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Riemann–Roch type inequality and an upper bound for the generalized greatest common divisor

by

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Thesis submitted to the Faculty of Mathematics
of Pontificia Universidad Católica de Chile,
as one of the requirements to qualify for
the academic Master's degree in Mathematics.

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September, 2024

Santiago, Chile.

Acknowledgments

I would like to thank to the professor Héctor Pastén for his patience and his useful comments during my master's, especially in the development of this thesis.

Also, I would like to thank to the professors Siddarth Mathur and Ricardo Menares for being part of the committee of this thesis.

Additionally, I want to thank to Karla Sepúlveda, Sandra Conejeros, Jaime Barrios and all my family. Their presence and support have been fundamental in all this process.

Finally, I would like to thank to my friends, in particular to Jorge Acuña, Vicente Monreal, Sebastián Muñoz-Thon, Eduardo Reyes, Ignacio Rojas, Juan Rojas, María Alejandra Schild, Dievo Vásquez and Felipe Vásquez. Their company and advice have helped me a lot.

My master's studies were financed by the Master's Fellowship ANID Folio 22221062.

Contents

Introduction

The height of a rational number is a way to measure its complexity, for example the numbers 1/3 and 100/299 are close to each other in terms of the Euclidean metrics, but 1/3 is simpler. The concept of height can be extend to arbitrary number fields via absolute values and furthermore, the height can be defined for points in the projective space. Moreover, since very ample divisors define an embedding from a projective variety to some projective space, one can define the height of a point of a variety with respect to a divisor on this variety, see for instance [HS00, Part B].

Silverman defined the height of a point in a variety with respect to arbitrary closed sub-schemes in [Sil87], and in [Sil05] he studied how this concept is related to the greatest common divisor. Then the generalized greatest common divisor is defined as the height of a point with respect to a closed sub-scheme.

The goal of this thesis is to give an upper bound for the generalized greatest common divisor of a point with respect to an irreducible sub-variety of a given variety, see Definition 2.2.3.

In the literature there are several bounds for the greatest common divisors in several cases. Bugeaud, Corvaja and Zannier [BCZ03] give an upper bound for the greatest common divisor of a^n-1 and b^n-1 where a, b are multiplicatively independent integers. Later, Corvaja and Zannier [CZ05] generalize the previous result when a^n and b^n are replaced by elements of a fixed finitely generated subgroup of $\overline{\mathbb{Q}}^*$. In this way, Levin [Lev19] gives a GCD bound for polynomials in several variables with algebraic coefficients, generalizing the result obtained by Corvaja and Zannier.

Bounds for the generalized greatest common divisor in varieties are given by Grieve in [Gri20] and by Wang and Yasukufu in [WY21] under integrality conditions of the points. Recently, García-Fritz and Pastén [GFP23] give an upper bound when the closed subscheme is reduced and consists on d geometric points, without integrality conditions. In fact they have:

Theorem 1. *Let X be a smooth projective variety defined over a number field K of dimension n and $\mathcal{A}$ be an ample line sheaf on X. Let Y be a reduced closed sub-scheme consisting on d geometric points. Then given any $\varepsilon > 0$ there is a properly contained Zariski closed set $Z_\varepsilon \subset X$ such that*

$$h_{\mathrm{gcd}}(x; Y) \leq \left(\sqrt[n]{\frac{d}{(\mathcal{A}^n)}} + \varepsilon \right) h(\mathcal{A}, x) + O(1)$$

as x varies in $(X - Z_\varepsilon)(\overline{K})$.

In this thesis we follow the ideas of García-Fritz and Pastén to give an upper bound for the GCD when the closed sub-scheme is a higher dimensional irreducible sub-variety. Explicitly we obtain:

Theorem 2. *Let X be a smooth projective variety defined over a number field K of dimension n and Y be an irreducible sub-variety of dimension d, also defined over K. Let $X' = X \setminus Y$. Then given any ample line sheaf $\mathcal{A}$ and $\varepsilon > 0$, there is a properly contained Zariski closed set $Z_\varepsilon \subset X'$ such that, for all $x \in (X' \setminus Z_\varepsilon)(\overline{K})$ the formula:*

$$h_{\mathrm{gcd}}(x;Y) \leq \left(\left(\frac{(\mathcal{A}^d \cdot Y)}{(\mathcal{A}^n)} \cdot \frac{n!}{c!} \right)^{\frac{1}{c}} + \varepsilon \right) h(\mathcal{A}, x) + O_\varepsilon(1)$$

holds, where $c = n - d$.

