf{c})$. $\square$

Remark 2.2.21. By applying the last proposition and the same argument as the one used in the the proof of Lemma 2.2.16, we get that for a function to be convex it suffices for points in adjacent cones to satisfy (5). Furthermore, if the function is positive homogeneous, it suffices for a function to satisfy (9) for points in adjacent cones.

Finally, sometimes it is useful to consider polytopes up to translations, so the next proposition gives a characterization of the support function of a translated polytope and a way of obtaining a translated polytope such that its support function is non-negative.

Proposition 2.2.22. *Let P be a polytope, then for every $\mathbf{a} \in \mathbb{R}$, $\mathsf{P}+\mathbf{a}$ has support function $h_{\mathsf{P}+\mathbf{a}}(\mathbf{c}) = h_{\mathsf{P}}(\mathbf{c}) + \mathbf{a}^t \mathbf{c}$ and for every $\mathsf{C} \in \mathcal{N}(\mathsf{P})$ there is an $\mathsf{a} \in \mathbb{R}^d$ such that $h_{\mathsf{P}+\mathbf{a}}(\mathbf{c}) \geq 0$ for every $\mathbf{c} \in \mathbb{R}^d$ and $h_{\mathsf{P}+\mathbf{a}}(\mathbf{c}) = 0$ for every $\mathbf{c} \in \mathsf{C}$.*

Proof. Let $\mathbf{a} \in \mathbb{R}^d$, then for every $\mathbf{c} \in \mathbb{R}^d$

$$\begin{aligned} h_{\mathsf{P}+\mathbf{a}}(\mathbf{c}) &= \max\{\mathbf{c}^t(\mathbf{p}+\mathbf{a}) : \mathbf{p} \in \mathsf{P}\} \\ &= \max\{\mathbf{c}^t(\mathbf{p}) : \mathbf{p} \in \mathsf{P}\} + \mathbf{c}^t \mathbf{a} \\ &= h_{\mathsf{P}}(\mathbf{c}) + \mathbf{c}^t \mathbf{a}. \end{aligned}$$

Now, let C be a cone of $\mathcal{N}(\mathsf{P})$ corresponding to the face F of P and $\mathbf{a} \in \operatorname{relint}(\mathsf{F})$, then $\mathbf{0} \in \mathsf{P} - \mathbf{a}$, so

$$h_{\mathsf{P}-\mathbf{a}}(\mathbf{c}) = \max_{\mathbf{x} \in \mathsf{P}-\mathbf{a}} \{\mathbf{c}^t \mathbf{x}\} \geq \mathbf{c}^t \mathbf{0} = 0.$$

Let $\mathbf{c} \in \mathsf{C}$, then

$$\begin{aligned} h_{\mathsf{P}-\mathbf{a}}(\mathbf{c}) &= h_{\mathsf{P}}(\mathbf{c}) - \mathbf{c}^t\mathbf{a} \\ &= \max_{\mathbf{p}\in\mathsf{P}}\{\mathbf{c}^t\mathbf{p}\} - \mathbf{c}^t\mathbf{a} \\ &= \mathbf{c}^t\mathbf{a} - \mathbf{c}^t\mathbf{a} \\ &= 0. \end{aligned}$$

□

2.3 Minkowski Sum

Definition 2.3.1. Let P and Q be polytopes in $\mathbb{R}^d$, then we define their *Minkowski sum* by

$$\mathsf{Q} + \mathsf{R} = \{\mathbf{q} + \mathbf{r} : \mathbf{q} \in \mathsf{Q}, \mathbf{r} \in \mathsf{R}\}.$$

Given a polytope P, Q is called a *summand of* P, denoted $\mathsf{Q} \leq \mathsf{P}$, if there is a polytope R such that $\mathsf{P} = \mathsf{Q} + \mathsf{R}$.

If there is a $\lambda \geq 0$ and a polytope R such that $\mathsf{P} = \lambda\mathsf{Q} + \mathsf{R}$, then Q is a *weak Minkowski summand of* P, which is denoted by $\mathsf{Q} \preceq \mathsf{P}$.

Taking $\lambda = 1$, we get that every Minkowski summand is also a weak Minkowski summand.

Example 2.3.2. Let P be a polytope, then $\lambda\mathsf{P} \preceq \mathsf{P}$ for every $\lambda > 0$. To understand why, it suffices to see that we can take $\mu = \frac{1}{\lambda}$ and Q the polytope consisting only of the point $\mathbf{0}$, and $\mathsf{P} = \mu(\lambda\mathsf{P}) + \mathsf{Q}$.

In addition, if P has at least 2 vertices, then $\lambda\mathsf{P} \leq \mathsf{P}$ if and only if $\lambda \in (0, 1]$. This is given by the fact that if $\lambda \in (0, 1]$, then $\mathsf{P} = \lambda\mathsf{P} + (1 - \lambda)\mathsf{P}$ and if λ is greater than 1, we will later see in Example 2.3.14 that there cannot be a polytope R such that $\mathsf{P} = \lambda\mathsf{P} + \mathsf{R}$.

Proposition 2.3.3. *The Minkowski sum of two polytopes is a polytope. Moreover, if* Q *and* R *are polytopes, the following are equivalent:*

(i) $\mathsf{P} = \mathsf{Q} + \mathsf{R}$.

(ii) If h_{Q} *and* h_{R} *are the support functions of* Q *and* R *respectively,* P *is the polytope defined by*

$$h_{\mathsf{P}}(\mathbf{c}) = h_{\mathsf{Q}}(\mathbf{c}) + h_{\mathsf{R}}(\mathbf{c}).$$

(iii) If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *are the vertices of* Q *and* $\mathbf{w}_1, \ldots, \mathbf{w}_m$ *are the vertices of* R *then*

$$\mathsf{P} = \operatorname{conv}\{\mathbf{v}_i + \mathbf{w}_j : i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}\}. \tag{12}$$

Proof. $(i \Rightarrow ii)$ If $\mathsf{P} = \mathsf{Q} + \mathsf{R}$, then $\mathsf{P} = \{\mathbf{q} + \mathbf{r} : \mathbf{q} \in \mathsf{Q}, \mathbf{r} \in \mathsf{R}\}$ so

$$\begin{aligned} h_{\mathsf{P}}(\mathbf{c}) &= \max_{\mathbf{p}\in\mathsf{P}}\{\mathbf{c}^t\mathbf{p}\} \\ &= \max_{\substack{\mathbf{r}\in\mathsf{R}\\ \mathbf{q}\in\mathsf{Q}}}\{\mathbf{c}^t(\mathbf{r} + \mathbf{q})\} \\ &= \max_{\substack{\mathbf{r}\in\mathsf{R}\\ \mathbf{q}\in\mathsf{Q}}}\{\mathbf{c}^t\mathbf{r} + \mathbf{c}^t\mathbf{q}\} \\ &= \max_{\mathbf{r}\in\mathsf{R}}\{\mathbf{c}^t\mathbf{r}\} + \max_{\mathbf{q}\in\mathsf{Q}}\{\mathbf{c}^t\mathbf{q}\} \\ &= h_{\mathsf{R}}(\mathbf{c}) + h_{\mathsf{Q}}(\mathbf{c}). \end{aligned}$$

$(ii \Rightarrow i)$ Let P, Q, R be polytopes such that $h_{\mathsf{P}}(\mathbf{c}) = h_{\mathsf{Q}}(\mathbf{c}) + h_{\mathsf{R}}(\mathbf{c})$ then

$$\begin{aligned}\mathsf{P} &= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{c}^t\mathbf{x} \leq h_{\mathsf{P}}(\mathbf{c})\ \forall \mathbf{c} \in \mathbb{R}^n\} \\ &= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{c}^t\mathbf{x} \leq h_{\mathsf{Q}}(\mathbf{c}) + h_{\mathsf{R}}(\mathbf{c})\ \forall \mathbf{c} \in \mathbb{R}^n\} \\ &= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{c}^t\mathbf{x} \leq h_{\mathsf{Q+R}}(\mathbf{c})\ \forall \mathbf{c} \in \mathbb{R}^n\} \\ &= \mathsf{Q} + \mathsf{R}.\end{aligned}$$

$(i \Leftrightarrow iii)$ To show the equivalence, it suffices to show that if Q and R are polytopes with vertices $\mathbf{v}_1, \dots, \mathbf{v}_n$ and $\mathbf{w}_1 \dots, \mathbf{w}_m$ respectively, then

$$\mathsf{Q} + \mathsf{R} = \operatorname{conv}\{\mathbf{v}_i + \mathbf{w}_i : i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\} \tag{13}$$

for every $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$ the point $\mathbf{v}_i + \mathbf{w}_j$ is in $\mathsf{Q} + \mathsf{R}$, so its convex hull is also in $\mathsf{Q} + \mathsf{R}$. Thus obtaining the right to left inclusion in (13).

On the other hand, if $\mathbf{q} \in \mathsf{Q}$ and $\mathbf{r} \in \mathsf{R}$, then there are $\lambda_i, \mu_j \in \mathbb{R}_{\geq 0}$ such that $\mathbf{q} = \sum_{i=1}^n \lambda_i \mathbf{v}_i$, $\mathbf{r} = \sum_{j=1}^n \mu_j \mathbf{w}_j$ and $\sum_{i=1}^n \lambda_i = \sum_{j=1}^m \mu_j = 1$, so

$$\begin{aligned}\mathbf{q} + \mathbf{r} &= \sum_{i=1}^n \lambda_i \mathbf{v}_i + \sum_{j=1}^n \mu_j \mathbf{w}_j \\ &= \sum_{j=1}^n \mu_j \sum_{i=1}^n \lambda_i \mathbf{v}_i + \sum_{i=1}^n \lambda_i \sum_{j=1}^n \mu_j \mathbf{w}_j \\ &= \sum_{j=1}^n \sum_{i=1}^n \mu_j \lambda_i \mathbf{v}_i + \sum_{i=1}^n \sum_{j=1}^n \lambda_i \mu_j \mathbf{w}_j \\ &= \sum_{i=1}^n \sum_{j=1}^m \lambda_i \mu_j (\mathbf{v}_i + \mathbf{w}_j).\end{aligned}$$

Also, $\sum_{i=1}^n \sum_{j=i}^m \lambda_i \mu_j = \sum_{i=1}^n \lambda_i \sum_{i=1}^m \mu_j = 1$, so

$$\mathbf{q} + \mathbf{r} \in \operatorname{conv}\{\mathbf{v}_i + \mathbf{w}_i : i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\}$$

obtaining (13).

□

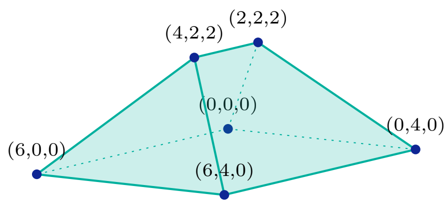
Remark 2.3.4. It is important to note that when calculating the Minkowski sum of polytopes given their vertices, it suffices to see the sum of the vertices, whereas in the case of the H-representation it is not enough to add up the values of the inequalities, it is necessary to add the support functions of the polytopes.

Example 2.3.5. Let Q and R be the polytopes in Figures 3a and 3b. Then, $\mathsf{P} := \mathsf{Q} + \mathsf{R}$ is the polytope seen in Figure 3c.

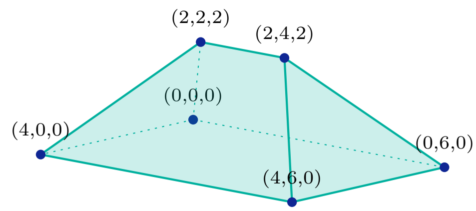
On the other hand, if we define

$$A := \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \qquad \mathbf{b}_{\mathsf{Q}} := \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix} \qquad \mathbf{b}_{\mathsf{R}} := \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

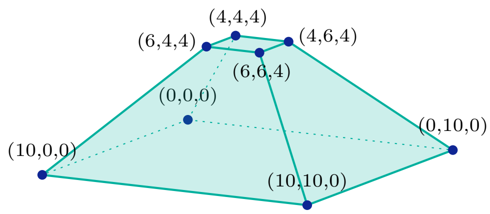
then Q and R can be represented by the inequalities $A\mathbf{x} \leq \mathbf{b}_{\mathsf{Q}}$ and $A\mathbf{x} \leq \mathbf{b}_{\mathsf{R}}$ respectively. Nevertheless, the polytope defined by $A\mathbf{x} \leq \mathbf{b}_{\mathsf{Q}} + \mathbf{b}_{\mathsf{R}}$ is not P, but the polytope show on Figure 3d.



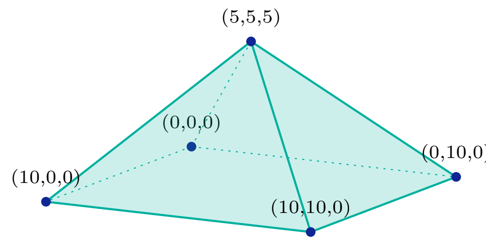
(a) Polytope Q



(b) Polytope R



(c) Polytope $\mathsf{P} = \mathsf{Q} + \mathsf{R}$



(d) Polytope defined by $A\mathbf{x} \leq \mathbf{b}_{\mathsf{Q}} + \mathbf{b}_{\mathsf{R}}$

Figure 3

Proposition 2.3.6. *Let* P, Q *and* R *be polytopes in* $\mathbb{R}^n$ *such that* $\mathsf{P} = \mathsf{Q} + \mathsf{R}$. *Then for every* $\mathbf{c} \in \mathbb{R}^n$,

$$\mathsf{P}^{\mathbf{c}} = \mathsf{Q}^{\mathbf{c}} + \mathsf{R}^{\mathbf{c}}. \tag{14}$$

Moreover, this decomposition of $\mathsf{P}^{\mathbf{c}}$ *as the sum of faces in* Q *and* R *is unique.*

Proof. Let $\mathsf{P} = \mathsf{Q} + \mathsf{R}$, then for every $\mathbf{c} \in \mathbb{R}^d$,

$$\begin{aligned} \mathsf{P}^{\mathbf{c}} &= \left\{ \mathbf{p} \in \mathsf{P} : \mathbf{c}^t \mathbf{p} = h_{\mathsf{P}}(\mathbf{c}) \right\} \\ &= \left\{ \mathbf{q} + \mathbf{r} : \mathbf{q} \in \mathsf{Q}, \, \mathbf{r} \in \mathsf{R}, \, \mathbf{c}^t(\mathbf{q} + \mathbf{r}) = h_{\mathsf{Q}}(\mathbf{c}) + h_{\mathsf{R}}(\mathbf{c}) \right\}. \end{aligned}$$

Since $\mathbf{c}^t\mathbf{q} \leq h_{\mathsf{Q}}(\mathbf{c})$ for every $\mathbf{q} \in \mathsf{Q}$ and $\mathbf{c}^t\mathbf{r} \leq h_{\mathsf{R}}(\mathbf{c})$ for every $\mathbf{r} \in \mathsf{R}$, then

$$\begin{aligned} \mathsf{P}^{\mathbf{c}} &= \left\{ \mathbf{q} + \mathbf{r} : \mathbf{q} \in \mathsf{Q}, \, \mathbf{r} \in \mathsf{R}, \, \mathbf{c}^t\mathbf{q} = h_{\mathsf{Q}}(\mathbf{c}), \, \mathbf{c}^t\mathbf{r} = h_{\mathsf{R}}(\mathbf{c}) \right\} \\ &= \{ \mathbf{q} + \mathbf{r} : \mathbf{q} \in \mathsf{Q}^{\mathbf{c}}, \, \mathbf{r} \in \mathsf{R}^{\mathbf{c}} \} \\ &= \mathsf{Q}^{\mathbf{c}} + \mathsf{R}^{\mathbf{c}}. \end{aligned}$$

To prove the uniqueness of this decomposition, let $\mathbf{b}, \mathbf{c} \in \mathbb{R}^d$ such that $\mathsf{P}^{\mathbf{b}} = \mathsf{P}^{\mathbf{c}}$, but $\mathsf{Q}^{\mathbf{b}} \neq \mathsf{Q}^{\mathbf{c}}$. Then without loss of generality, we can take $\mathbf{q} \in \mathsf{Q}^{\mathbf{c}}$ such that $\mathbf{q} \notin \mathsf{Q}^{\mathbf{b}}$, this means that $\mathbf{b}^t\mathbf{q} < h_{\mathsf{Q}}(\mathbf{b})$. On the other hand, $\mathbf{q} + \mathsf{R}^{\mathbf{c}} \subset \mathsf{Q}^{\mathbf{c}} + \mathsf{R}^{\mathbf{c}} = \mathsf{P}^{\mathbf{c}} = \mathsf{P}^{\mathbf{b}}$, but for every $\mathbf{r} \in \mathsf{R}^{\mathbf{c}}$

$$\begin{aligned} \mathbf{b}^t(\mathbf{q} + \mathbf{r}) &= \mathbf{b}^t\mathbf{q} + \mathbf{b}^t\mathbf{r} \\ &\leq \mathbf{b}^t\mathbf{q} + h_{\mathsf{R}}(\mathbf{b}) \\ &< h_{\mathsf{Q}}(\mathbf{b}) + h_{\mathsf{R}}(\mathbf{b}) \\ &= h_{\mathsf{P}}(\mathbf{b}) \end{aligned}$$

thus, arriving to a contradiction. The case for R is analogous. ☐

Proposition 2.3.7. *If* $\mathsf{Q} + \mathsf{R} = \mathsf{P}$, *then* $\mathcal{N}(\mathsf{P})$ *is the common refinement of* $\mathcal{N}(\mathsf{Q})$ *and* $\mathcal{N}(\mathsf{R})$.