To prove the result, we study certain geometric conditions. Explicitly, we give a Riemann–Roch type inequality for a particular type of coherent sheaves on X. Using the ideas of Matsusaka in [Mat72] we obtain:

Theorem 3. *Let X be a projective variety defined over K and $Y \subset X$ be an irreducible sub-variety of dimension d. Let $\mathcal{I}_Y$ be the ideal sheaf associated to Y. Then given any very ample line sheaf $\mathcal{A}$ on X, we have:*

$$h^0(X, \mathcal{A}^{\otimes m} \otimes \mathcal{O}_X/\mathcal{I}_Y^r) \leq \frac{r^c \cdot m^d}{c!} e_X(Y) \cdot (\mathcal{A}^d \cdot Y) + O(r^{c-1})$$

for $r \gg 1$, where $c = \dim X - d$, $e_Y(X)$ is the algebraic multiplicity of X along Y and the constant in $O(r^{c-1})$ only depends on X, Y and $\mathcal{A}$.

Let us comment the structure of this thesis. In the first chapter of this thesis we review certain geometric concepts that appear in the proof of the main theorem. Section 1.1 is a brief review about multiplicities in varieties. Then in section 1.2 we talk about the intersection number in varieties and finally in section 1.3 we study the Riemann–Roch theorem and we give the proof of Theorem 3.

The second chapter contains the proof of the main result, Theorem 2. Section 2.1 contains the principal concepts and properties about heights. Then in section 2.2 we define the generalized greatest common divisors for algebraic points in varieties defined over number fields. In section 2.3 we give the proof of the GCD bound Theorem 2 and finally Section 2.4 contains three applications of Theorem 2. The first one is an upper bound for the greatest common divisor of rational numbers. The second one is an upper bound for the greatest common divisor of two algebraic numbers. And the third one is an upper bound for the greatest common divisor of points in the projective space with respect to arbitrary sub-varieties.

Chapter 1

Geometric preliminaries

1.1 Multiplicities

This section is an approach to the concept of multiplicities in varieties. For a more general definition of multiplicities see [Ful98].
For a review of the concepts about sheaves and schemes, see for instance [Har77, Chapter II].

Definition 1.1.1. Let K be a field. A variety is an integral separated scheme of finite type over K (i.e. over $\operatorname{Spec} K$).

Definition 1.1.2. Let D be an effective Cartier divisor on a variety X and x be a point in X. Let f be a local equation for D at x. We define the multiplicity of D at x, denoted by $m_x(D)$, to be the largest integer r such that $f \in \mathfrak{m}_x^r$, where $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is the maximal ideal of the local ring.

Definition 1.1.3. Let D and X be as in Definition 1.1.2. For an irreducible sub-variety $Y \subset X$ we define the multiplicity of D at Y, denoted by $m_Y(D)$, as the multiplicity of D at a general point x in Y.

Proposition 1.1.4 ([Ful98, Example 4.3.9])**.** *Let D be an effective Cartier divisor on X and $Y \subset X$ be an irreducible sub-variety of X. Consider $\pi : \tilde{X} \to X$ be the blow-up of X at Y and E_Y be the exceptional divisor of π. If $\tilde{D}$ be the strict transform of D, then*

$$\tilde{D} = \pi^* D - m_Y(D) \cdot E_Y$$

Lemma 1.1.5. *Let Y be an irreducible sub-variety of a variety X. Let $\mathfrak{m}_{X,Y} \subset \mathcal{O}_{X,Y}$ be the maximal ideal of the local ring. Then for $t \gg 0$, $\operatorname{lenght}_{\mathcal{O}_{X,Y}}(\mathcal{O}_{X,Y}/\mathfrak{m}^t_{X,Y})$ is a polynomial of degree $c = \operatorname{codim}(Y, X)$.*

Proof. [AM69, Proposition 11.4]. □

Definition 1.1.6. Let Y and X as in Lemma 1.1.5. The algebraic multiplicity of X along Y, denoted by $e_Y(X)$, is $c!$ times the leader term of $\operatorname{length}_{\mathcal{O}_{X,Y}}(\mathcal{O}_{X,Y}/\mathfrak{m}^t_{X,Y})$ for $t \gg 0$ where $c = \operatorname{codim}(Y, X)$.