Proof. Lets show that $\mathcal{N}(\mathsf{P}) \preceq \mathcal{N}(\mathsf{Q}) \wedge \mathcal{N}(\mathsf{R})$. Let C be a cone in $\mathcal{N}(\mathsf{Q}) \wedge \mathcal{N}(\mathsf{R})$, then it is a subset of a cone in $\mathcal{N}(\mathsf{Q})$ and of a cone in $\mathcal{N}(\mathsf{R})$, meaning that every $\mathbf{c} \in \text{relint}(\mathsf{C})$ defines the same face $\mathsf{Q}^{\mathbf{c}}$ in Q and $\mathsf{R}^{\mathbf{c}}$ in R. Then, as per proposition 2.3.6, every $\mathbf{c} \in \mathsf{C}$ defines the same face $\mathsf{P}^{\mathbf{c}} = \mathsf{Q}^{\mathbf{c}} + \mathsf{R}^{\mathbf{c}}$ in P, and therefore, is in the same cone in $\mathcal{N}(\mathsf{P})$.

On the other hand, let C be a cone in $\mathcal{N}(\mathsf{P})$, then every $\mathbf{c} \in \text{relint}(\mathsf{C})$ defines the same face $\mathsf{P}^{\mathbf{c}}$ in P, again using the previous proposition, we get that $\mathsf{Q}^{\mathbf{c}} + \mathsf{R}^{\mathbf{c}} = \mathsf{P}^{\mathbf{c}}$, and by the uniqueness of the decomposition, we can conclude that the face defined in Q and R by every element in relint(C) is the same, so the cone must be contained in the cones defined by $\mathsf{Q}^{\mathbf{c}}$ and $\mathsf{R}^{\mathbf{c}}$ in their respective normal fans. Thus concluding that $\mathcal{N}(\mathsf{P}) = \mathcal{N}(\mathsf{Q}) \wedge \mathcal{N}(\mathsf{R})$. $\square$

Now we will define the difference of polytopes so it is an inverse of the addition when possible. This is not the only way Minkowski difference is defined in the literature, but it is the definition currently implemented in Sage. However, the documentation does not provide a reference for these basic properties.

Definition 2.3.8. Let P and Q be polytopes. We define their *Minkowski difference* as

$$\mathsf{P} - \mathsf{Q} = \bigcap_{\mathbf{q} \in Q} \mathsf{P} - \mathbf{q}$$

where $\mathsf{P} - \mathbf{q}$ is the polytope obtained from translating P by $-\mathbf{q}$.

The problem with this definition is that it is not immediately clear that it should be a polytope, to make it clearer, it is possible to define the difference using just the vertices.

Proposition 2.3.9. *Let P and Q be polytopes. The following are equivalent:*

(i) $\mathsf{R} = \mathsf{P} - \mathsf{Q}$.

(ii) If $\mathbf{v}_1, \dots, \mathbf{v}_n$ are the vertices of Q, then $\mathsf{R} = \bigcap_{i=1}^{n} \mathsf{P} - \mathbf{v}_i$.

(iii) $\mathsf{R} = (\mathsf{P}^c + (-\mathsf{Q}))^c$ *where* $\mathsf{P}^c = \mathbb{R}^d \setminus \mathsf{P}$

Proof. ($i \Leftrightarrow ii$) Proving this equivalence is the same as showing that

$$\bigcap_{\mathbf{q} \in Q} \mathsf{P} - \mathbf{q} = \bigcap_{i=1}^{n} \mathsf{P} - \mathbf{v}_i. \tag{15}$$

Vertices are points in the polytope, so the left to right inclusion is clear. On the other hand, if $\mathbf{p} \in \bigcap_{i=1}^{n} \mathsf{P} - \mathbf{v}_i$ then $\mathbf{p} \in \mathsf{P} - \mathbf{v}_i$ for all $i \in \{1, \dots, n\}$, so there are points $\mathbf{v}_i \in \mathsf{P}$ for every $i \in \{1, \dots, n\}$ such that $\mathbf{p} = \mathbf{p}_i - \mathbf{v}_i$. Now let $\mathbf{q}$ be any point in Q, then there are $\lambda_i \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$ and $\mathbf{q} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i$. So

$$\begin{aligned}
\mathbf{p} &= \sum_{i=1}^{n} \lambda_i \mathbf{p} \\
&= \sum_{i=1}^{n} \lambda_i (\mathbf{p}_i - \mathbf{q}_i) \\
&= \sum_{i=1}^{n} \lambda_i \mathbf{p}_i - \sum_{i=1}^{n} \lambda_i \mathbf{q}_i \\
&= \sum_{i=1}^{n} \lambda_i \mathbf{p}_i - \mathbf{q}.
\end{aligned}$$

So $\mathbf{p} \in \mathsf{P} - \mathbf{q}$ and we can conclude the equality on (15).

($i \Leftrightarrow iii$) For this equivalence notice that

$$\begin{aligned}(\mathsf{P}^c + (-\mathsf{Q}))^c &= \{\mathbf{x} : \mathbf{x} \neq \mathbf{p} - \mathbf{q} \; \forall \mathbf{p} \in \mathsf{P}^c, \; \forall \mathbf{q} \in \mathsf{Q}\} \\ &= \{\mathbf{x} : \mathbf{x} + \mathbf{q} \neq \mathbf{p} \; \forall \mathbf{p} \in \mathsf{P}^c, \; \forall \mathbf{q} \in \mathsf{Q}\} \\ &= \{\mathbf{x} : \mathbf{x} + \mathbf{q} \in \mathsf{P} \; \forall \mathbf{q} \in \mathsf{Q}\} \\ &= \bigcap_{\mathbf{q} \in \mathsf{Q}} \mathsf{P} - \mathbf{q}.\end{aligned}$$

Therefore, $(\mathsf{P}^c + (-\mathsf{Q}))^c = \mathsf{P} - \mathsf{Q}$. □

The second characterization of the proposition gives a way of computing the difference between two polytopes. Also, from that definition we can conclude that the difference between polytopes is a finite intersection of polytopes, so it's a polytope.

Taking the Minkowski difference of two polytopes does not always produce an inverse under Minkowski sum, this only happens in particular cases.

Proposition 2.3.10. *Let* P *and* Q *be polytopes. Then,*

(i) $(\mathsf{P} + \mathsf{Q}) - \mathsf{Q} = \mathsf{P}$*,*

(ii) $(\mathsf{P} - \mathsf{Q}) + \mathsf{Q} \subseteq \mathsf{P}$*,*

(iii) $(\mathsf{P} - \mathsf{Q}) + \mathsf{Q} = \mathsf{P}$ *if and only if* Q *is a summand of* P*.*

Proof. (i) Let P and Q be polytopes, then $h_{\mathsf{P}+\mathsf{Q}} = h_{\mathsf{P}} + h_{\mathsf{Q}}$, so

$$\begin{aligned}(\mathsf{P} + \mathsf{Q}) - \mathsf{Q} &= \bigcap_{\mathbf{q} \in \mathsf{Q}} (\mathsf{P} + \mathsf{Q}) - \mathbf{q} \\ &= \bigcap_{\mathbf{q} \in \mathsf{Q}} \{\mathbf{x} - \mathbf{q} : \mathbf{x} \in \mathsf{P} + \mathsf{Q}, \, \mathbf{q} \in \mathsf{Q}\} \\ &= \{\mathbf{x} : \mathbf{x} + \mathbf{q} \in \mathsf{P} + \mathsf{Q} \, \forall \mathbf{q} \in \mathsf{Q}\} \\ &= \left\{\mathbf{x} : \mathbf{c}^t(\mathbf{x} + \mathbf{q}) \leq h_{\mathsf{P}+\mathsf{Q}}(\mathbf{c}) \; \forall \mathbf{q} \in \mathsf{Q}\right\}.\end{aligned}$$

Now it is possible to apply the second characterization of Proposition 2.3.3 and write the set in terms of the support function of P and Q.

$$\begin{aligned}(\mathsf{P} + \mathsf{Q}) - \mathsf{Q} &= \left\{\mathbf{x} : \mathbf{c}^t(\mathbf{x} + \mathbf{q}) \leq h_{\mathsf{P}}(\mathbf{c}) + h_{\mathsf{Q}}(\mathbf{c}) \; \forall \mathbf{q} \in \mathsf{Q}\right\} \\ &= \left\{\mathbf{x} : \mathbf{c}^t\mathbf{x} \leq h_{\mathsf{P}}(\mathbf{c}) + h_{\mathsf{Q}}(\mathbf{c}) - \mathbf{c}^t\mathbf{q} \; \forall \mathbf{q} \in \mathsf{Q}\right\} \\ &= \left\{\mathbf{x} : \mathbf{c}^t\mathbf{x} \leq h_{\mathsf{P}}(\mathbf{c}) + h_{\mathsf{Q}}(\mathbf{c}) - \max_{\mathbf{q} \in \mathsf{Q}}\{\mathbf{c}^t\mathbf{q}\}\right\} \\ &= \left\{\mathbf{x} : \mathbf{c}^t\mathbf{x} \leq h_{\mathsf{P}}(\mathbf{c}) + h_{\mathsf{Q}}(\mathbf{c}) - h_{\mathsf{Q}}(\mathbf{c})\right\} \\ &= \left\{\mathbf{x} : \mathbf{c}^t\mathbf{x} \leq h_{\mathsf{P}}(\mathbf{c})\right\} \\ &= \mathsf{P}.\end{aligned}$$

(ii) If $\mathbf{x} \in (\mathsf{P} - \mathsf{Q}) + \mathsf{Q}$ then $\mathbf{x} = \mathbf{r} + \mathbf{q}$ for some $\mathbf{r} \in \mathsf{P} - \mathsf{Q}$ and $\mathbf{q} \in \mathsf{Q}$, which means $\mathbf{r} \in \mathsf{P} - \mathbf{q}'$ for every $\mathbf{q}' \in \mathsf{Q}$, in particular for $\mathbf{q}' = \mathbf{q}$. Then, there is a $\mathbf{p} \in \mathsf{P}$ such that $\mathbf{x} = (\mathbf{p} - \mathbf{q}) + \mathbf{q} \in \mathsf{P}$, and $(\mathsf{P} - \mathsf{Q}) + \mathsf{Q} \subseteq \mathsf{P}$.

(iii) If $(\mathsf{P} - \mathsf{Q}) + \mathsf{Q} = \mathsf{P}$, then we can define the polytope $\mathsf{R} = \mathsf{P} - \mathsf{Q}$. Then $\mathsf{R} + \mathsf{Q} = \mathsf{P}$, so Q is a summand of P.

If Q is a summand of P, then there exists a polytope R such that $\mathsf{P} = \mathsf{R} + \mathsf{Q}$, so we can apply (i) and obtain

$$(\mathsf{P} - \mathsf{Q}) + \mathsf{Q} = ((\mathsf{R} + \mathsf{Q}) - \mathsf{Q}) + \mathsf{Q} = \mathsf{R} + \mathsf{Q} = \mathsf{P}.$$

□

Remark 2.3.11. So far, the difference of polytopes has been described in terms of vertices and intersections, but it is also possible to do so in terms of the support function of both polytopes. Let P and Q be polytopes in $\mathbb{R}^d$, then

$$\begin{aligned}
\mathsf{P} - \mathsf{Q} &= \bigcap_{\mathbf{q} \in \mathsf{Q}} \left\{\mathbf{x} : \mathbf{c}^t\mathbf{x} \leq h_{\mathsf{P}-\mathbf{q}}(\mathbf{x})\ \forall \mathbf{c} \in \mathbb{R}^d\right\} \\
&= \left\{\mathbf{x} : \mathbf{c}^t\mathbf{x} \leq h_{\mathsf{P}}(\mathbf{c}) - \mathbf{c}^t\mathbf{q}\ \forall \mathbf{c} \in \mathbb{R}^d\ \forall \mathbf{q} \in \mathsf{Q}\right\} \\
&= \left\{\mathbf{x} : \mathbf{c}^t\mathbf{x} \leq h_{\mathsf{P}}(\mathbf{c}) - \max_{\mathbf{q} \in \mathsf{Q}}\{\mathbf{c}^t\mathbf{q}\}\ \forall \mathbf{c} \in \mathbb{R}^d\right\} \\
&= \left\{\mathbf{x} : \mathbf{c}^t\mathbf{x} \leq h_{\mathsf{P}}(\mathbf{c}) - h_{\mathsf{Q}}(\mathbf{c})\ \forall \mathbf{c} \in \mathbb{R}^d\right\}.
\end{aligned}$$

Therefore, for every $\mathbf{c} \in \mathbb{R}^d$, $h_{\mathsf{P}-\mathsf{Q}}(\mathbf{c}) \leq h_{\mathsf{P}}(\mathbf{c}) - h_{\mathsf{Q}}(\mathbf{c})$.

This doesn't mean $\mathsf{P} - \mathsf{Q}$ will always have support function $h_{\mathsf{P}} - h_{\mathsf{Q}}$, this will only happen when for every $\mathbf{c} \in \mathbb{R}^d$, the inequality defined is tight.

Lemma 2.3.12. *The equality $h_{\mathsf{P}-\mathsf{Q}} = h_{\mathsf{P}} - h_{\mathsf{Q}}$ holds if and only if $h_{\mathsf{P}} - h_{\mathsf{Q}}$ is the support function for a polytope.*

Proof. Let $h_{\mathsf{P}-\mathsf{Q}} = h_{\mathsf{P}} - h_{\mathsf{Q}}$, then

$$h_{(\mathsf{P}-\mathsf{Q})+\mathsf{Q}} = h_{\mathsf{P}-\mathsf{Q}} + h_{\mathsf{Q}} = h_{\mathsf{P}} - h_{\mathsf{Q}} + h_{\mathsf{Q}} = h_{\mathsf{P}},$$

so $\mathsf{P} = (\mathsf{P} - \mathsf{Q}) + \mathsf{Q}$ and by the third statement of Proposition 2.3.10, Q is a summand of P.

The other direction is consequence of the previous remark. □

Finally, applying Theorem 2.2.17, we obtain the following result.

Corollary 2.3.13. *The polytope Q is a summand of P if and only if $h_{\mathsf{P}} - h_{\mathsf{Q}}$ is subadditive.*

Notice that it is possible to ignore the other two conditions of support functions because $h_{\mathsf{P}} - h_{\mathsf{Q}}$ will be the difference of two piecewise linear functions over $\mathcal{N}(\mathsf{P})$ and $\mathcal{N}(\mathsf{Q})$, so it is a piecewise linear function over $\mathcal{N}(\mathsf{P}) \wedge \mathcal{N}(\mathsf{Q})$. On the other hand, it will be positive homogeneous because for every $\lambda > 0$ and $\mathbf{c} \in \mathbb{R}^d$,

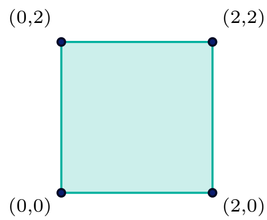
$$\begin{aligned}
h_{\mathsf{P}} - h_{\mathsf{Q}}(\lambda\mathbf{c}) &= h_{\mathsf{P}}(\lambda\mathbf{c}) - h_{\mathsf{Q}}(\lambda\mathbf{c}) \\
&= \lambda h_{\mathsf{P}}(\mathbf{c}) - \lambda h_{\mathsf{Q}}(\mathbf{c}) \\
&= \lambda(h_{\mathsf{P}} - h_{\mathsf{Q}}(\mathbf{c})).
\end{aligned}$$

Example 2.3.14. If P is a polytope and $\lambda > 1$, then $h_{\mathsf{P}} - h_{\lambda\mathsf{P}} = (1 - \lambda)h_{\mathsf{P}}$. Since h_{P} is subadditive,

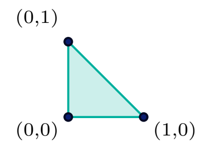
$$\begin{aligned}
&h_{\mathsf{P}}(\mathbf{c}) + h_{\mathsf{P}}(-\mathbf{c}) \geq h_{\mathsf{P}}(\mathbf{0}) = 0 \quad \forall \mathbf{c} \in \mathbb{R}^d \\
\Rightarrow &(1 - \lambda)h_{\mathsf{P}}(\mathbf{c}) + (1 - \lambda)h_{\mathsf{P}}(-\mathbf{c}) \leq 0 \quad \forall \mathbf{c} \in \mathbb{R}^d
\end{aligned}$$

If $\lambda\mathsf{P}$ was a summand of P, then $(1 - \lambda)h_{\mathsf{P}}$ would be subadditive, so

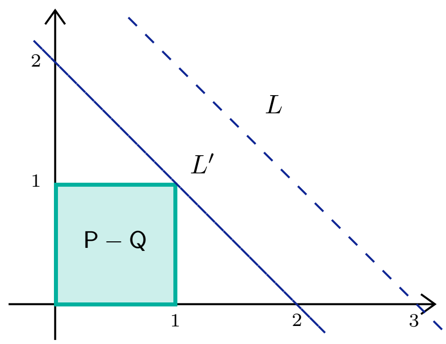
$$(1 - \lambda)h_{\mathsf{P}}(\mathbf{c}) + (1 - \lambda)h_{\mathsf{P}}(-\mathbf{c}) \geq 0 \quad \forall \mathbf{c} \in \mathbb{R}^d$$



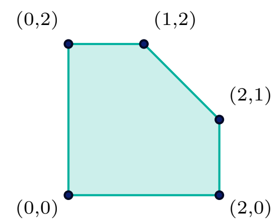
(a) Polytope P



(b) Polytope Q



(c) Polytope $\mathsf{P}-\mathsf{Q}$



(d) Polytope $(\mathsf{P}-\mathsf{Q})+\mathsf{Q}$

Figure 4

So $h_{\mathsf{P}}(\mathbf{c}) = h_{\mathsf{P}}(-\mathbf{c})$ for every $\mathbf{c} \in \mathbb{R}^d$, so P is contained in an affine linear subspace orthogonal to $\mathbf{c}$ for every $\mathbf{c} \in \mathbb{R}^d$. Then P is a point.

Conversely, if P has at least 2 vertices, then $(1-\lambda)h_{\mathsf{P}}$ is not subadditive, so $\lambda\mathsf{P}$ is not a summand of P.

Example 2.3.15. An example where there is a $\mathbf{c} \in \mathbb{R}^d$ such that $h_{\mathsf{P}}(\mathbf{c}) - h_{\mathsf{Q}}(\mathbf{c}) > h_{\mathsf{P}-\mathsf{Q}}(\mathbf{c})$ and $\mathsf{P}-\mathsf{Q}$ is not empty, is if P and Q are the polytopes in figure 4a and 4b respectivelly. Then, if $\mathbf{c} = (1,1)$, a vertex in $\mathbf{p}$ that maximizes $\mathbf{c}$ is (2,2), so $h_{\mathsf{P}}(\mathbf{c}) = 4$ and a vertex in Q that maximizes $\mathbf{c}$ is $(0,1)$, so $h_{\mathsf{Q}}(\mathbf{c}) = 1$. However, in $\mathsf{P}-\mathsf{Q}$ it is maximized in (1,1) so

$$h_{\mathsf{P}-\mathsf{Q}}(\mathbf{c}) = 2 < 4 - 1 = h_{\mathsf{P}}(\mathbf{c}) - h_{\mathsf{Q}}(\mathbf{c}).$$

In fact, $h_{\mathsf{P}} + h_{\mathsf{Q}}$ is not subbaditive, because $h_{\mathsf{P}}((0,1)) = h_{\mathsf{P}}((1,0)) = 2$ and $h_{\mathsf{Q}}((0,1)) = h_{\mathsf{Q}}((1,0)) = 1$, so

$$h_{\mathsf{P}} - h_{\mathsf{Q}}((0,1)) + h_{\mathsf{P}} - h_{\mathsf{Q}}((1,0)) = 2 < 4 = h_{\mathsf{P}} - h_{\mathsf{Q}}((0,1)+(1,0)).$$

The difference between $h_{\mathsf{P}} - h_{\mathsf{Q}}$ and $h_{\mathsf{P}-\mathsf{Q}}$ can be seen in Figure 4c, where L is the line defined by $\mathbf{c}^t\mathbf{x} = h_{\mathsf{P}} - h_{\mathsf{Q}}(\mathbf{c})$ and L' is the line defined by $\mathbf{c}^t\mathbf{x} = h_{\mathsf{P}-\mathsf{Q}}(\mathbf{x})$.

Consequently, $(\mathsf{P}-\mathsf{Q})+\mathsf{Q} \neq \mathsf{P}$, but rather, $\mathsf{P}+\mathsf{Q}$ is the polytope shown in Figure 4d.

In general, any piecewise linear function can be obtained as the difference of two support functions [14, Proposition 5.13], so in most cases, taking the difference of support functions will not provide a support function.

3 Polytope decomposition

In this section we will explore how a polytope can be decomposed in Minkowski summands. This will be done firstly by presenting a way of detecting when a polytope can or cannot be written as the of sum of polytopes that are not scalings of itself. After this, we will construct the set of summands of a polytope in two different ways, based on the length of the edges, and on the support function. Finally, we will define a polytope's graph and use that to detect when a polytope is indecomposable.

3.1 Shephard's criterion

Shephard's criterion provides a way of detecting whether a polytope is the summand of another polytope by analyzing the dimensions of their faces and the length of their edges. To prove it, we first describe the dimension of the faces in terms of the normal fan and then analyze the difference between their support functions.

Lemma 3.1.1. *Let* P *and* Q *be polytopes, then the following are equivalent:*

(i) $\dim(\mathsf{P}^{\mathbf{c}}) \geq \dim(\mathsf{Q}^{\mathbf{c}})$ *for every* $\mathbf{c} \in \mathbb{R}^d$*,*

(ii) $\mathcal{N}(\mathsf{P}) \preceq \mathcal{N}(\mathsf{Q})$*,*

(iii) $h_{\mathsf{P}} - \lambda h_{\mathsf{Q}}$ *is subadditive for some* $\lambda > 0$*.*

Proof. ($i \Rightarrow ii$) We'll prove that given two points $\mathbf{c}, \mathbf{c}' \in \operatorname{relint}(\mathsf{C})$ for a $\mathsf{C} \in \mathcal{N}(\mathsf{P})_d$, then they will always be in the same cone in $\mathcal{N}(\mathsf{Q})$, this is enough to prove $\mathcal{N}(\mathsf{P}) \preceq \mathcal{N}(\mathsf{Q})$ because full dimensional cones define the fan, see Remark 2.2.6.

If $\dim(\mathsf{P}^{\mathbf{c}}) \geq \dim(\mathsf{Q}^{\mathbf{c}})$ for every $\mathbf{c} \in \mathbb{R}^d$ and $\mathbf{c}, \mathbf{c}' \in \operatorname{relint}(\mathsf{C})$, then $\mathbf{c}$ and $\mathbf{c}'$ also define vertices in Q. This means $\mathbf{c}$ and $\mathbf{c}'$ are both in the relative interior of full dimensional cones D and D' in $\mathcal{N}(\mathsf{Q})$, if $\mathsf{D} \neq \mathsf{D}'$, then the segment $[\mathbf{c}, \mathbf{c}'] := \operatorname{conv}\{\mathbf{c}, \mathbf{c}'\}$ must go through the intersection of two full dimensional cones, which is a face of both of them, so it must go through a cone of dimension at most $d-1$. Let $\mathbf{d} = \lambda\mathbf{c} + (1-\lambda)\mathbf{c}'$ be such a point, then $\mathbf{d} \in \operatorname{relint}(\mathsf{C})$ and $\dim(\mathsf{P}^{\mathbf{d}}) = 0$, but $\dim(\mathsf{Q}^{\mathbf{d}}) \geq 1$, which goes against $\dim(\mathsf{P}^{\mathbf{d}}) \geq \dim(\mathsf{Q}^{\mathbf{d}})$. In conclusion, D and D' are the same cone.

($ii \Rightarrow iii$) Let $\mathcal{N}(\mathsf{P}) \preceq \mathcal{N}(\mathsf{Q})$. Let $\mathsf{C}_1, \mathsf{C}_2 \in \mathcal{N}(\mathsf{P})_d$ be adjacent cones, then there are cones $\mathsf{D}_1, \mathsf{D}_2 \in \mathcal{N}(\mathsf{Q})_d$ such that $\mathsf{C}_1 \subset \mathsf{D}_1$ and $\mathsf{C}_2 \subset \mathsf{D}_2$. Notice that D_1 and D_2 could be the same cone or adjacent cones. On the other hand, h_{P} and h_{Q} are convex, so by Proposition 2.2.20, for every $\mathbf{c} \in \mathsf{C}_1$, $h_{\mathsf{P}}|_{\mathsf{C}_1}(\mathbf{c}) \geq h_{\mathsf{P}}|_{\mathsf{C}_2}(\mathbf{c})$ and $h_{\mathsf{Q}}|_{\mathsf{C}_1}(\mathbf{c}) \geq h_{\mathsf{Q}}|_{\mathsf{C}_2}(\mathbf{c})$.

If D_1 and D_2 are the same cone, then $h_{\mathsf{Q}}|_{\mathsf{C}_1} = h_{\mathsf{Q}}|_{\mathsf{C}_2}$, so for any $\lambda \in \mathbb{R}$ and $\mathbf{c} \in \mathsf{C}_1$,

$$\begin{aligned}(h_{\mathsf{P}} - \lambda h_{\mathsf{Q}})|_{\mathsf{C}_1}(\mathbf{c}) - (h_{\mathsf{P}} - \lambda h_{\mathsf{Q}})|_{\mathsf{C}_2}(\mathbf{c}) &= (h_{\mathsf{P}}|_{\mathsf{C}_1}(\mathbf{c}) - h_{\mathsf{P}}|_{\mathsf{C}_2}(\mathbf{c})) - \lambda(h_{\mathsf{Q}}|_{\mathsf{C}_1}(\mathbf{c}) - h_{\mathsf{Q}}|_{\mathsf{C}_2}(\mathbf{c})) \\ &= h_{\mathsf{P}}|_{\mathsf{C}_1}(\mathbf{c}) - h_{\mathsf{P}}|_{\mathsf{C}_2}(\mathbf{c}) \\ &\geq 0.\end{aligned}$$

Let D_1 and D_2 be adjacent cones. There are vertices $\mathbf{v}_1$ and $\mathbf{v}_2$ in P and $\mathbf{w}_1$ and $\mathbf{w}_2$ in Q such that $h_{\mathsf{P}}|_{\mathsf{C}_1}(\mathbf{c}) = \mathbf{c}^t\mathbf{v}_1$, $h_{\mathsf{P}}|_{\mathsf{C}_2}(\mathbf{c}) = \mathbf{c}^t\mathbf{v}_2$, $h_{\mathsf{Q}}|_{\mathsf{C}_1}(\mathbf{c}) = \mathbf{c}^t\mathbf{w}_1$, $h_{\mathsf{Q}}|_{\mathsf{C}_2}(\mathbf{c}) = \mathbf{c}^t\mathbf{w}_2$ for every $\mathbf{c} \in \mathbb{R}^d$. Lets define

$$\mu := \frac{\|\mathbf{v}_1 - \mathbf{v}_2\|}{\|\mathbf{w}_1 - \mathbf{w}_2\|} > 0,$$

since $h_{\mathsf{P}}|_{\mathsf{C}_1} - h_{\mathsf{P}}|_{\mathsf{C}_2}$ and $h_{\mathsf{Q}}|_{\mathsf{C}_1} - h_{\mathsf{Q}}|_{\mathsf{C}_2}$ are 0 in the hyperplane spanned by $\mathsf{C}_1 \cap \mathsf{C}_2$ and they are positive at the same side of said hyperplane, then $\mathbf{v}_1 - \mathbf{v}_2 = \mu(\mathbf{w}_1 - \mathbf{w}_2)$ and for every $\mathbf{c} \in \mathbb{R}^d$,

$$\begin{aligned} h_{\mathsf{P}}|_{\mathsf{C}_1} - h_{\mathsf{P}}|_{\mathsf{C}_2}(\mathbf{c}) &= \mathbf{c}^t(\mathbf{v}_1 - \mathbf{v}_2) \\ &= \mu(\mathbf{c}^t(\mathbf{w}_1 - \mathbf{w}_2)) \\ &= \mu(h_{\mathsf{Q}}|_{\mathsf{C}_1} - h_{\mathsf{Q}}|_{\mathsf{C}_2}(\mathbf{c})). \end{aligned}$$

Since there is a finite pair of adjacent cones in $\mathcal{N}(\mathsf{Q})$, we can define λ as the minimum of all the μ's defined this way, then $\lambda > 0$ and for every $\mathbf{c} \in \mathsf{C}_1$,

$$h_{\mathsf{P}}|_{\mathsf{C}_1} - h_{\mathsf{P}}|_{\mathsf{C}_2}(\mathbf{c}) \geq \lambda(h_{\mathsf{Q}}|_{\mathsf{C}_1} - h_{\mathsf{Q}}|_{\mathsf{C}_2}(\mathbf{c})).$$

So now, for every $\mathbf{c} \in \mathsf{C}_1$,

$$\begin{aligned} (h_{\mathsf{P}} - \lambda h_{\mathsf{Q}})|_{\mathsf{C}_1}(\mathbf{c}) - (h_{\mathsf{P}} - \lambda h_{\mathsf{Q}})|_{\mathsf{C}_2}(\mathbf{c}) &= (h_{\mathsf{P}}|_{\mathsf{C}_1}(\mathbf{c}) - h_{\mathsf{P}}|_{\mathsf{C}_2}(\mathbf{c})) - \lambda(h_{\mathsf{Q}}|_{\mathsf{C}_1}(\mathbf{c}) - h_{\mathsf{Q}}|_{\mathsf{C}_2}(\mathbf{c})) \\ &\geq (h_{\mathsf{P}}|_{\mathsf{C}_1}(\mathbf{c}) - h_{\mathsf{P}}|_{\mathsf{C}_2}(\mathbf{c})) - (h_{\mathsf{P}}|_{\mathsf{C}_1}(\mathbf{c}) - h_{\mathsf{P}}|_{\mathsf{C}_2}(\mathbf{c})) \\ &= 0. \end{aligned}$$

So by Proposition 2.2.20, $h_{\mathsf{P}} - \lambda h_{\mathsf{Q}}$ is subbaditive.

($iii \Rightarrow i$) Let $h_{\mathsf{P}} - \lambda h_{\mathsf{Q}}$ be subadditive for a $\lambda > 0$ and $\mathbf{c} \in \mathbb{R}^d$, then by Proposition 2.2.7 the normal cone C of $\mathsf{P}^{\mathbf{c}}$ has dimension $d - \dim(\mathsf{P}^{\mathbf{c}})$. Let $\mathbf{c}_1, \ldots, \mathbf{c}_n$ be the generators of C. By subadditivity we get

$$\begin{aligned} &\sum_{i=1}^{n} h_{\mathsf{P}} - \lambda h_{\mathsf{Q}}(\mathbf{c}_i) \leq h_{\mathsf{P}} - \lambda h_{\mathsf{Q}}\left(\sum_{i=1}^{n} \mathbf{c}_i\right) \\ \Rightarrow &\sum_{i=1}^{n}(h_{\mathsf{P}}(\mathbf{c}_i) - \lambda h_{\mathsf{Q}}(\mathbf{c}_i)) \leq h_{\mathsf{P}}\left(\sum_{i=1}^{n} \mathbf{c}_i\right) - h_{\mathsf{Q}}\left(\sum_{i=1}^{n} \mathbf{c}_i\right) \\ \Rightarrow &\sum_{i=1}^{n} \lambda h_{\mathsf{Q}}(\mathbf{c}_i) - \lambda h_{\mathsf{Q}}\left(\sum_{i=1}^{n} \mathbf{c}_i\right) \leq \sum_{i=1}^{n} h_{\mathsf{P}}(\mathbf{c}_i) - h_{\mathsf{P}}\left(\sum_{i=1}^{n} \mathbf{c}_i\right) \\ \Rightarrow &\lambda \sum_{i=1}^{n} h_{\mathsf{Q}}(\mathbf{c}_i) - \lambda h_{\mathsf{Q}}\left(\sum_{i=1}^{n} \mathbf{c}_i\right) \leq 0 \\ \Rightarrow &\sum_{i=1}^{n} h_{\mathsf{Q}}(\mathbf{c}_i) - h_{\mathsf{Q}}\left(\sum_{i=1}^{n} \mathbf{c}_i\right) \leq 0. \end{aligned}$$

Finally, as h_{Q} is subadditive, $\sum_{i=1}^{n} h_{\mathsf{Q}}(\mathbf{c}_i) - h_{\mathsf{Q}}\left(\sum_{i=1}^{n} \mathbf{c}_i\right) = 0$, so $\mathbf{c}_1, \ldots \mathbf{c}_n$ must all be in the same normal cone D in $\mathcal{N}(\mathsf{Q})$ by Proposition 2.2.18. Therefore, $\dim(\mathsf{C}) \leq \dim(\mathsf{D})$, and since $\mathbf{c} \in \mathsf{D}$, $\dim(\mathsf{P}^{\mathbf{c}}) \geq \dim(\mathsf{Q}^{\mathbf{c}})$.

□

The following theorem is Shephard's criterion. The original proof, written by Shephard, uses arguments similar to the ones that will be presented next section to prove that summands are in correspondence with Minkowski weights. Here we provide a proof using only the properties of support functions and normal fans.