An important example of algebraic multiplicity along a sub-variety is the following:

Proposition 1.1.7. *If $Y \subset X$ is an irreducible sub-variety such that X is regular at the generic point of Y, then $e_Y(X) = 1$.*

Proof. Let ξ be the the generic point of Y. Then

$$\operatorname{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{X,\xi}/\mathfrak{m}^t_{X,\xi}) = \dim_{k(\xi)}\left(\sum_{i=1}^{t}\mathfrak{m}^{i-1}_{X,\xi}/\mathfrak{m}^{i}_{X,\xi}\right).$$

Since $\mathcal{O}_{X,\xi}$ is a regular local ring, [Har77, Theorem 8.21A(e), Chapter II] says that this dimension is equal to $\binom{c+t}{c} = \frac{t^c}{c!} + O(t^{c-1})$ for $t \gg 0$. Thus $e_Y(X) = 1$. □

Remark. The algebraic multiplicity of X along Y is a measure of how regular X is in Y.

1.2 Intersection number

This section follows [Kol96, Chapter VI.2]. For a more detailed review about intersection number see [Kle66] and [Sna60].
Again, for a review of the concepts about sheaves and schemes see [Har77, Chapter II].

Definition 1.2.1. Let X be a Noetherian scheme over a field K. Let $K(X)$ denote the abelian group generated by the symbols $[\mathcal{F}]$ where $\mathcal{F}$ is a coherent sheaf, modulo the relations $[\mathcal{F}_2] = [\mathcal{F}_1] + [\mathcal{F}_3]$ for every short exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0.$$

$K(X)$ is called the Grothendieck Group of X. For each $r \geq 0$, let $K_r(X) \subseteq K(X)$ denote the subgroup generated those $\mathcal{F}$ whose support has dimension at most r.
We only use $\mathcal{F}$ to denote this class in $K(X)$.

Definition 1.2.2. Let $\mathcal{L}$ be a line sheaf on X. The map $\mathcal{F} \mapsto \mathcal{L} \otimes \mathcal{F}$ is an automorphism of the abelian group $K(X)$. We define an endomorphism of $K(X)$ by the formula:

$$c_1(\mathcal{L}) \cdot \mathcal{F} := \mathcal{F} - \mathcal{L}^\vee \otimes \mathcal{F}.$$

Proposition 1.2.3 (Properties of $c_1(\mathcal{L})$)**.** *Let $\mathcal{L}$ be an line sheaf on X.*

(a) $F \mapsto c_1(\mathcal{L}) \cdot \mathcal{F}$ *is a well defined additive endomorphism of* $K(X)$.

(b) $c_1(\mathcal{L}) \cdot K_r(X) \subset K_{r-1}(X)$.

(c) $c_1(\mathcal{L}_1)$ *and* $c_1(\mathcal{L}_2)$ *commute for any two line bundles* $\mathcal{L}_1, \mathcal{L}_2$.

(d) If $Z \subset X$ *is integral of dimension* r *and* $\mathcal{L}|_Z \cong \mathcal{O}_Z(D)$ *where* D *is an effective Cartier divisor, then* $c_1(\mathcal{L}) \cdot \mathcal{O}_Z = \mathcal{O}(D)$.

(e) If $Z \subset X$ *is integral of dimension* r *and* $\mathcal{L}|_Z \cong \mathcal{O}_Z(D_1 - D_2)$ *where* D_i *are effective Weil divisors, then* $c_1(\mathcal{L}) \cdot \mathcal{O}_Z \equiv \mathcal{O}_{Z_1} - \mathcal{O}_{Z_2} \mod K_{r-2}(X)$.

(f) $c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) \cdot \mathcal{F} \equiv c_1(\mathcal{L}_1) \cdot \mathcal{F} + c_1(\mathcal{L}_2) \cdot \mathcal{F} \mod K_{r-2}(X)$.

Proof. [Kol96, Proposition 2.5, Chapter VI.2]. ☐

We recall that the Euler–Poincaré characteristic of a line sheaf $\mathcal{L}$ on X is

$$\chi(X, \mathcal{L}) = \sum_{i=0}^{\dim X} (-1)^i h^i(X, \mathcal{L}).$$

Definition 1.2.4. Let X/S be a Noetherian scheme and $\mathcal{F}$ a coherent sheaf on X. Assume that $m \geq r = \dim \operatorname{Supp} \mathcal{F}$. The intersection number of $\mathcal{L}_1, \ldots, \mathcal{L}_m$ on $\mathcal{F}$ is defined by

$$(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_m \cdot \mathcal{F}) := \chi(X, c_1(\mathcal{L}_1) \cdot \ldots \cdot c_1(\mathcal{L}_m) \cdot \mathcal{F}).$$

Remarks.

(i) If $\mathcal{L} = \mathcal{L}_1 = \ldots = \mathcal{L}_m$, then we write $(\mathcal{L}^m \cdot \mathcal{F})$ instead of $(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_m \cdot \mathcal{F})$.

(ii) If $Z \subset X$ is a closed sub-scheme, then we write $(\mathcal{L}^m \cdot Z)$ instead of $(\mathcal{L}^m \cdot \mathcal{O}_Z)$.

(iii) $(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_m)$ means $(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_m \cdot X)$.

(iv) If $\mathcal{L}_i = \mathcal{O}(D_i)$ for some Cartier divisors $D_1, \ldots, D_m$ then $(D_1 \cdot \ldots \cdot D_m \cdot \mathcal{F})$ stands for $(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_m \cdot \mathcal{F})$.

Some properties of the intersection number are the following:

Proposition 1.2.5. *Notation as in Definition 1.2.4.*

(a) $(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_m \cdot \mathcal{F}) = 0$ *if* $m > \dim \operatorname{Supp}\mathcal{F}$.

(b) $(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_m \cdot \mathcal{F})$ *is a symmetric m-linear function on* $\operatorname{Pic}(X)$ *for fixed* $\mathcal{F}$.

(c) Let Y_i be the m-dimensional irreducible components of $\operatorname{Supp}\mathcal{F}$ *and y_i the generic point of Y_i. Then*

$$(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_m \cdot \mathcal{F}) = \sum \operatorname{length}_{y_i}(\mathcal{F}) \cdot (\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_m \cdot Y_i).$$

Proof. [Kol96, Proposition 2.7, Chapter VI.2]. □

Theorem 1.2.6. *Let X be a scheme defined over a field K and $\mathcal{F}$ a coherent sheaf on X such that* $\operatorname{Supp}\mathcal{F}$ *is proper and has dimension at most m. Let $\mathcal{L}_1, \ldots, \mathcal{L}_m$ be line sheaves on X such that $\mathcal{L}_i = \mathcal{O}_X(D_i)$ for some effective Cartier divisors D_i. Assume that for every $x \in \cap D_i$ the sheaf $\mathcal{F}$ is locally free at x and $D_1, \ldots, D_m$ (or rather their local equations $f_{i,x}$) form a regular sequence in $\mathcal{O}_{X,x}$. Then*

$$(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_m \cdot \mathcal{F}) = \sum_{x \in \cap D_i} \operatorname{rank}_x \mathcal{F} \cdot \operatorname{length}(\mathcal{O}_{X,x}/(f_{1,x}, \ldots, f_{m,x})).$$

Proof. [Kol96, Theorem 2.8, Chapter VI.2]. □

This result show that this definition recover the classical definition of intersection number for algebraic varieties. See for instance [Sil87, Section A.2.3].