Theorem 3.1.2 (Shephard's Criterion)**.** *[11, Theorem 15.2] Let* P *and* Q *be polytopes. Then* Q *is a summand of* P *if and only if the following two conditions are met*

1. $\dim(\mathsf{P}^{\mathbf{c}}) \geq \dim(\mathsf{Q}^{\mathbf{c}})$ *for every* $\mathbf{c} \in \mathbb{R}^d$*.*

2. If $\mathbf{c} \in \mathbb{R}^d$ *is such that* $\mathsf{P}^{\mathbf{c}}$ *is an edge, then* $\mathsf{Q}^{\mathbf{c}}$ *is a vertex or an edge of smaller length.*

Proof. Lets first prove the forward direction. If Q is a summand of P, then by Proposition 2.3.7, we get that $\mathcal{N}(\mathsf{P}) \preceq \mathcal{N}(\mathsf{Q})$, therefore for every $\mathbf{c} \in \mathbb{R}^d$, the cone in $\mathcal{N}(\mathsf{P})$ defined by $\mathsf{P}^{\mathbf{c}}$ is at most of the same dimension as the cone defined by $\mathsf{Q}^{\mathbf{c}}$ in $\mathcal{N}(\mathsf{Q})$, so $\dim(\mathsf{P}^{\mathbf{c}}) \geq \dim(\mathsf{Q}^{\mathbf{c}})$. Now, if $\mathsf{P}^{\mathbf{c}}$ is an edge, then $\mathsf{Q}^{\mathbf{c}}$ must be of dimension 0 or 1, meaning a vertex or an edge. Since there is a polytope R such that $\mathsf{P} = \mathsf{Q} + \mathsf{R}$, then $\mathsf{P}^{\mathbf{c}} = \mathsf{Q}^{\mathbf{c}} + \mathsf{R}^{\mathbf{c}}$. If $\mathsf{P}^{\mathbf{c}}$ were an edge and $\mathbf{r} \in \mathsf{R}^{\mathbf{c}}$, then $\mathbf{r} + \mathsf{Q}^{\mathbf{c}} \subset \mathsf{P}^{\mathbf{c}}$ so $\mathsf{P}^{\mathbf{c}}$ must at least be of the same length as $\mathsf{Q}^{\mathbf{c}}$.

For the backward direction, notice that if (1) holds, then by Lemma 3.1.1, there is a $\lambda > 0$ such that $h_{\mathsf{P}} - h_{\lambda\mathsf{Q}}$ is subadditive. From the proof of Lemma 3.1.1, we know that if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are the vertices of P and $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are the vertices of Q such that the cone defined by $\mathbf{v}_i$ in $\mathcal{N}(\mathsf{P})$ is contained in the cone defined by $\mathbf{w}_i$ in $\mathcal{N}(\mathsf{Q})$, then

$$\lambda = \min \left\{ \frac{\|\mathbf{v}_i - \mathbf{v}_j\|}{\|\mathbf{w}_i - \mathbf{w}_j\|} : \text{ there is an edge between } \mathbf{w}_i \text{ and } \mathbf{w}_j \right\},$$

so if (2) holds, $\lambda \geq 1$ and $h_{\mathsf{P}} - \lambda h_{\mathsf{Q}}$ is subadditive, so by Corollary 2.3.13 and Proposition 2.2.19, we have that $\lambda\mathsf{Q}$ is a summand of P, so there is a polytope R such that $\mathsf{P} = \lambda\mathsf{Q} + \mathsf{R}$.

As seen in Example 2.3.2, $\lambda\mathsf{Q} = \mathsf{Q} + (1-\lambda)\mathsf{Q}$, so $\mathsf{P} = \mathsf{Q} + ((1-\lambda)\mathsf{Q} + \mathsf{R})$. This means, Q is a summand of P. ☐

Corollary 3.1.3. *A polytope* Q *is a weak summand of* P *if and only if* $\mathcal{N}(\mathsf{P}) \preceq \mathcal{N}(\mathsf{Q})$*.*

Proof. For the forward direction, notice that there is a $\lambda > 0$ such that $\lambda\mathsf{Q} \leq \mathsf{P}$ so applying Shephard's criterion, we get that $\mathcal{N}(\mathsf{P}) \preceq \mathcal{N}(\lambda\mathsf{Q}) = \mathcal{N}(\mathsf{Q})$. For the backward direction, we use the third statement form Lemma 3.1.1, so there is a $\lambda > 0$ such that $h_{\mathsf{P}} - \lambda h_{\mathsf{Q}}$ is subadditive, and we recall that $h_{\lambda\mathsf{Q}} = \lambda h_{\mathsf{Q}}$, so $h_{\mathsf{P}} - h_{\lambda\mathsf{Q}}$ is subadditive and by Proposition 2.3.13 we conclude that $\lambda\mathsf{Q} \preceq \mathsf{P}$. ☐

3.2 Type cone

Now that we have a way of detecting when a polytope is a summand of another, we can construct the set of all summands of a polytope. We will first do this by choosing the length of the edges.

Given an edge E of a polytope with vertices $\mathbf{w}$ and $\mathbf{u}$, the *edge vector* associated to E can be either $\mathbf{u} - \mathbf{w}$ or $\mathbf{w} - \mathbf{u}$, a *cyclic orientation* of the edge vectors of a 2-polytope is a choice of this vectors such that the endpoint of an edge vector correspond to the stating point of the adyacent edge vector.

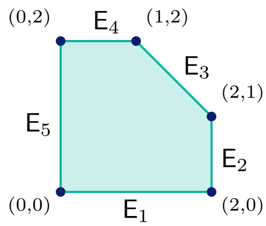
Definition 3.2.1. Let P be a polytope. A *1-Minkowski weight* on P is a function $\omega : \mathcal{F}_1(\mathsf{P}) \to \mathbb{R}$ such that for each $\mathsf{F} \in \mathcal{F}_2(\mathsf{P})$ choosing a cyclic orientation $\mathbf{v}_{\mathsf{E}}$ of its edge vectors gives

$$\sum_{\mathsf{E} \in \mathcal{F}_1(\mathsf{F})} \omega(\mathsf{E}) \mathbf{v}_{\mathsf{E}} = \mathbf{0}. \tag{16}$$

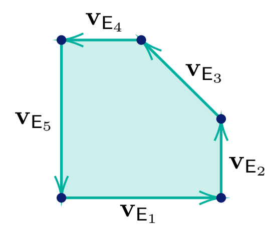
Equation (16) is called the *balancing condition.* The set of all 1-Minkowski weights on P is denoted $\Omega_1(\mathsf{P})$

The set of all 1-Minkowski weights with non negative values is called *type cone* and is denoted as

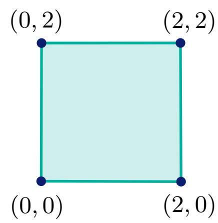
$$\mathbb{TC}(\mathsf{P}) = \{\omega \in \Omega_1(\mathsf{P}) : \omega(\mathsf{E}) \geq 0, \forall \mathsf{E} \in \mathcal{F}_1(\mathsf{P})\}. \tag{17}$$



(a) P



(b) Cyclic orientation of its edge vectors



(c) Weak summand of P

Figure 5

Clearly, $\mathbb{TC}(\mathsf{P})$ is a cone, because it is defined by a set of equations (balancing conditions) and the inequalities to make the values non negative. This cone does not contain lines because it is a subset of the cone defined by the inequalities

$$\omega(\mathsf{E}) \geq 0 \tag{18}$$

for every edge E of P, which is a pointed cone. It is also non empty because every constant non negative map is a 1-Minkowski weight, this is given by the fact that the edges of every 2-face of a polytope form a cycle, so if we choose a cyclic orientation, we have that,

$$\sum_{\mathsf{E} \in \mathcal{F}_1(\mathsf{F})} \lambda \mathbf{v}_{\mathsf{E}} = \mathbf{0} \tag{19}$$

for every $\lambda \geq 0$.

Example 3.2.2. Every 2-polytope has a unique 2-face, if P is the polytope in Figure 5a, then a possible cyclic orientation is presented in Figure 5b. A valid choice of 1-Minkowski weight is $\omega(\mathsf{E}_1) = 1$, $\omega(\mathsf{E}_2) = 2$, $\omega(\mathsf{E}_3) = 0$, $\omega(\mathsf{E}_4) = 2$ and $\omega(\mathsf{E}_5) = 1$ because it satisfies the balancing condition,

$$1(2,0) + 2(0,1) + 0(-1,1) + 2(-1,0) + 1(0,-2) = (0,0).$$

Lemma 3.2.3. *Let P be a polytope and ω a 1-Minkowski weight that takes only non negative values. If $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is a sequence of vertices such that there are edges between consecutive vertices and $\mathbf{v}_1 = \mathbf{v}_m$ then*

$$\sum_{i=1}^{m-1} \omega(\mathsf{E}_{i+1,i})(\mathbf{v}_{i+1} - \mathbf{v}_i) = \mathbf{0} \tag{20}$$

where $\mathsf{E}_{i+1,i}$ is the edge between $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$.

Proof. This result will be proven by induction on the amount of faces of dimension 2 that are inside the cycle. If $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are the vertices of a 2-face, then by definition of 1-Minkowski weights we obtain (20). On the other hand, if $\mathbf{v}_1, \ldots, \mathbf{v}_m$ surrounds more than one 2-face then let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be the vertices such that they all share a 2-face F, but $\mathbf{v}_{k+1}$ is no longer in the face. Then, let $\mathbf{v}'_1, \ldots, \mathbf{v}'_{k'}$ be the vertices of C such that $\mathbf{v}_1 = \mathbf{v}'_1$, $\mathbf{v}_k = \mathbf{v}'_k$ and $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}'_{k-1}, \ldots, \mathbf{v}'_1$ induce a cyclic orientation on the edges. Then if $\mathsf{E}'_{i,i+1}$ is the edge between $\mathbf{v}'_i$ and $\mathbf{v}'_{i+1}$ by definition of 1-Minkowski weight,

$$\begin{aligned}
&\sum_{i=1}^{k-1} \omega(\mathsf{E}_{i+1,i})(\mathbf{v}_{i+1} - \mathbf{v}_i) + \sum_{i=1}^{k'-1} \omega(\mathsf{E}'_{i,i+1})(\mathbf{v}'_i - \mathbf{v}'_{i+1}) = 0 \\
\Rightarrow &\sum_{i=1}^{k-1} \omega(\mathsf{E}_{i+1,i})(\mathbf{v}_{i+1} - \mathbf{v}_i) = -\sum_{i=1}^{k'-1} \omega(\mathsf{E}'_{i,i+1})(\mathbf{v}'_i - \mathbf{v}'_{i+1}) \\
\Rightarrow &\sum_{i=1}^{k-1} \omega(\mathsf{E}_{i+1,i})(\mathbf{v}_{i+1} - \mathbf{v}_i) = \sum_{i=1}^{k'-1} \omega(\mathsf{E}'_{i+1,i})(\mathbf{v}'_{i+1} - \mathbf{v}'_i).
\end{aligned}$$

Then,

$$\sum_{i=1}^{m-1} \omega(\mathsf{E}_{i+1,i})(\mathbf{v}_{i+1} - \mathbf{v}_i) = \sum_{i=1}^{k'-1} \omega(\mathsf{E}'_{i+1,i})(\mathbf{v}'_{i+1} - \mathbf{v}'_i) + \sum_{i=k}^{m-1} \omega(\mathsf{E}_{i+1,i})(\mathbf{v}_{i+1} - \mathbf{v}_i).$$

So now we are calculating the sum for a cycle that does not contain C, so it surrounds one 2-face less. Repeating this for every surrounded face returns us to the case where there is only one 2-face surrounded and the sum is 0. □

Lemma 3.2.4. *Let P be a polytope and $\mathbf{c} \in \mathbb{R}^d$, then given a vertex $\mathbf{v}$ of P such that $\mathbf{v} \notin \mathsf{P}^{\mathbf{c}}$, there must be another vertex $\mathbf{w}$ in P such that there is an edge between $\mathbf{v}$ and $\mathbf{w}$ and $\mathbf{c}^t\mathbf{v} < \mathbf{c}^t\mathbf{w}$.*

Proof. Let $\mathbf{v}$ be a vertex that is not in $\mathsf{P}^{\mathbf{c}}$ and $\mathbf{w}$ a vertex in $\mathsf{P}^{\mathbf{c}}$, and let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be the vertices connected to $\mathbf{v}$ by an edge, then we can take $\lambda > 0$ such that $\mathbf{p} := \lambda\mathbf{v} + (1-\lambda)\mathbf{w}$ is in $\operatorname{conv}(\{\mathbf{v}_i : 1 \le i \le m\} \cup \{\mathbf{v}\})$. If $\mathbf{c}^t\mathbf{v}_i \le \mathbf{c}^t\mathbf{v} < \mathbf{c}^t\mathbf{w}$ for every $i \in \{1, \ldots, n\}$, then

$$\mathbf{c}^t\mathbf{p} \le \mathbf{c}^t\mathbf{v} < \lambda\mathbf{c}^t\mathbf{v} + (1-\lambda)\mathbf{c}^t\mathbf{w}$$

which is a contradiction with the definition of $\mathbf{p}$. So there must be a $\mathbf{v}_i$ such that $\mathbf{c}^t\mathbf{v} < \mathbf{c}^t\mathbf{v}_i$. □

Theorem 3.2.5. *[16, Lemma 8.1] Let P be a polytope, then the points in $\mathbb{TC}(\mathsf{P})$ are in correspondence with its weak summands (up to translation).*

Proof. If Q is a weak summand of P, then Q has the same normal fan as a summand of P, meaning that $\mathcal{N}(\mathsf{P}) \preceq \mathcal{N}(\mathsf{Q})$, lets name the vertices of P $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Every full dimensional cone of $\mathcal{N}(\mathsf{P})$ is the union of cones in $\mathcal{N}(\mathsf{Q})$, so we can label the vertices of Q by $\mathbf{w}_1, \ldots, \mathbf{w}_n$ so that the cone defined by $\mathbf{v}_i$ in $\mathcal{N}(\mathsf{P})$ is contained in the cone defined by $\mathbf{w}_i$ in $\mathcal{N}(\mathsf{Q})$. This means every vertex of Q is labeled at least once (but could appear multiple times).

Notice that for every pair of vertices $\mathbf{v}_i$ and $\mathbf{v}_j$ of P such that there is an edge between them, the intersection between their respective cones in P must be a cone of codimension 1 we'll call C, this means the intersection between the cones of $\mathbf{w}_i$ and $\mathbf{w}_j$ in $\mathcal{N}(\mathsf{Q})$ must also contain C, so they are the same cone or they intersect in a facet. If they intersect in a facet, there is an edge between $\mathbf{w}_i$ and $\mathbf{w}_j$. Also, the affine space defined by both edges is perpendicular to the space spanned by C, so the edge between $\mathbf{v}_i$ and $\mathbf{v}_j$ and the edge between $\mathbf{w}_i$ and $\mathbf{w}_j$ must be parallel. Therefore, there is a $\lambda_{i,j} \in \mathbb{R}$ such that

$$\mathbf{w}_i - \mathbf{w}_j = \lambda_{i,j}(\mathbf{v}_i - \mathbf{v}_j).$$

Also, if $\mathbf{c} \in \mathbb{R}^d$ is such that $\mathbf{v}_i = \mathsf{P}^{\mathbf{c}}$, then $\mathbf{w}_i = \mathsf{Q}^{\mathbf{c}}$. So $\mathbf{c}^t\mathbf{v}_i > \mathbf{c}^t\mathbf{v}_j$ and $\mathbf{c}^t\mathbf{w}_i \geq \mathbf{c}^t\mathbf{w}_j$ and if $\lambda_{i,j} \neq 0$

$$\begin{aligned} &\mathbf{c}^t(\mathbf{v}_i - \mathbf{v}_j) > 0 \\ \Rightarrow &\mathbf{c}^t\left(\frac{1}{\lambda_{i,j}}(\mathbf{w}_i - \mathbf{w}_j)\right) > 0 \\ \Rightarrow &\frac{1}{\lambda_{i,j}}\mathbf{c}^t(\mathbf{w}_i - \mathbf{w}_j) > 0 \\ \Rightarrow &\frac{1}{\lambda_{i,j}} > 0. \end{aligned}$$

So $\lambda_{i,j} \geq 0$. Lets define $\varphi : \{\mathsf{Q}\ d-\text{polytope} : \mathsf{Q} \preceq \mathsf{P}\} \to \mathbb{TC}(\mathsf{P})$ by $\varphi(\mathsf{Q}) := \omega_{\mathsf{Q}}$ with

$$\omega_{\mathsf{Q}}(\mathsf{E}_{i,j}) = \lambda_{i,j}$$

where $\mathsf{E}_{i,j}$ is the edge between $\mathbf{v}_i$ and $\mathbf{v}_j$, ω_{Q} is well defined because $\lambda_{i,j} = \lambda_{j,i}$. Notice that if F is a 2-face of P, with vertices $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_s}$ in cyclic order and $\mathbf{v}_{i_1} = \mathbf{v}_{i_s}$, then $\mathbf{w}_{i_1}, \ldots, \mathbf{w}_{i_s}$ are the vertices of the corresponding face in Q in cyclic order, because $\mathbf{w}_{i_j}$ and $\mathbf{w}_{i_{j+1}}$ always are the same vertex or share an edge. Therefore,

$$\begin{aligned} \sum_{j=1}^{s-1} \omega_{\mathsf{Q}}(\mathsf{E}_{i_{j+1},i_j})(\mathbf{v}_{i_{j+1}} - \mathbf{v}_{i_j}) &= \sum_{j=1}^{s-1} \lambda_{i_{j+1},i_j}(\mathbf{v}_{i_{j+1}} - \mathbf{v}_{i_j}) \\ &= \sum_{j=1}^{s-1}(\mathbf{w}_{i_{j+1}} - \mathbf{w}_{i_j}) \\ &= 0. \end{aligned}$$

So ω_{Q} is a 1-Minkowski weight such that $\omega_{\mathsf{Q}}(\mathsf{E}) \geq 0$ for every edge of P, so it is a point in $\mathbb{TC}(\mathsf{P})$.