Proposition 1.2.7. *Let $\mathcal{L}_i$ and $\mathcal{L}'_i$ be line sheaves on X such that $\mathcal{L}_i$ and $\mathcal{L}'_i$ are algebraically equivalents for every i. Then for every coherent sheaf $\mathcal{F}$*

$$(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_m \cdot \mathcal{F}) = (\mathcal{L}'_1 \cdot \ldots \cdot \mathcal{L}'_m \cdot \mathcal{F}).$$

Proof. [Kol96, Corollary 2.9.1, Chapter VI.2]. □

Theorem 1.2.8. *(Projection Formula) Let $f : Y \to X$ be a morphism between K-schemes, $\mathcal{L}_i$ line sheaves on X and $\mathcal{F}$ a coherent sheaf on Y. Let $m \geq \dim \operatorname{Supp}\mathcal{F}$. Then*

$$(f^*\mathcal{L}_1 \cdot \ldots \cdot f^*\mathcal{L}_m \cdot \mathcal{F}) = (\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_m \cdot f_*\mathcal{F}).$$

Proof. [Kol96, Proposition 2.11, Chapter VI.2]. □

1.3 Riemann–Roch

Riemann–Roch is an important result that relates the geometry of a scheme and the algebra of the line sheaves. In this section we will give an estimate of the dimension of certain space that will be useful for the GCD bound 2.3.

We recall the Riemann–Roch theorem:

Theorem 1.3.1. *(Asymptotic Riemann–Roch) Let X be a projective variety of dimension n, $\mathcal{L}$ be an ample line sheaf on X and $\mathcal{F}$ be a coherent sheaf on X. Then*

$$h^i(X, \mathcal{L}^{\otimes m} \otimes \mathcal{F}) = O(m^{n-1}) \text{ for } i > 0 \text{ and}$$
$$h^0(X, \mathcal{L}^{\otimes m} \otimes \mathcal{F}) = \frac{(\mathcal{L}^n \cdot \mathcal{F})}{n!} m^n + O(m^{n-1})$$

where the constant in $O(m^{n-1})$ depends on $X, \mathcal{F}$ and $\mathcal{L}$.

Suppose that X is a projective variety over a field K and $Y \subset X$ is a sub-variety of X also defined over K. Then we have the ideal sheaf $\mathcal{I}_Y$ associated to Y, and for $r \geq 1$ we consider the coherent sheaf $\mathcal{O}_X/\mathcal{I}_Y^r$. In Section 2.3 we will need to estimate $h^0(X, \mathcal{L}^{\otimes m} \otimes \mathcal{O}_X/\mathcal{I}_Y^r)$ but we will have to know how this dimension changes as r increases, which Theorem 1.3.1 does not give us.
Nevertheless, the following two results will help us in to proceed.

Lemma 1.3.2. *Let Z be a projective and of finite type scheme over a number field K of dimension d. Let $\mathcal{A}$ be an ample base point free line sheaf on Z. Then*

$$h^0(Z, \mathcal{A}) \leq (\mathcal{A}^d) + d.$$

Proof. Let $q : Z \to \mathbb{P}^n$ be the morphism defined by $\mathcal{A}$ and let $Y = q(Z)$ be its image. Then $h^0(Z, \mathcal{A}) = n + 1$. Let n_Y be the dimension of Y.
First, we will proof that $n + 1 \leq (\mathcal{O}_{\mathbb{P}^n}(1)^{n_Y} \cdot Y) + n_Y$. We will do induction on n_Y.
If $n_Y = 0$ then the inequality holds.
If $n_Y \geq 1$, let $H \subset \mathbb{P}^n$ be a hyperplane meeting Y properly. Consider $Y' = Y \cap H \subset \mathbb{P}^{n-1}$. Then there is a properly contained $Z' \subset Z$ and a morphism $Z' \to Y' \subset \mathbb{P}^n$. By [Har77, Theorem 7.1, II] this morphism comes from an ample and base point free line sheaf $\mathcal{A}'$ on Z'. Then, by induction

$$(n-1) + 1 \leq (\mathcal{O}_{\mathbb{P}^{n-1}}(1)^{n_Y - 1} \cdot Y') + (n_Y - 1) \Rightarrow n + 1 \leq (\mathcal{O}_{\mathbb{P}^{n-1}}(1)^{n_Y - 1} \cdot Y') + n_Y.$$