On the other hand, let $\omega : \mathcal{F}_1(\mathsf{P}) \to \mathbb{R}$ be a 1-Minkowski weight that only takes non negative values. For every $i > 1$ we can define a sequence of vertices $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_s}$ such that $i_1 = 1$, $i_s = i$ and there is an edge between every consecutive pair of vertices. We define $\mathbf{w}_1 := \mathbf{v}_1$ and

$$\mathbf{w}_i = \mathbf{v}_1 + \sum_{j=1}^{s-1} \omega(\mathsf{E}_{i_{j+1},i_j})(\mathbf{v}_{i_{j+1}} - \mathbf{v}_{i_j}),$$

then let $\mathsf{Q} := \text{conv}\{\mathbf{w}_i : 0 \leq i \leq n\}$.

This doesn't depend on the choice of path for each $\mathbf{w}_i$ because given two different paths from $\mathbf{v}_1$ to $\mathbf{v}_i$, $\mathbf{v}_{i_1}, \ldots \mathbf{v}_{i_s}$ and $\mathbf{v}'_{i_1}, \ldots \mathbf{v}'_{i_{s'}}$, then $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_s}, \mathbf{v}'_{i_{s'-1}}, \ldots, \mathbf{v}'_{i_1}$ forms a closed cycle on P so if $\mathbf{w}_i$ is the vertex defined with the same path and $\mathbf{w}'_i$ is the vertex defined with the second path, by Lemma 3.2.3 we obtain

$$\begin{aligned} \mathbf{w}_i - \mathbf{w}'_i &= \mathbf{v}_1 + \sum_{j=1}^{s-1} \omega(\mathsf{E}_{i_{j+1},i_j})(\mathbf{v}_{i_{j+1}} - \mathbf{v}_{i_j}) - \left(\mathbf{v}_1 + \sum_{j=1}^{s'-1} \omega(\mathsf{E}_{i_{j+1},i_j})(\mathbf{v}'_{i_{j+1}} - \mathbf{v}'_{i_j})\right) \\ &= \sum_{j=1}^{s-1} \omega(\mathsf{E}_{i_{j+1},i_j})(\mathbf{v}_{i_{j+1}} - \mathbf{v}_{i_j}) + \sum_{j=1}^{s'-1} \omega(\mathsf{E}_{i_j,i_{j+1}})(\mathbf{v}'_{i_j} - \mathbf{v}'_{i_{j+1}}) \\ &= \mathbf{0}. \end{aligned}$$

Finally, lets prove that Q is actually a weak summand of P using the corollary of Shephard's criterion.

Let $\mathbf{c} \in \mathbb{R}^d$, and $\mathbf{v}_i \in \mathsf{P}^{\mathbf{c}}$, then for any other vertex $\mathbf{v}_j$, $\mathbf{c}^t\mathbf{v}_i \geq \mathbf{c}^t\mathbf{v}_j$. If $\mathbf{w}_i = \mathbf{w}_j$, then clearly $\mathbf{c}^t\mathbf{w}_i = \mathbf{c}^t\mathbf{w}_j$, if $\mathbf{w}_i \neq \mathbf{w}_j$ then by Lemma 3.2.4 there is a path from $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_s}$ such that $i_1 = j$, $i_s = i$ and $\mathbf{c}^t\mathbf{v}_{i_k} \leq \mathbf{c}^t\mathbf{v}_{i_{k+1}}$ for every $1 \leq k \leq s-1$. Then, since the path chosen to define $\mathbf{w}_i$ didn't matter, we could choose the path to $\mathbf{w}_j$ and then add the vertices $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_s}$. Then, since every $\omega(\mathsf{E}_{i_{j+1},i_j})$ is non negative

$$\begin{aligned}
\mathbf{c}^t(\mathbf{w}_i - \mathbf{w}_j) &= \mathbf{c}^t\left(\sum_{k=0}^{s-1}\omega(\mathsf{E}_{i_{k+1},i_k})(\mathbf{v}_{i_{k+1}} - \mathbf{v}_{i_k})\right)\\
&= \sum_{k=0}^{s-1}\omega(\mathsf{E}_{i_{k+1},i_k})(\mathbf{c}^t\mathbf{v}_{i_{k+1}} - \mathbf{c}^t\mathbf{v}_{i_k})\\
&\geq 0.
\end{aligned}$$

Then $\mathbf{w}_i \in \mathsf{Q}^{\mathbf{c}}$. This means that the cone corresponding to $\mathbf{v}_i$ in $\mathcal{N}(\mathsf{P})$ is contained in the cone corresponding to $\mathbf{w}_i$ in $\mathcal{N}(\mathsf{Q})$. So we have that $\mathcal{N}(\mathsf{P}) \preceq \mathcal{N}(\mathsf{Q})$, and by Corollary 3.1.3 we can conclude that Q is a weak summand of P.

This correspondence is up to translation because we could have chosen $\mathbf{w}_0$ to be any in $\mathbb{R}^d$ and that would have yielded the polytope $\mathsf{Q} - \mathbf{v}_0 + \mathbf{w}_0$, which is the polytope Q translated. □

Example 3.2.6. The summands that corresponds to the 1-Minkowski weights chosen in Example 3.2.2 if we set the first vertex to stay in (0,0) is the polytope in Figure 5c.

3.3 Nef Cone

Another way of defining the cone of weak summands of a full-dimensional polytope is by analyzing their support functions, this gives rise to a new polyhedron we will call the Nef cone.

Definition 3.3.1. Let $\mathcal{F}$ be a complete fan of dimension d and W a cone of codimension 1 separating two full dimensional cones C and C' of $\mathcal{F}$. Let $\mathsf{R}_1, \ldots \mathsf{R}_{d-1}$ any $d-1$ collection of linearly independent rays of W and Q a ray in C and Q' a ray in C' such that neither of them is in W. Then there is a unique choice, up to scaling, of $c_1, \ldots, c_{d-1} \in \mathbb{R}$ and $c, c' > 0$ such that

$$c\mathbf{v}_{\mathsf{Q}} + c'\mathbf{v}_{\mathsf{Q}'} = \sum_{i=1}^{d-1} c_i \mathbf{v}_{\mathsf{R}_i}$$

where $\mathbf{v}_{\mathsf{R}}$ is a generator of R. Then we will denote $c\mathbf{v}_{\mathsf{Q}} + c'\mathbf{v}_{\mathsf{Q}'} - \sum_{i=1}^{d-1} c_i \mathbf{v}_{\mathsf{R}_i}$ by $I_{\mathcal{F},\mathsf{W}}$ and the *wall-crossing inequality* associated to W is

$$I_{\mathcal{F},\mathsf{W}}(h) := ch(\mathbf{v}_{\mathsf{Q}}) + c'h(\mathbf{v}_{\mathsf{Q}'}) - \sum_{i=1}^{d-1} c_i h(\mathbf{v}_{\mathsf{R}_i}) \geq 0$$

for a piecewise linear function $h : \mathbb{R}^d \to \mathbb{R}$ over $\mathcal{F}$.

Lemma 3.3.2. *[1, Lemma 2.11] A piecewise linear function over $\mathcal{F}$, $h : \mathbb{R}^d \to \mathbb{R}$ satisfies $I_{\mathcal{F},\mathsf{W}}(h) \geq 0$ for every W cone of codimension 1 of $\mathcal{F}$ if and only if it is convex.*

Proof. For the backward direction, let $\mathsf{R}_1, \ldots \mathsf{R}_{d-1}$ any $d-1$ collection of linearly independent rays of a wall W that separates C and C' and Q a ray in C and Q' a ray in C' such that neither of them is in W. Let $c_1, \ldots, c_{d-1} \in \mathbb{R}$ and $c, c' > 0$ be such that

$$c\mathbf{v}_{\mathsf{Q}} + c'\mathbf{v}_{\mathsf{Q}'} = \sum_{i=1}^{d-1} c_i \mathbf{v}_{\mathsf{R}_i}.$$

Define $b := \frac{c}{c+c'}$ and $b_i := \frac{c_i}{c+c'}$ for every $1 \leq i \leq d-1$, so $b \in [0,1]$ and $b' = 1-b$. Then, if h is convex,

$$\begin{aligned} bh(\mathbf{v}_{\mathsf{Q}}) + b'h(\mathbf{v}_{\mathsf{Q}'}) &\geq h(b\mathbf{v}_{\mathsf{Q}} + b'\mathbf{v}_{\mathsf{Q}'}) \\ &= h\left(\sum_{i=1}^{d-1} b_i \mathbf{v}_{\mathsf{R}_i}\right), \end{aligned}$$

and since h is linear in W,

$$h\left(\sum_{i=1}^{d-1} b_i \mathbf{v}_{\mathsf{R}_i}\right) = \sum_{i=1}^{d-1} b_i h(\mathbf{v}_{\mathsf{R}_i}).$$

So multiplying everything by $c+c'$ results in

$$ch(\mathbf{v}_{\mathsf{Q}}) + c'h(\mathbf{v}_{\mathsf{Q}'}) \geq \sum_{i=1}^{d-1} c_i h(\mathbf{v}_{\mathsf{R}_i}),$$

then, the inequality $I_{\mathcal{F},\mathsf{W}}(h) \geq 0$ holds.

For the forward direction, it is enough to show that if h satisfies that $I_{\mathcal{F},\mathsf{W}}(h) \geq 0$ for every wall W, then it is convex in full-dimensional adjacent cones. Let $\mathsf{C}, \mathsf{C}' \in \mathcal{F}_d$ be adjacent cones, then there is a wall W between them and a correspoding wall-crossing inequality

$$ch(\mathbf{v}_{\mathsf{Q}}) + c'h(\mathbf{v}_{\mathsf{Q}'}) - \sum_{i=1}^{d-1} c_i h(\mathbf{v}_{\mathsf{R}_i}) \geq 0.$$

We have that $c\mathbf{v}_{\mathsf{Q}} + c'\mathbf{v}_{\mathsf{Q}'} = \sum_{i=1}^{d-1} c_i \mathbf{v}_{\mathsf{R}_i}$, $h|_{\mathsf{C}'}$ is linear and $\mathbf{v}_{\mathsf{Q}'} \in \mathsf{C}'$ and $\mathbf{v}_{\mathsf{R}_i} \in \mathsf{C}'$, so

$$h|_{\mathsf{C}'}\left(c\mathbf{v}_{\mathsf{Q}} + c'\mathbf{v}_{\mathsf{Q}'} - \sum_{i=1}^{d-1} c_i \mathbf{v}_{\mathsf{R}_i}\right) = 0$$

and

$$\begin{aligned} ch(\mathbf{v}_{\mathsf{Q}}) + c'h(\mathbf{v}_{\mathsf{Q}'}) - \sum_{i=1}^{d-1} c_i h(\mathbf{v}_{\mathsf{R}_i}) &= ch(\mathbf{v}_{\mathsf{Q}}) + c'h(\mathbf{v}_{\mathsf{Q}'}) - \sum_{i=1}^{d-1} c_i h(\mathbf{v}_{\mathsf{R}_i}) \\ &\qquad - h|_{\mathsf{C}'}\left(c\mathbf{v}_{\mathsf{Q}} + c'\mathbf{v}_{\mathsf{Q}'} - \sum_{i=1}^{d-1} c_i \mathbf{v}_{\mathsf{R}_i}\right) \\ &= ch(\mathbf{v}_{\mathsf{Q}}) + c'h(\mathbf{v}_{\mathsf{Q}'}) - \sum_{i=1}^{d-1} c_i h(\mathbf{v}_{\mathsf{R}_i}) \\ &\qquad - ch|_{\mathsf{C}'}(\mathbf{v}_{\mathsf{Q}}) + c'h|_{\mathsf{C}'}(\mathbf{v}_{\mathsf{Q}'}) - \sum_{i=1}^{d-1} c_i h|_{\mathsf{C}'}(\mathbf{v}_{\mathsf{R}_i}) \\ &= ch(\mathbf{v}_{\mathsf{Q}}) - ch|_{\mathsf{C}'}(\mathbf{v}_{\mathsf{Q}}). \end{aligned}$$

So $h|_{\mathsf{C}}(\mathbf{v}_{\mathsf{Q}}) - h|_{\mathsf{C}'}(\mathbf{v}_{\mathsf{Q}}) \geq 0$. Notice that $h|_{\mathsf{C}} - h|_{\mathsf{C}'}$ is a linear functional that is 0 for every point in W, so it must be of the form $h|_{\mathsf{C}} - h|_{\mathsf{C}'}(\mathbf{x}) = \mathbf{a}^t\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^d$ and some $\mathbf{a} \in \mathbb{R}^d$ orthogonal to W. Since the function is positive for a point $\mathbf{v}_{\mathsf{Q}} \in \mathsf{C}$, $\mathbf{a}$ must be an inner normal vector for the cone C so for every $\mathbf{c} \in \mathsf{C}$, $h|_{\mathsf{C}} - h|_{\mathsf{C}'}(\mathbf{c}) \geq 0$ and by Proposition 2.2.20, we obtain that h must be convex. □

Definition 3.3.3. Let P be a polytope, its *Def cone*, denoted $\mathbb{DC}(\mathsf{P})$ is the cone of piecewise linear functions over $\mathcal{N}(\mathsf{P})$, $h : \mathbb{R}^d \to \mathbb{R}$ such that $I_{\mathcal{N}(\mathsf{P}),\mathsf{W}}(h) \geq 0$ for every $\mathsf{W} \in \mathcal{N}(\mathsf{P})_{d-1}$.

Notice that if P is full dimensional, then its fan is complete and pointed, so piecewise linear functions over $\mathcal{N}(\mathsf{P})$ are completely defined by the value they take on the rays, this means $\mathbb{DC}(\mathsf{P})$ is of dimension at most $|\mathcal{N}(\mathsf{P})_1|$. Also, if $h \in \mathbb{DC}(\mathsf{P})$, then it is convex, so it is the support function of a polytope Q such that $\mathcal{N}(\mathsf{Q}) \preceq \mathcal{N}(\mathsf{P})$, so by Corollary 3.1.3, we obtain that Q is a weak summand of P.

Conversely, if $\mathsf{Q} \preceq \mathsf{P}$, then its support function is piecewise linear over $\mathcal{N}(\mathsf{P})$ and convex, so $h_{\mathsf{Q}} \in \mathbb{DC}(\mathsf{P})$. In conclusion, since a polytope is uniquely determined by its support function, the points in $\mathbb{DC}(\mathsf{P})$ are in correspondence with the weak summands of P.

This correspondence is not up to translation as with $\mathbb{TC}(\mathsf{P})$, so we will define an alternative cone that represents summands up to translation

Definition 3.3.4. Let $h \in \mathbb{DC}(\mathsf{P})$, then we will say $h \sim h'$ when $h - h'$ is a linear function. Then the *Nef cone* of P, denoted $\mathbb{NC}(\mathsf{P})$, is the cone defined by $\mathbb{DC}(\mathsf{P})/\sim$.

Theorem 3.3.5. *The points in* $\mathbb{NC}(\mathsf{P})$ *are in correspondence with the weak summands of* P *up to translation.*

Proof. The only thing left to prove is that if Q and Q' are weak summands of P then $h_{\mathsf{Q}} \sim h_{\mathsf{Q}'}$ if and only if Q is a translation of Q'. Let $\mathbf{a} \in \mathbb{R}^d$, then by Proposition 2.2.22,

$$\begin{aligned}\mathsf{Q}' = \mathsf{Q} + \mathbf{a} &\Leftrightarrow h_{\mathsf{Q}'}(\mathbf{c}) = h_{\mathsf{Q}+\mathbf{a}}(\mathbf{c}) \quad \forall \mathbf{c} \in \mathbb{R}^d \\ &\Leftrightarrow h_{\mathsf{Q}'}(\mathbf{c}) = h_{\mathsf{Q}}(\mathbf{c}) + \mathbf{c}^t\mathbf{a} \quad \forall \mathbf{c} \in \mathbb{R}^d \\ &\Leftrightarrow h_{\mathsf{Q}} \sim h_{\mathsf{Q}'}.\end{aligned}$$

□

3.4 Batyrev's Criterion

During this whole section we will be working with fans that are the normal fan for some polytope.

Definition 3.4.1. Let $\mathcal{F}$ be a fan. A *non-face* is a set of rays $\mathcal{S} \subset \mathcal{F}_1$ such that they don't form a cone, a *primitive collection* is a non-face such that every proper subset of $\mathcal{S}$ forms a cone.

Notice that in the case of simplicial fans, every subset of the set of rays of a cone is a cone, so to check if a set is a primitive collection we just need to verify that the set obtained by removing any ray is the set of rays of a cone.

Since in this section it will always be clear the fan in which we are working in, we will write $I_{\mathsf{W}}(h)$ to refer to $I_{\mathcal{F},\mathsf{W}}(h)$. Let P be a simple polytope, then the coefficients of the wall-crossing inequalities are unique up to scaling, to get rid of the scaling we will assume that if Q and Q' are the rays outside of W with corresponding positive coefficients, then the smallest coefficient between the two of them is 1. For every ray $\mathsf{R} \in \mathcal{N}(\mathsf{P})_1$, $(I_{\mathsf{W}})_{\mathsf{R}}$ will denote the coefficient of $\mathbf{v}_{\mathsf{R}}$ in I_{W}.