Furthermore $(\mathcal{O}_{\mathbb{P}^{n-1}}(1) \cdot Y') = (\mathcal{O}_{\mathbb{P}^{n-1}} \cdot H \cap Y) = (\mathcal{O}_{\mathbb{P}^n}(1) \cdot Y)$.
Therefore

$$n + 1 \leq (\mathcal{O}_{\mathbb{P}^n})(1) \cdot Y) + n_Y.$$

Finally, by [Laz04, Corollary 1.2.15] the morphism $q : Z \to Y$ is finite. Then $q_* Z = \deg(q) \cdot Y$ in $K_d(\mathbb{P}^n)$. By Theorem 1.2.8 $(A^d \cdot Z) = t(\mathcal{O}_{\mathbb{P}^n}(1) \cdot Y)$. Since $n_y \leq d$ we conclude the desired. □

Theorem 1.3.3. *(Riemann–Roch type inequality) Let X be a projective variety defined over a number field K and $Y \subset X$ be an irreducible sub-variety of dimension d, also defined over K. Let $\mathcal{I}_Y$ be the ideal sheaf associated to Y. Then given any very ample line sheaf $\mathcal{A}$ on X,*

$$h^0(X, \mathcal{A}^{\otimes m} \otimes \mathcal{O}_X/\mathcal{I}_Y^r) \leq \frac{r^c \cdot m^d}{c!} e_X(Y) \cdot (\mathcal{A}^d \cdot Y) + O(r^{c-1})$$

for $r \gg 1$, where $c = \dim X - d$.

Proof. Since $\mathcal{A}$ is very ample, then $\mathcal{A}|_{Y_r}$ is very ample on Y_r. By Lemma 1.3.2 then

$$h^0(X, \mathcal{A}^{\otimes m} \otimes \mathcal{O}_X/\mathcal{I}_Y^r) \leq (\mathcal{A}^d \cdot \mathcal{O}_X/\mathcal{I}_Y^r)m^d + d. \tag{1.1}$$

Let ξ_Y be the generic point of Y, by Lemma 1.2.5

$$(\mathcal{A}^d \cdot \mathcal{O}_X/\mathcal{I}_Y^r) = \operatorname{length}_{\xi_Y} (\mathcal{O}_X/\mathcal{I}_Y^r)_{\xi_Y} (\mathcal{A}^d \cdot Y)$$

By [Ful98, Example 4.3.4] this length is equal to

$$e_Y(X) \cdot \frac{r^c}{c!} + O(r^{c-1})$$

for $r \gg 1$. Replacing this in (1.1) give the result. $\square$

Chapter 2

Arithmetic results

2.1 Heights

The goal of this section is to understand the concept of *height of a point on a variety.* Before that, we have to introduce the notion of absolute values and height for elements in a number field.
Consider a number field K and its ring of integers $\mathcal{O}_K$.

Definition 2.1.1. The set of places M_K is the set consisting of each nonzero prime ideal $\mathfrak{p} \subset \mathcal{O}_K$, the real embeddings $\sigma : K \hookrightarrow \mathbb{R}$ and the embeddings $\sigma : K \hookrightarrow \mathbb{C}$, such that $\sigma(K) \not\subset \mathbb{R}$, without the conjugate of the complex embeddings. Such elements of M_K are called non-archimedean places, real places and complex places, respectively. Furthermore, M_K^0 will denote the set of the non-archimedean places and M_K^∞ the set of archimedean places, consist on the real and complex places.

Example 2.1.2. If $K = \mathbb{Q}$ then $M_{\mathbb{Q}} = \{\infty, 2, 3, 5, \ldots\}$ where ∞ represents the one real place corresponding to the inclusion $\mathbb{Q} \subset \mathbb{R}$ and the elements $2, 3, 5, \ldots$ the infinitely many non-archimedean places corresponding to the primes rational integers.

Definition 2.1.3. Let v be a place in M_K, we define the (normed) absolute value $|\cdot|_v$ defined by $|0|_v = 0$ and for $x \neq 0$:

$$|x|_v = \begin{cases} (\mathcal{O}_K : \mathfrak{p})^{-\operatorname{ord}_{\mathfrak{p}}(x)} & \text{if } v \text{ corresponds to } \mathfrak{p} \subset \mathcal{O}_K \\ |\sigma(x)| & \text{if } v \text{ corresponds to a real place} \\ |\sigma(x)|^2 & \text{if } v \text{ corresponds to a complex place} \end{cases}$$

where $\operatorname{ord}_{\mathfrak{p}}(x)$ means the exponent of $\mathfrak{p}$ in the factorization of the fractional ideal (x) with the convention that $\operatorname{ord}_{\mathfrak{p}}(0) = -\infty$.