Lemma 3.4.2. *Let* $\mathcal{F}$ *be a simplicial fan, then* $\mathcal{S} = \{\mathsf{R} \in \mathcal{N}(\mathsf{P}) : (I_{\mathsf{W}})_{\mathsf{R}} > 0\}$ *is a non-face.*

Proof. Let P be a simple polytope, $\mathsf{W} \in \mathcal{N}(\mathsf{P})_{d-1}$ and let $\mathcal{R}$ be the set of rays in the cones that have W as a face. Then

$$\mathbf{x} := \sum_{\mathsf{R} \in \mathcal{S}} (I_{\mathsf{W}})_{\mathsf{R}} \mathbf{v}_{\mathsf{R}} = - \sum_{\mathsf{R} \in (\mathcal{R} \setminus \mathcal{S})} (I_{\mathsf{W}})_{\mathsf{R}} \mathbf{v}_{\mathsf{R}}.$$

Since all the rays in $\mathcal{R} \setminus \mathcal{S}$ are rays is W, they form a cone. We also know that $-(I_\mathsf{W})_\mathsf{R} \geq 0$ for every $\mathsf{R} \in \mathcal{R} \setminus \mathcal{S}$ so $\mathbf{x}$ is in the cone formed by the rays $\mathcal{R} \setminus \mathcal{S}$. If the rays in $\mathcal{S}$ formed a cone, then $\mathbf{x}$ would also be in the cone formed by those rays, but there are no rays in common between $\mathcal{R} \setminus \mathcal{S}$ and $\mathcal{S}$, so $\mathcal{S}$ is a non-face. □

Definition 3.4.3. Let $\mathcal{F}$ be a simplicial fan and $\mathcal{S} \subset \mathcal{F}_1$ a primitive collection, then there is a minimal cone in $\mathcal{F}$ that contains $\sum_{\mathsf{R} \in \mathcal{S}} \mathbf{v}_\mathsf{R}$. If $\mathcal{T}$ is the set if rays of said cone, then there is a unique choice of coefficients $c_\mathsf{R} > 0$ for every $\mathsf{R} \in \mathcal{T}$ such that

$$\sum_{\mathsf{R} \in \mathcal{S}} \mathbf{v}_\mathsf{R} = \sum_{\mathsf{R} \in \mathcal{T}} c_\mathsf{R} \mathbf{v}_\mathsf{R}.$$

We will define the *primitive relation* by

$$I_\mathcal{S}(h) := \sum_{\mathsf{R} \in \mathcal{S}} h(\mathbf{v}_\mathsf{R}) - \sum_{\mathsf{R} \in \mathcal{T}} c_\mathsf{R} h(\mathbf{v}_\mathsf{R})$$

and its coefficients will be

$$(I_\mathcal{S})_\mathsf{R} = \begin{cases} 1 & if \quad \mathsf{R} \in \mathcal{S} \setminus \mathcal{T} \\ 1 - c_\mathsf{R} & if \quad \mathsf{R} \in \mathcal{S} \cap \mathcal{T} \\ -c_\mathsf{R} & if \quad \mathsf{R} \in \mathcal{T} \setminus \mathcal{S} \\ 0 & \quad a.o.c. \end{cases}$$

Lemma 3.4.4. *If $\mathcal{F}$ is a simplicial fan and $\mathcal{S} \subset \mathcal{F}_1$ a primitive collection, then $(I_\mathcal{S})_\mathsf{R} > 0$ if and only if $\mathsf{R} \in \mathcal{S}$.*

Proof. From the definition, $(I_\mathcal{S})_\mathsf{R} \leq 0$ for every $\mathsf{R} \notin \mathcal{S}$ and $(I_\mathcal{S})_\mathsf{R} > 0$ for every $\mathsf{R} \in \mathcal{S} \setminus \mathcal{T}$, so the only thing left to prove is that $c_\mathsf{R} < 1$ for every $\mathsf{R} \in \mathcal{T} \cap \mathcal{S}$. Suppose there is an $\mathsf{R}' \in \mathcal{T} \cap \mathcal{S}$ such that $c_{\mathsf{R}'} \geq 1$, then

$$\mathbf{x} := \sum_{\mathsf{R} \in \mathcal{S} \setminus \{\mathsf{R}'\}} \mathbf{v}_\mathsf{R} = (c_{\mathsf{R}'} - 1)\mathbf{v}_{\mathsf{R}'} + \sum_{\mathsf{R} \in \mathcal{T} \setminus \{\mathsf{R}'\}} c_\mathsf{R} \mathbf{v}_\mathsf{R}. \tag{21}$$

Since $\mathcal{S}$ is a primitive collection, the rays in $\mathcal{S} \setminus \{\mathsf{R}'\}$ form a cone $\mathsf{C} \in \mathcal{F}$. Also the rays in $\mathcal{T}$ form a cone $\mathsf{C}' \in \mathcal{F}$. If $c_{\mathsf{R}'} = 1$, then $\mathbf{x}$ lies in the interior of C and the cone spanned by $\mathcal{T} \setminus \{\mathsf{R}'\}$, so they are both the same cone, and $\mathcal{S} = (\mathcal{S} \setminus \{\mathsf{R}'\}) \cup \{\mathsf{R}'\} = (\mathcal{T} \setminus \{\mathsf{R}'\}) \cup \{\mathsf{R}'\} = \mathcal{T}$. If $c_{\mathsf{R}'} > 1$, all coefficients in (21) are positive, so $\mathbf{x}$ lies in the interior of C and C'. This means, C and C' are the same cone, so the rays in $\mathcal{S}$ are the rays of the cone C. In both cases $\mathcal{S} = \mathcal{T}$, which is a contradiction with the fact that $\mathcal{S}$ is a non-face. □

Lemma 3.4.5. *If an inequality is facet defining for a cone, then it can't be written as conical combinations of other inequalities.*

Proof. Let C be a cone and $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^d \setminus \{0\}$, $b_1, \ldots, b_n \in \mathbb{R}$ such that they are the facet defining inequalities. Then by Farkas' lemma (Lemma 2.2.14), we know that any valid inequality for C can be written as conical combination of the facet defining inequalities, so to prove the lemma it suffices to prove that $\mathbf{a}_1^t \mathbf{x} \leq b_1$ cannot be written as a conical combination of the others. Let $\mathbf{x} \in \text{relint}(\mathsf{F})$ where F is the facet defined by the inequality $\mathbf{a}_1^t \mathbf{x} \leq b_1$. The relative interior of facets is disjoint, so for any $i \in \{2, \ldots, n\}$, $\mathbf{a}_i^t \mathbf{x} < b_i$. Then, any choice of non negative coefficients c_i is such that,

$$\sum_{i=2}^n c_i \mathbf{a}_i^t \mathbf{x} < \sum_{i=2}^n c_i b_i.$$

So $\mathbf{a}_1^t\mathbf{x} \leq b_1$ cannot be written as conical combination of other inequalities. $\square$

Theorem 3.4.6 (Batyrev's Criterion)**.** *Let $\mathcal{F}$ be a simplicial fan. A piecewise-linear function h over $\mathcal{F}$ is the supporting function of a polytope if and only if*

$$I_{\mathcal{S}}(h) \geq 0 \tag{22}$$

for every primitive collection $\mathcal{S}$ of $\mathcal{F}$.

Proof. To prove the backward direction, we will show that every wall crossing inequality $I_{\mathsf{W}}(h) \geq 0$ that is facet defining for the Def cone is also a primitive inequality for some primitive collection. Let $\mathsf{W} \in \mathcal{F}_{d-1}$, then by Lemma 3.4.2, we know that $\{\mathsf{R} \in \mathcal{F}_{d-1} : (I_{\mathsf{W}})_{\mathsf{R}} > 0\}$ is a non-face, then it has a subset $\mathcal{S}$ that is a primitive collection. Lets define $\lambda > 0$ such that

$$\lambda < \min\left\{\frac{(I_{\mathsf{W}})_{\mathsf{R}}}{|(I_{\mathcal{S}})_{\mathsf{R}}|} : (I_{\mathsf{W}})_{\mathsf{R}} > 0, (I_{\mathsf{S}})_{\mathsf{R}} \neq 0\right\}$$

and lets define

$$(I)_{\mathsf{R}} = (I_{\mathsf{W}})_{\mathsf{R}} - \lambda(I_{\mathcal{S}})_{\mathsf{R}}.$$

If $(I_{\mathsf{W}})_{\mathsf{R}} = 0$, then $\mathsf{R} \notin \{\mathsf{R} \in \mathcal{F}_{d-1} : (I_{\mathsf{W}})_{\mathsf{R}} > 0\}$, so $\mathsf{R} \notin \mathcal{S}$, and we can conclude that $(I_{\mathcal{S}})_{\mathsf{R}} \leq 0$ and $(I)_{\mathsf{R}} = -\lambda(I_{\mathcal{S}})_{\mathsf{R}} \geq 0$.

If $(I_{\mathsf{W}})_{\mathsf{R}} > 0$, then

$$(I)_{\mathsf{R}} = (I_{\mathsf{W}})_{\mathsf{R}} - \lambda(I_{\mathcal{S}})_{\mathsf{R}} \geq (I_{\mathsf{W}})_{\mathsf{R}} - \frac{(I_{\mathsf{W}})_{\mathsf{R}}}{|(I_{\mathcal{S}})_{\mathsf{R}}|}(I_{\mathcal{S}})_{\mathsf{R}} \geq 0.$$

In conclusion,

$$(I_{\mathsf{W}})_{\mathsf{R}} \geq 0 \Rightarrow (I)_{\mathsf{R}} \geq 0,$$

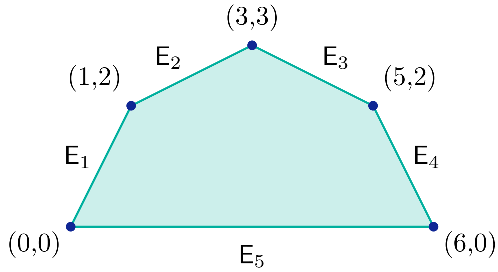
in particular, $(I)_{\mathsf{R}} \geq 0$ for every R outside of W.

Now let h be any convex function over $\mathcal{F}$, by Proposition 2.2.22, there is an $\mathbf{a} \in \mathbb{R}^d$ such that $h(\mathbf{x}) = g(\mathbf{x}) + \mathbf{a}^t\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^d$ where $g : \mathbb{R}^d \to \mathbb{R}$ is a convex function over $\mathcal{F}$ such that $g(\mathbf{x}) = 0$ for every $\mathbf{x} \in \mathsf{W}$ and $g(\mathbf{x}) \geq 0$ for every $\mathbf{x} \in \mathbb{R}^d$. Then,

$$I(g) := \sum_{\mathsf{R}\in\mathcal{F}_1} (I)_{\mathsf{R}} g(\mathbf{v}_{\mathsf{R}}) = \sum_{\substack{\mathsf{R}\in\mathcal{F}_1\\ \mathsf{R}\not\subseteq\mathsf{W}}} (I)_{\mathsf{R}} g(\mathbf{v}_{\mathsf{R}}) \geq 0.$$

Finally, from the definition of wall-crossing inequalities and primitive relations we obtain that for a linear functional they always equal 0, so

$$\begin{aligned}
I(h) &= \sum_{\mathsf{R}\in\mathcal{F}_1} (I)_{\mathsf{R}} h(\mathbf{v}_{\mathsf{R}})\\
&= \sum_{\mathsf{R}\in\mathcal{F}_1} (I)_{\mathsf{R}} g(\mathbf{v}_{\mathsf{R}}) + \mathbf{a}^t\mathbf{v}_{\mathsf{R}}\\
&= \sum_{\mathsf{R}\in\mathcal{F}_1} \left((I_{\mathsf{W}})_{\mathsf{R}} - \lambda(I_{\mathcal{S}})_{\mathsf{R}}\right)\left(g(\mathbf{v}_{\mathsf{R}}) + \mathbf{a}^t\mathbf{v}_{\mathsf{R}}\right)\\
&= \sum_{\mathsf{R}\in\mathcal{F}_1} (I_{\mathsf{W}})_{\mathsf{R}} g(\mathbf{v}_{\mathsf{R}}) - \lambda \sum_{\mathsf{R}\in\mathcal{F}_1} (I_{\mathcal{S}})_{\mathsf{R}} g(\mathbf{v}_{\mathsf{R}}) + \sum_{\mathsf{R}\in\mathcal{F}_1} (I_{\mathsf{W}})_{\mathsf{R}} \mathbf{a}^t\mathbf{v}_{\mathsf{R}} - \lambda \sum_{\mathsf{R}\in\mathcal{F}_1} (I_{\mathcal{S}})_{\mathsf{R}} \mathbf{a}^t\mathbf{v}_{\mathsf{R}}\\
&= I(g) + I_{\mathsf{W}}(\mathbf{a}) - I_{\mathcal{S}}(\mathbf{a})\\
&= I(g).
\end{aligned}$$



(a) (b)

Figure 6

So $I(h) \geq 0$ is a valid inequality for the Def cone. Now, we can write

$$I(h) + \lambda(I_{\mathcal{S}'})(h) = I_{\mathsf{W}}(h),$$

so by Lemma 3.4.5, we obtain that $I(h)$ and $\lambda(I_{\mathcal{S}'})(h)$ are just a scaling of $I_{\mathsf{W}}(h)$. This means that $\mathcal{S} = \{\mathsf{R} \in \mathcal{F}_{d-1} : (I_{\mathsf{W}})_{\mathsf{R}} > 0\}$ and if a piecewise lineal function $h : \mathbb{R}^d \to \mathbb{R}$ over $\mathcal{F}$ satisfies $I_{\mathcal{S}}(h) \geq 0$ for every primitive collection $\mathcal{S}$, then it satisfies all the inequalities $I_{\mathsf{W}}(h) \geq 0$ that are facet defining, so by Lemma 3.3.2, h is convex.

The other direction can be proven inductively using that convex functions over F are subadditive. □

Notice that from the proof we can conclude that if $\mathsf{W} \in \mathcal{F}_{d-1}$ is such that the set $\{\mathsf{R} \in \mathcal{N}(\mathsf{P}) : (I_{\mathsf{W}})_{\mathsf{R}} > 0\}$ is not a primitive collection, then $I_{\mathsf{W}}(h) \geq 0$ is not a facet-defining inequality. This gives a way of reducing the amount of inequalities necessary to build the cone.

Example 3.4.7. As with the other methods, Batyrev's Criterion might return some redundant inequalities. An example of this is the polytope in Figure 6a, notice that the sets of rays in $\mathcal{F}(\mathsf{P})$ induced by $\{\mathsf{E}_1, \mathsf{E}_3\}$, $\{\mathsf{E}_2, \mathsf{E}_4\}$, $\{\mathsf{E}_3, \mathsf{E}_5\}$, $\{\mathsf{E}_1, \mathsf{E}_4\}$ and $\{\mathsf{E}_2, \mathsf{E}_5\}$ are primitive collections because every pair of edges have empty intersection, so they are not rays of the same cone and every proper set is a ray, which is a cone in P. It is also easy to check that all of them result in a distinct inequality (22).

On the other hand, if we construct the Type cone for P, we get that it is a cone in $\mathbb{R}^5$ that satisfies two equations,

$$\mathbf{x}_1 + 2\mathbf{x}_2 + 2\mathbf{x}_3 + \mathbf{x}_4 = 0$$
$$2\mathbf{x}_1 + \mathbf{x}_2 - \mathbf{x}_3 - 2\mathbf{x}_4 - 6\mathbf{x}_5 = 0,$$

so it has dimension 3. Also, the facets of that cone correspond to the rays of $\mathcal{N}(\mathsf{P})$ such that there is a summand of P that doesn't have that ray in its normal fan. This is only possible for the edges $\mathsf{E}_1, \mathsf{E}_2, \mathsf{E}_3, \mathsf{E}_4$, so the cone has four facets, meaning not every inequality defined earlier is facet defining.

The amount of redundant inequalities can be decreased by taking first the wall-crossing inequalities and then only considering those which have positive coefficients in a primitive collection of rays. However, this can also lead to redundant inequalities as in the following example.

Example 3.4.8. By modifying the previous example a little bit, as seen in Figure 6b, we can make the redundant inequality have positive coefficients only in a primitive collection. Again, we can conclude the Nef cone of this polytope is 3 by analyzing the equations provided by the 1-Minkowski weights.

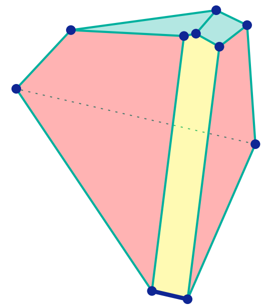


Figure 7

The ray generators in $\mathcal{N}(\mathsf{P})$ are $\mathbf{v}_1 = (-1, 0)$, $\mathbf{v}_2 = (-1, 2)$, $\mathbf{v}_3 = (1, 2)$, $\mathbf{v}_4 = (1, 0)$ and $\mathbf{v}_5 = (0, -1)$, so the wall-crossing inequality associated with E_5 is

$$h(\mathbf{v}_1) + h(\mathbf{v}_4) \geq 0, \tag{23}$$

so it has positive coefficients only in the rays associates with E_1 and E_4 which is a primitive collection. However, the wall-crossing inequalities associated to E_2 and E_3 are

$$\mathbf{v}_3 + 2\mathbf{v}_1 - \mathbf{v}_2 \geq 0$$
$$\mathbf{v}_2 + 2\mathbf{v}_4 - \mathbf{v}_3 \geq 0$$

which added up, are equivalent to the inequality in (23), so that is not a facet defining inequality.

To end this section, we provide an example found using Sage where there is a redundant inequality that has positive coefficients only in a primitive collection, but that doesn't present parallel facets as in the previous example.

Example 3.4.9. The polytope in Figure 7 is defined by

$$\begin{bmatrix} -6 & -6 & 4 \\ -6 & 6 & 4 \\ 5 & -5 & -5 \\ -4 & 0 & -6 \\ -3 & 6 & 2 \\ 1 & -1 & 0 \\ 0 & 4 & -6 \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

The edge in blue corresponds to a wall in $\mathcal{N}(\mathsf{P})$ such that the rays with positive coefficients form a primitive collection(their respective facets are shown in red), and there is a ray with a negative coefficient (corresponding to the facet in yellow). However, this is not a facet defining inequality.

3.5 Indecomposable Polytopes and Examples

In this section, we define indecomposable polytopes and show some examples that have interesting cones of summands. We will use the term *deformation cone* to refer to the cone of weak Minkowski summands of a polytope P (either parametrized by support functions or 1-Minkowski weights).

Definition 3.5.1. Let P be a polytope. It is *indecomposable* if its only summands are $\lambda\mathbf{P}$ for a $\lambda \in [0, 1]$.

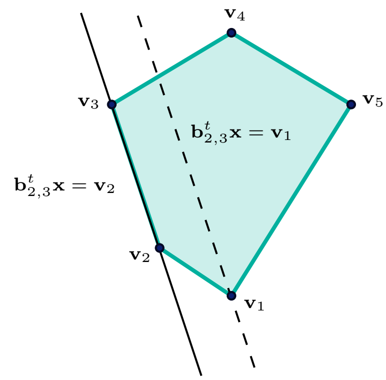


Figure 8: Constructing a summand for a pentagon

Remark 3.5.2. A polytope is indecomposable if and only if its deformation cone is 1-dimensional. This happens because for every polytope P and $\lambda \geq 0$, $\lambda\mathsf{P}$ is a weak summand.

Example 3.5.3. Let P be the d-simplex, then it has $d+1$ vertices $\mathbf{v}_1, \dots, \mathbf{v}_{d+1}$ and there is an edge $\mathsf{E}_{i,i+1}$ between $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$ for every $1 \leq i \leq d$ and between $\mathbf{v}_{d+1}$ and $\mathbf{v}_1$. Since the vectors of the edges mentioned span $\mathbb{R}^d$, there must be d independent vectors, we can assume that the independent vectors are the ones corresponding to $\mathsf{E}_{i,i+1}$ for every $1 \leq i \leq d$, if they are not, we can just shift the indices. Then,

$$\sum_{i=1}^{d} c_i(\mathbf{v}_{i+1} - \mathbf{v}_i) = \mathbf{v}_{d+1} - \mathbf{v}_1$$

if and only if $c_i = 1$ for every $1 \leq i \leq d$.

If Q is a summand of P, then it has a corresponding Minkowsky weight $\omega : \mathcal{F}_1(\mathsf{P}) \to \mathbb{R}$. By Lemma 3.2.3, we know that

$$\sum_{i=1}^{d} \omega(\mathsf{E}_{i+1,i})(\mathbf{v}_{i+1} - \mathbf{v}_i) = \omega(\mathsf{E}_{d+1,1})(\mathbf{v}_{d+1} - \mathbf{v}_1),$$

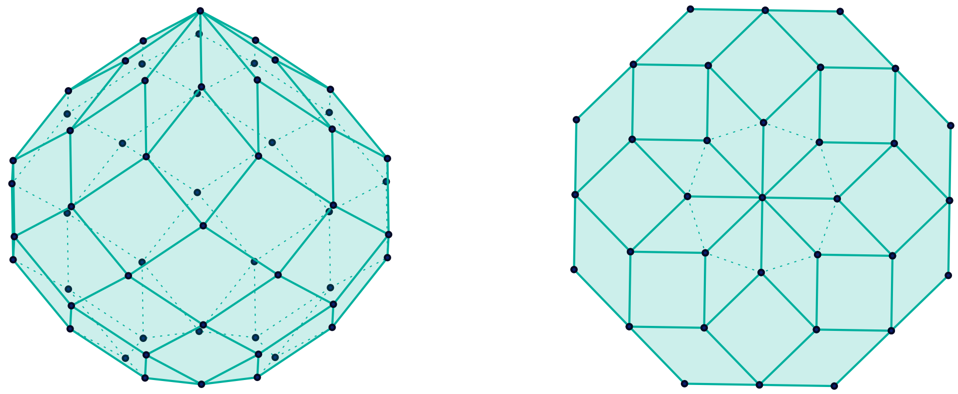
so for every $1 \leq i \leq d$, we have that $\omega(\mathsf{E}_{i+1,i}) = \omega(\mathsf{E}_{d+1,1})$. Since we can do this for every hamiltonian path in the polytope, ω must be constant. Therefore, $\mathsf{Q} = \lambda\mathsf{Q}$ for a $\lambda \geq 0$ and P is indecomposable.

Example 3.5.4. In $\mathbb{R}^2$, the last example shows that triangles are indecomposable. Now let P be a polytope in $\mathbb{R}^2$ with vertices $\mathbf{v}_1, \dots \mathbf{v}_n$ in cyclic order with $n \geq 4$ and let $\mathbf{b}_{i,i+1}^t\mathbf{x} \leq \mathbf{b}_{i,i+1}^t\mathbf{v}_i$ and $\mathbf{b}_{n,1}^t\mathbf{x} \leq \mathbf{b}_{n,1}^t\mathbf{v}_n$ be the facet defining inequalities. Without loss of generality, lets assume that $\mathbf{b}_{2,3}^t\mathbf{v}_1 \geq \mathbf{b}_{2,3}^t\mathbf{v}_4$ then we can define the polytope Q that satisfies the same inequalities as P, but it also satisfies $\mathbf{b}_{2,3}^t\mathbf{x} \leq \mathbf{b}_{2,3}^t\mathbf{v}_1$. Notice that Q still contains $\mathbf{v}_1$ and $\mathbf{v}_4$, so it has at least 2 vertices. Also, its fan has at most the rays spanned by $\mathbf{b}_{i,i+1}$ and $\mathbf{b}_{n,1}$, which are all rays of $\mathcal{N}(\mathsf{P})$, so $\mathcal{N}(\mathsf{Q}) \preceq \mathcal{N}(\mathsf{P})$. Finally, notice that $\mathbf{v}_1$ maximizes the linear functional defined by $\mathbf{b}_{n,1}$, $\mathbf{b}_{1,2}$ and $\mathbf{b}_{2,3}$ in Q, but in $\mathbb{R}^2$ a vertex only touches two facets, so one of the inequalities defined by $\mathbf{b}_{n,1}$, $\mathbf{b}_{1,2}$ and $\mathbf{b}_{2,3}$ must be redundant and thus $\mathcal{N}(\mathsf{Q})$ has less rays that $\mathcal{N}(\mathsf{P})$ and $\mathsf{Q} \neq \lambda\mathsf{P}$ for every $\lambda \geq 0$.

Figure 8 shows how this process would look like in a pentagon. In this case we would be replacing $\mathbf{b}_{2,3}^t\mathbf{x} \leq \mathbf{b}_{2,3}^t\mathbf{v}_2$ by $\mathbf{b}_{2,3}^t\mathbf{x} \leq \mathbf{b}_{2,3}^t\mathbf{v}_1$ and the redundant inequality resulting from this is $\mathbf{b}_{1,2}^t\mathbf{x} \leq \mathbf{b}_{1,2}^t\mathbf{v}_1$.

Therefore, the only indecomposable 2-polytopes are triangles.

In the book "Polytopes and graphs" [18] another way of deciding whether a polytope is indecomposable is presented by extending the concept of indecomposability to graphs and applying the following theorem.



(a) Side view (b) Top view

Figure 9

Theorem 3.5.5. *[18, Theorem 6.8] The graph of a polytope is indecomposable if and only if it contains an indecomposable subgraph that touches every facet of the polytope.*

This criterion is useful when a polytope has a lot of simplicial faces. However, in general, it does not provide an algorithmic way to decide if a polytope is indecomposable. An example of when finding and indecomposable subgraph may be difficult is the following.

Example 3.5.6. Let P be the 3-polytope obtained by adding the eight segments that start at $(0, 0, 0)$ and end at $(0, 3, 2), (3, 0, 2), (0, -3, 2), (-3, 0, 2), (2, 2, 2), (2, -2, 2), (-2, -2, 2)$ and $(-2, 2, 2)$ and then removing the vertex at $(0, 0, 0)$. The obtained polytope is shown in Figure 9. By using SageMath to calculate its Nef cone, we obtain that it is one dimensional, so it is indecomposable. However, of its 49 facets only 8 are indecomposable and they do not touch every vertex, so there isn't an obvious way to prove its indecomposability by using Theorem 3.5.5.

Now that we have seen some examples of indecomposable polytopes, there is a natural question that arises for polytopes in general ¿Can we always write a polytope as sum if indecomposable polytopes? and ¿How could we find such decomposition? To answer them, we provide the following result.

Proposition 3.5.7. *Given a polytope* P*, the rays of the deformation cone represent indecomposable summands of* P.

Proof. Let Q be the polytope represented by a point in a ray of the deformation cone. If it was decomposable, then it could be written as the sum of two polytopes that are not a scaling of Q and that are also summands of P, so the corresponding point in the deformation cone could be written as the sum of the points corresponding to its summands, thus not being part of a ray. □

Also, since the point representing P in the deformation cone can be written as conical combination of ray generators, there is a way of writing P as the sum of indecomposable polytopes. However, this decomposition may not be unique, as with the example in Figure 10.

Another interesting example of decomposable polytope is the permutohedron. This consists of the polytope obtained by taking the convex hull of all the points in $\mathbb{R}^d$ that are permutations of $(1, \dots, d)$. The summands of a permutohedron correspond to submodular functions [17, Section 4], so counting the rays of the deformation cone is equivalent to counting the extremal submodular functions. This amount is known for submodular functions on 5 elements, but it is only estimated for submodular functions on 6 elements [7].

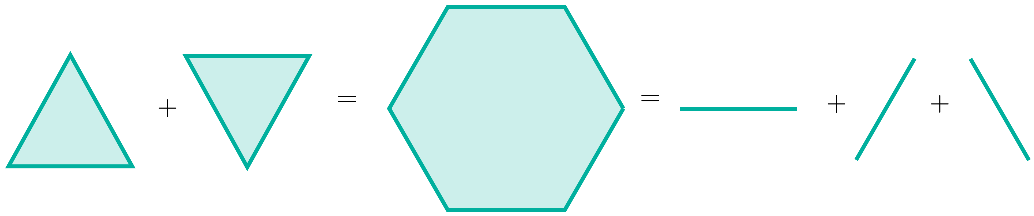


Figure 10: Regular hexagon as two different sums

Finally, we have seen that in general counting rays of a polytope is not trivial; moreover, there isn't even an easy way of telling the dimension of the deformation cone of a polytope, only that it is bounded by the amount of edges and by the amount of facets. However, in the case of simple polytopes, the dimension is known and it can even be used to calculate the amount of rays in certain contexts, as it will be seen in Theorem 3.5.9.

Theorem 3.5.8. *[15, Theorem 11] If* P *is a simple* d*-polytope, then the dimension of the deformation cone is* $|\mathcal{F}_{d-1}(\mathsf{P})| - d$*.*

Proof. Firstly, notice that $\dim(\mathbb{DC}(\mathsf{P})) \leq \mathcal{F}_{d-1}(\mathsf{P})$ because submodular functions are completely defined by the value they take on the rays of the normal fan of the polytope. Furthermore, to get the cone of summands up to translation, we can set the position of a specific vertex to always be $\mathbf{0}$, since the polytope is simple, this means we are defining the value of the support function to be 0 in d rays, so $\dim(\mathbb{NC}(\mathsf{P})) \leq \mathcal{F}_{d-1}(\mathsf{P}) - d$. To prove that this is exactly the dimension of the cone, for every ray R we have not set to 0, we will construct a weak summand of P we will call Q, such that h_{P} is equal to h_{Q} in every ray except for R.

Let $\mathbf{a}_0^t\mathbf{x} \leq b_0$ be a facet defining inequality that is satisfied in the facet F and $\mathcal{V}$ the vertices of P that are not in F, then we can define $b_0' < b_0$ such that

$$b_0' > \min\{\mathbf{a}_0^t\mathbf{v} : \mathbf{v} \in \mathcal{V}\},$$

and the polytope Q such that it satisfies the same facet defining inequalities as P, but changing $\mathbf{a}_0^t\mathbf{x} \leq b_0$ for $\mathbf{a}_0^t\mathbf{x} \leq b_0'$. The new inequality will also be facet defining in Q because the hyperplane H defined by $\mathbf{a}_0^t\mathbf{x} = b_0'$ is not a supporting hyperplane for P and it has non empty intersection with P, so the intersection must be of dimension $d-1$. Then, $\mathcal{N}(\mathsf{P})$ and $\mathcal{N}(\mathsf{Q})$ have the same rays and the support funtions are the same in every ray except for the ray spanned by $\mathbf{a}_0$. Notice that $\mathsf{Q} \subset \mathsf{P}$, so for every $\mathbf{c} \in \mathbb{R}^d$, $h_{\mathsf{P}}(\mathbf{c}) \geq h_{\mathsf{Q}}(\mathbf{c})$. Also, every $\mathbf{v} \in \mathcal{V}$ satisfies all the inequalities that define Q, so they are also points in Q; moreover, if the linear functional $\mathbf{c}$ is maximized in $\mathbf{v} \in \mathcal{V}$ in P then $h_{\mathsf{Q}}(\mathbf{c}) \geq \mathbf{c}^t\mathbf{v} = h_{\mathsf{P}}(\mathbf{c})$, so $\mathbf{v}$ also maximizes $\mathbf{c}$ in Q. Then, the cone corresponding to $\mathbf{v}$ in $\mathcal{N}(\mathsf{Q})$ contains the corresponding cone in $\mathcal{N}(\mathsf{P})$ and $\mathbf{v}$ is also a vertex of Q.

Now, in order to conclude that $\mathcal{N}(\mathsf{P}) = \mathcal{N}(\mathsf{Q})$, we just need to find vertices in Q not in $\mathcal{V}$ that have the same normal cone as the vertices in P not in $\mathcal{V}$. Let $\mathbf{v}$ be a vertex in F, then since the polytope is simple, there is a unique vertex $\mathbf{w} \in \mathcal{V}$ such that there is an edge E between $\mathbf{v}$ and $\mathbf{w}$. Let $\mathbf{v}'$ be the intersection between E and the hyperplane H, and let $\mathbf{c}$ be a point in the cone C corresponding to $\mathbf{v}$ in $\mathcal{N}(\mathsf{P})$. Notice that the ray spanned by $\mathbf{a}_0$ is a ray of C, and the cone is simplicial, so there are $\mathbf{a}_1, \ldots, \mathbf{a}_{d-1} \in \mathbb{R}^d$ and $b_1, \ldots, b_{d-1} \in \mathbb{R}$ such that $\mathbf{a}_i^t\mathbf{x} \leq b_i$ is a facet defining inequality for every $i \in \{1, \ldots, d-1\}$ and the rays of C are exactly those generated by $\mathbf{a}_0, \ldots, \mathbf{a}_{d-1}$. The rays generated by $\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}$ are the rays of the cone in $\mathcal{N}(\mathsf{P})$ corresponding to E.

There is a unique choice of coefficients $\lambda_0, \ldots, \lambda_{d-1}$ such that $\mathbf{c} = \sum_{i=0}^{d-1} \lambda_i\mathbf{a}_i$, and since the support

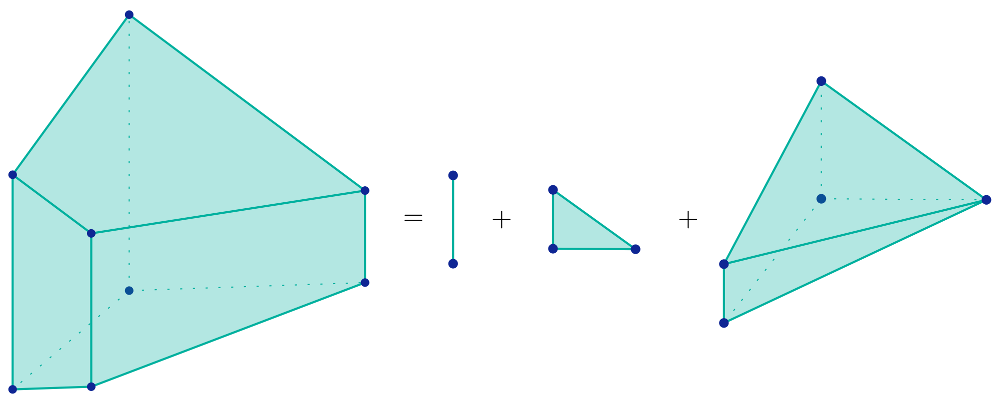


Figure 11

function is convex, we have that

$$\begin{aligned} h_{\mathsf{Q}}(\mathbf{c}) &\leq \sum_{i=0}^{d-1} \lambda_i h_{\mathsf{Q}}(\mathbf{a}_i) \\ &\leq \lambda_0 h_{\mathsf{Q}}(\mathbf{a}_0) + \sum_{i=1}^{d-1} \lambda_i h_{\mathsf{P}}(\mathbf{a}_i). \end{aligned}$$

Notice that since $\mathbf{v} \in \mathsf{H}$, then $h_{\mathsf{Q}}(\mathbf{a}_0) = b' = \mathbf{a}_0^t \mathbf{v}'$ and since $\mathbf{v}' \in \mathsf{E}$, then $h_{\mathsf{P}}(\mathbf{a}_i) = \mathbf{a}_i^t \mathbf{v}'$. So

$$\begin{aligned} h_{\mathsf{Q}}(\mathbf{c}) &\leq \lambda_0 \mathbf{a}_0^t \mathbf{v}' + \sum_{i=1}^{d-1} \mathbf{a}_i^t \mathbf{v}' \\ &= \sum_{i=0}^{d-1} \lambda_i \mathbf{a}_i^t \mathbf{v}' \\ &= \mathbf{c}^t \mathbf{v}', \end{aligned}$$

and $\mathbf{v}'$ is a vertex in Q with the same normal cone as $\mathbf{v}$ in P. This means, that $\mathcal{N}(\mathsf{Q}) = \mathcal{N}(\mathsf{P})$ and Q is a weak summand of P. □

With this theorem, we can now proceed to compute the deformation cone of any $d-$cube. This result was published in [4, Theorem 5.10], but it was proven using rainbow configurations.