Example 2.1.4. If $K = \mathbb{Q}$ then we obtain the absolute values $|\cdot|_\infty$ which is the usual absolute value on $\mathbb{R}$ restricted to $\mathbb{Q}$ and if $v = p$ is a non-archimedean place then $|\cdot|_p$ is the p-adic absolute value in $\mathbb{Q}$.

Definition 2.1.5. Let L be a finite extension of K, and consider places $w \in M_L$ and $v \in M_K$. We say that w lies over v, and write $w|v$ if v arise from w in one of the following ways:

- If w is finite, then corresponds to a nonzero prime ideal $\mathfrak{q} \subset \mathcal{O}_L$, thus $\mathfrak{p} := \mathfrak{q} \cap \mathcal{O}_K$ is a nonzero prime ideal of $\mathcal{O}_K$, and gives rise to a non-archimedean place $v \in M_K$.

- If w is real or complex, then corresponds to an embedding $\sigma : L \hookrightarrow \mathbb{C}$, and its restriction $\sigma_K : K \hookrightarrow \mathbb{C}$ gives arise to a unique real or complex place $v \in M_K$

Proposition 2.1.6. *Let L be a finite extension of K. Then for all $v \in M_K$ and for all $y \in L$ we have*

$$\prod_{\substack{w\in M_L \\ w|v}} |y|_w = \left|N_{L/K}(y)\right|_v .$$

Proof. [Lan94, Corollary 2, Chapter II]. ☐

Definition 2.1.7. For an element $x \in K$. The height of x in K is

$$h_K(x) = \sum_{v\in M_K} \log^+ |x|_v$$

where $\log^+(x) = \max\{\log x, 0\}$.

Proposition 2.1.8. *Let L be a finite extension of K. For all $x \in K$ we have*

$$h_L(x) = [L : K]h_K(x).$$

Proof. Follows from Proposition 2.1.6. ☐

Definition 2.1.9. The height of an element $x \in \overline{\mathbb{Q}}$ is

$$h(x) = \frac{1}{[L : \mathbb{Q}]} h_L(x),$$

for any number field $L \supseteq \mathbb{Q}(x)$.

Remark. By Proposition 2.1.8 $h(x)$ is independent of the choice of L.

Before defining the height of a point in a variety, we will introduce the height of an element in $\mathbb{P}^n(\mathbb{Q})$. To this end, we need an important result:

Theorem 2.1.10. *(Product formula) Let K be a number field and let $x \in K^*$. Then*

$$\prod_{v\in M_K} |x|_v = 1.$$

Proof. If $K = \mathbb{Q}$ the product formula follows from the unique factorization in $\mathbb{Z}$.
In the general case follows from Proposition 2.1.6 and the product formula in $\mathbb{Q}$. ☐

Definition 2.1.11. The field of definition of a point $x = [x_0 : x_1 : \ldots : x_n] \in \mathbb{P}^n(\overline{\mathbb{Q}})$ is $Q(x) = Q(x_0/x_j, x_1/x_j, \ldots, x_n/x_j)$ for any j with $x_j \neq 0$.

Definition 2.1.12. Let $x = [x_0 : x_1 : \ldots : x_n] \in \mathbb{P}^n(\overline{\mathbb{Q}})$, the height of x is defined by

$$h(x) = \frac{1}{[L : \mathbb{Q}]} h_L(x)$$

for any number field $L \supseteq \mathbb{Q}(x)$ where

$$h_L(x) = \sum_{v\in M_L} \log\max\{|x_0|_v , |x_1|_v , \ldots, |x_n|_v\}.$$

By Theorem 2.1.10, this height is independent of the choice of homogeneous coordinates of x.

Remark. Note that for $x \in \overline{\mathbb{Q}}$ we have $h(x) = h([x : 1])$.