Theorem 3.5.9. *The deformation cone of a $d-$cube is a simplicial cone of dimension d.*

Proof. Let P be the $0-1$ d-cube, then the the full-dimensional cones correspond to the orthants of $\mathbb{R}^d$. Pairs of opposite rays are primitive collections, and if we take any collection of rays that don't contain an opposite pair, then they generate a cone, the pairs of opposite rays are the only primitive collections. Moreover, there are d pairs of opposite rays, so by Batyrev's criterion, the deformation cone of P is generated by d inequalities. On the other hand, by Theorem 3.5.8, the deformation cone has dimension $2d - d = d$, so it must be a simplicial cone.

Notice that if Q is any other d-cube, then it has the same primitive collections as P because primitive collections only depend on the inclusion relations in the normal fan, which are completely determined by the face lattice of the polytope. So the same argument holds to prove that its deformation cone is simplicial.

□

In general, the deformation cones of combinatorically equivalent polytopes are not combinatorically equivalent. An example of this is the pentagon in Figure 6a, whose deformation cone has 4 rays. However, the deformation cone of the regular pentagon has 5 rays.

It is also interesting to note that despite the fact that all $d-$cubes have combinatorically equivalent deformation cones, its indecomposable summands are not combinatorically equivalent.

Example 3.5.10. Notice that the standard $d-$cube can be obtained by adding all the segments starting in the origin and ending on an element of the canonical basis of $\mathbb{R}^d$. Since all the segments are indecompodable and there are d of them, they must be the only indecomposable summands of the standard $d-$cube. However, if we take the $3-$cube with vertices $(0,0,0)$, $(0,0,9)$, $(0,9,3)$, $(0,9,6)$, $(9,3,1)$, $(9,3,8)$, $(9,6,2)$ and $(9,6,7)$ and use Sage to obtain its indecomposable summands, we get the decomposition seen in Figure 11.

4 Implementation

The construction of the deformation cone by 1-Minkowski weights has already been implemented in some mathematical software like Polymake and Macaulay2, however none of these software provide the implementation by constructing the cone of support functions. In the case of SageMath, it presents the possibility of calculating the Minkowski difference of polytopes and of deciding whether a given polytope is summand of another, but the construction of the deformation cone is yet to be implemented.

In the following section, we will describe what is currently implemented in SageMath [19] and how we have used it to construct the Type and Nef cones for polytopes. This work was conducted using SageMath version 10.5.beta2 and Python 3.12.4 and all the code can be found in GitHub at https://github.com/sofiaemd/Deformation-cones-SageMath.

4.1 Currently implemented

Currently there are a lot of tools implemented in Sage to work with polytopes, here we describe those that were used the most during the construction of the Nef and Type cone. In both cases, the inputs were polytopes created using `Polyhedron`.

When building the deformation cone parametrized by 1-Minkowski weights it suffices to use the methods implemented in `Polyhedron`, except for the cyclic ordering of the vertices, in which case the function `cyclic_sort_vertices_2d` had to be imported from `sage.geometry.polyhedron.plot`.

On the other hand, to construct the cone of support functions a lot of other tools had to be used, for instance, `NormalFan` was used to construct the normal fan of the input polytope along with various methods implemented within the class of Rational Polyhedral Fans, such as `primitive_collections`, to calculate all the primitive collections, `subdivide` to make the fan simplicial and `cone_lattice` to obtain the lattice of cones ordered by inclusion. Moreover, in the case of lattices, the method `upper_cover` was used to find out which full-dimensional cones contained each wall. Finally, matrix solving tools were used to find the coefficients for the wall-crossings and primitive collections.

4.2 Deformation cone implementation

4.2.1 Type cone

Let P be a d-polytope. To construct its type cone, we make a cone in $\mathbb{R}^{|\mathcal{F}_1(\mathsf{P})|}$ such that if $\mathbf{x} = (x_{\mathsf{E}})_{\mathsf{E}\in\mathcal{F}_1(\mathsf{P})} \in \mathbb{R}^{|\mathcal{F}_1(\mathsf{P})|}$ then its corresponding 1-Minkowski weight is $\omega(\mathsf{E}) = x_{\mathsf{E}}$. By Definition 3.2.1, the necessary equations to construct this cone are

$$\sum_{\mathsf{E}\in\mathcal{F}_1(\mathsf{F})} \omega(\mathsf{E})\mathbf{v}_{\mathsf{E}} = 0$$

for every $\mathsf{F} \in \mathcal{F}_2(\mathsf{P})$ with edges in cyclic order. This means that if $\mathbf{v}_{\mathsf{E}} = (v_{\mathsf{E},1}, \ldots, v_{\mathsf{E},d})$ is the edge vector for E, then we need to consider the equations

$$\sum_{\mathsf{E}\in\mathcal{F}_1(\mathsf{F})} \omega(\mathsf{E})v_{\mathsf{E},i} = 0$$

for every $i \in \{1, \ldots, d\}$. If n is the number of vertices of the polytope, the number of equations is $d|\mathcal{F}_2(\mathsf{P})|$, then by Upper Bound Theorem [20, Theorem 8.23] this means that the number of equations is at most $d\binom{n}{3}$, and exactly that amount in cyclic polytopes (although some of them could be repeated).

In addition, we need to add the necessary inequalities to make every coordinate positive, this adds $|\mathcal{F}_1(\mathsf{P})|$ inequalities which are all different, but not necessarily facet defining.

Notice that in this implementation, if n is the number of vertices, we take $\mathcal{O}(n^4)$ for each equation, because after choosing the edge orientation, for every edge in the 2-face we go through all the edges again to find the corresponding coefficient in the cone, and the amount of edges of a polytope could be quadratic on the number of vertices by Upper Bound Theorem. This means that generating a list of the equations needed to construct the cone takes $\mathcal{O}(d\binom{n}{3}n^4) + \mathcal{O}(\binom{n}{2}) = \mathcal{O}(dn^7)$. This could be improved by choosing a better data structure for the edges of the polytope so that obtaining the corresponding indices is faster.

Let P be a polytope with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$ such that every vertex is connected by an edge to at least one of the previous vertices, this can be done by ordering them by the sum of their coordinates. To construct a summand of P from a point in the Type cone corresponding to the 1-Minkowski weight $\omega : \mathcal{F}_2(\mathsf{P}) \to \mathbb{R}$ with $\omega(\mathsf{E}_{i,j}) = \lambda_{i,j}$, for every vertex $\mathbf{v}_i$ of P we will find the corresponding vertex $\mathbf{w}_j$ in the summand starting by assigning $(0, 0, 0)$ to $\mathbf{v}_1$, then for every $i \in \{2, \ldots, n\}$ we look for the first vertex $\mathbf{v}_j$ such that there is an edge between $\mathbf{v}_i$ and $\mathbf{v}_j$ and we assign $\mathbf{w}_j := \lambda_{i,j}(\mathbf{v}_j - \mathbf{v}_i) + \mathbf{w}_i$. Finally, we can take the convex hull of $\mathbf{w}_1, \ldots, \mathbf{w}_n$.

Given the graph polytope and a point in the deformation cone, the construction first orders the vertices and then for every vertex it checks to which vertex it is previously connected to. Thus, this construction takes $\mathcal{O}(n^2)$ if n is the number of vertices of the polytope.

4.2.2 Nef Cone

To construct the cone of piecewise linear functions over a given d-dimensional complete fan $\mathcal{F}$, notice that it is enough to know the value of the function in the generators of the rays, because every linear function is defined by its image in d points.

A first method to construct the Nef cone of a simple d-polytope P could be by its wall-crossing inequalities defined in Definition 3.3.1. This can be implemented by first creating the normal fan of P and its cone lattice. For every $d - 1$ dimensional wall W, we check its upper cover in the cone lattice, which correspond to the cones that contain W. Then, we take the two rays that span the cones adjacent to the wall and create a matrix with them and all the rays of W. An element in the kernel of this matrix provides a choice of coefficients for I_{W}, then we consider

$$\sum_{\mathsf{R} \in \mathcal{N}(\mathsf{P})_1} (I_{\mathsf{W}})_{\mathsf{R}} x_{\mathsf{R}} \geq 0.$$

The points $\mathbf{x} \in \mathbb{R}^{|\mathcal{N}(\mathsf{P})_1|}$ that satisfy those inequalities, correspond to possible values of support functions in each ray generator. This means, every point $\mathbf{x} = (x_{\mathsf{R}})_{\mathsf{R}}$ in the cone uniquely defines a piecewise linear function $h : \mathbb{R}^d \to \mathbb{R}$ such that $h(\mathbf{v}_{\mathsf{R}}) = x_{\mathsf{R}}$.

Finally, if we want the points to represent the polytopes up to translation, we can choose a vertex $\mathbf{v}$ of P and add equations such every point $\mathbf{x} = (x_{\mathsf{R}})_{\mathsf{R}}$ satisfies $x_{\mathsf{R}} = 0$ for every ray in the cone corresponding to $\mathbf{v}$ in $\mathcal{N}(\mathsf{P})$.

This means that the cone is created in $\mathbb{R}^{|\mathcal{F}_d(\mathsf{P})|}$ and is defined by $|\mathcal{N}_{d-1}(\mathsf{P})|$ inequalities and d equations, by Upper Bound Theorem, the number of inequalities is bounded by $\binom{n}{2}$ for n vertices. For every one of the cones of codimension 1 W, we first need to search for the rays that generate the full dimensional cones which intersect in W that takes $\mathcal{O}(n)$. Then, it is necessary to find the solution to a $d \times (d + 1)$ matrix, which is done in $\mathcal{O}(d^3)$ and finally we search the indices to construct the inequalities which takes $f(d + 1)$ where f is the number of facets of P. All together, this means the process takes $\mathcal{O}(\binom{n}{2}(n+d^3+f(d+1))) = \mathcal{O}(n^3+n^2d^3+n^2df)$, but by using Upper Bound theorem, we get that the number of facets is exponential in the dimension. Still, this could potentially be improved by choosing better data structures, so that it is not necessary to go through all the coordinates of the inequality to generate it.

To reduce the number of equations, it is possible to use the argument presented in the proof of Batyrev's Criterion (Theorem 3.4.6), and only consider the inequalities $I_{\mathsf{W}}(h) \geq 0$ such that $\{\mathsf{R} : (I_{\mathsf{W}})_{\mathsf{R}} > 0\}$ are primitive collections.

On the other hand, it is possible to define the cone using only the inequalities provided from Batyrev's criterion, meaning that we first construct the primitive collections of the normal fan of P, then for every collection $\mathcal{S}$ we consider the inequality

$$\sum_{\mathsf{R} \in \mathcal{N}(\mathsf{P})_1} (I_{\mathcal{S}})_{\mathsf{R}} x_{\mathsf{R}} \geq 0.$$

However, this could yield more inequalities than working with wall-crossings.

To make the construction for non-simple polytopes, it is possible to triangulate the normal fan of the polytope and construct all the wall-crossings, but considering the inequalities for the original walls of the fan and taking the equality in the case of the walls added in the triangulation. The equations are necessary to ensure that the function is linear in each cone, and the inequalities ensure that the function is convex.

To build a summand of P from a point in the cone, it is enough to consider the facet-defining inequalities of P and then changing the constant term to the one in the coordinate of the ray corresponding to said facet.

4.3 Computational experiments

To test the running time for the algorithms, the functions were tested over some classic examples of polytopes and polytopes generated by random facets and vertices. All the experiments were conducted on an Intel Core CPU i7-13620H with 16 GB of RAM, under Ubuntu 24.04.1.

Firstly, the methods were tested over some of the polytopes already implemented in SageMath, the table in Figure 12 shows the time (in seconds) taken to compute each of the cones, the result is the average of computing the cone 10 times. The first column shows the time spent when the cone is built using the 1-Minkowski weights, the second when it is built using wall-crossing inequalities, the third when it is built using all the inequalities obtained from primitive collections, and the final column shows the time spent when using only the wall-crossing inequalities that have positive coefficients in a primitive collection.

The polytopes chosen were:

1. Simplices: Previously defined, simplices always have a one dimensional deformation cone, so it is easy to check if the cone is being correctly computed.

2. Cubes: As shown in the end of the previous section, cubes present a lot of redundant inequalities when constructing the cone with wall-crossing inequalities, so it should be faster computing it that way rather than by 1-Minkowski weights.

3. Permutohedron: The rays of its deformation cone are extremal submodular functions, so it is interesting to study the possibility of computing the cone for permutahedra in dimension 6 or more. To construct the normal fan, Sage requires the polytope to be full-dimensional, so we had to project the polytope to a space of lower dimension. Currently it is only possible to do this for dimension up to 4, so for the 5-permutohedron, it was only possible to construct the

	1-Minkoski weights	Wall-crossings	Primitive collections	Batyrev
3−simplex	0.05	0.016	0.021	0.019
6−simplex	0.376	0.048	0.107	0.047
8−simplex	1.138	0.114	1.119	0.114
9−simplex	15.84	0.19	4.13	0.21
3−cube	0.025	0.013	0.008	0.014
5−cube	1.49	0.096	0.04	0.096
6−cube	11.574	0.356	0.125	0.28
3−permutahedron	0.017	0.016	0.028	0.022
4−permutahedron	0.308	0.192	0.391	0.184
5−permutahedron	5837.84	−	−	−
$(3,30)$−cyclic	0.86	2.12	−	−
$(4,20)$−cyclic	11.29	39.1	−	−
$(5,20)$−cyclic	22.5	320	−	−

Figure 12

cone of 1-Minkowski weights. Also, in the case of the 5-permutohedron we considered the time of running the function only once.

4. Cyclic polytopes: The (d,n)-cyclic polytope is defined as the polytope with vertices $(t,t^2,\dots,t^d)$ for $t \in \{1,\dots,n\}$. This polytope is famous for having a maximal f-vector, meaning it has the greatest amount possible of faces for each dimension. Since it is not simple, it was not possible to remove inequalities by studying the primitive collections of its normal fan.

To generate the random polytopes we created a function that takes as input the dimension, number of inequalities or vertices and the range of possible integer values for the coordinates of said vertices or inequalities. In the case of choosing vertices, it is then possible to just generate the convex hull and obtain a polytope, but when choosing inequalities, it is also necessary to check that the polyhedron created is bounded.

The table in Figure 13, shows the time (is seconds) taken to compute the deformation cone of 10 random H-polytopes, the column headers show the dimension and amount of inequalities used to define the polytopes, in all cases the range for the coordinates was $[-10,10]$ and it was checked whether the polytopes were simple, so we could apply Batyrev's criterion. It was also necessary to check whether the polytopes were full dimensional in order to compute their normal fan.

Finally, we compared the time over random V-polytopes, again the coordinates were considered between -10 and 10, the headers show the dimension and number of vertices. The time shown considers the computation of 10 different random polytopes in seconds.

By profiling, we can understand what in the process takes the longest time. In the case of 1-Minkowski weights, in every case we tested, most of the time was spent constructing the cone, not calculating the coefficients for the equations.

On the other hand, for of the cone of support functions, in every non simplicial polytope tested, at least 65% of the time was spent making the normal fan of the polytope simplicial and most of the time

	$(3,20)$	$(4,10)$	$(4,15)$
1-Minkowski weights	22.791	2.637	164.091
Wall-crossings	1.967	0.388	19.378
Primitive collections	26.397	0.877	17.803
Batyrev	1.282	0.356	4.06

Figure 13: Random facet polytopes

	$(3,20)$	$(4,10)$	$(4,20)$
1-Minkowski weights	0.917	2.083	5.217
Wall-crossings	3.554	13.047	28.42

Figure 14: Random vertices polytopes

left was spent building the cone lattice. For simplicial polytopes, as the dimension of the polytope increased, the steps that took the most time were generating the set of walls of the normal fan and the set of primitive collection sof rays. However, when we increased the range of for the coordinates of the inequalities in the generation of random polytopes, the step that took the longest was generating the cone with the inequalities previously calculated.

In conclusion, if we wanted to make the implementation of the deformation cone more efficient, probably the best thing we could do would be to find a way of reducing even more the number of inequalities or the dimension of the ambient space of the cone. However, the algorithms implemented give a reasonable tool to work with a variety of polytopes in lower dimensions and with bounded rational coefficients.

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