An important theorem is the following:

Theorem 2.1.13. *(Northcott's finiteness theorem) For any numbers $r \in \mathbb{Z}_{>0}$ and $C \in \mathbb{R}$, the set*

$$\{x \in \mathbb{P}^n(\overline{\mathbb{Q}}) : [\mathbb{Q}(x) : \mathbb{Q}] \leq r \text{ and } h(x) \leq C\}$$

is finite.

Proof. [HS00, Theorem B.2.3]. ☐

Now we are allowed to define the height of a point on a variety. Recall that a variety over K is an integral separated scheme of finite type over K (i.e., over $\operatorname{Spec} K$) 1.1.1.

Theorem 2.1.14. *(Weil's height machine) Let X be a smooth projective variety over K, there exist a map*

$$h_X : \operatorname{Div}(X) \to \{\text{functions } X(\overline{K}) \to \mathbb{R}\}$$

with the following properties:

(a) (Normalization) Let $H \subset \mathbb{P}^n$ be a hyperplane and let $h(x)$ be the height on $\mathbb{P}^n$ (Definition 2.1.12). Then

$$h_{\mathbb{P}^n,H}(x) = h(x) + O(1) \quad \text{for all } x \in \mathbb{P}^n(\overline{K}).$$

(b) (Functoriality) Let $\varphi : X \to Y$ be a morphism of K-varieties and let $D \in \operatorname{Div}(Y)$. Then

$$h_{X,\varphi^*D}(x) = h_{Y,D}(\varphi(x)) + O(1) \quad \text{for all } x \in X(\overline{K}).$$

(c) (Additivity) Let $D_1, D_2 \in \operatorname{Div}(X)$. Then

$$h_{X,D_1+D_2}(x) = h_{X,D_1}(x) + h_{X,D_2}(x) + O(1) \quad \text{for all } x \in X(\overline{K}).$$

(d) (Linear Equivalence) Let $D_1, D_2 \in \operatorname{Div}(X)$ with D_1 linearly equivalent to D_2. Then

$$h_{X,D_1}(x) = h_{X,D_2}(x) + O(1) \quad \text{for all } x \in X(\overline{K}).$$

(e) (Positivity) Let $D \in$ Div(X) be an effective divisor, and let B be the base locus of the linear system $|D|$. Then

$$h_{X,D}(x) \geq O(1) \quad \text{for all } x \in (X \setminus B)(\overline{K}).$$

(f) (Finiteness) Let $D \in \operatorname{Div}(X)$ be ample. Then for all constant $r \in \mathbb{Z}_{>0}$ and $C \in \mathbb{R}$, the set

$$\{x \in X(\overline{K}) : [K(x) : K] \leq r \text{ and } h_{X,D}(x) \leq C\}$$

is finite.

Proof. The important construction is that if D is a very ample divisor on X then it has an associated embedding ϕ_D from X to some projective space. Thus define $h_{X,D}(x) = h(\phi(x))$. Then given any $D \in \operatorname{Div}(X)$ write $D = D_1 - D_2$ with D_1, D_2 very ample and define $h_{X,D} = h_{X,D_1} - h_{X,D_2}$. For the details see [HS00, Theorem B.3.2]. ☐

Remarks.

(i) Given $D \in \mathrm{Div}(X)$, the function $h_{X,D}(x)$ is the height of X with respect to D, and by part (d) of Theorem 2.1.14 this function is unique up to linear equivalence.

(ii) The $O(1)$ constant that appear in the theorem depends on the varieties, divisors and morphism, but they are independent of the points on the varieties.

Definition 2.1.15. Given any line sheaf $\mathcal{L}$, if $\mathcal{L} \cong \mathcal{O}(D)$ for some $D \in \mathrm{Div}(X)$ we define $h_{X,\mathcal{L}} = h_{X,D}$.

2.2 Generalized greatest common divisor

For a smooth projective variety X defined over a number field K, we have defined the height of a point $x \in X$ with respect to a divisor $D \in \mathrm{Div}(X)$. In this section we will define the height of a point with respect to an irreducible subvariety Z of X. We will see how this height is related to the greatest common divisors of two integers numbers, and in a more general way the height with respect to an irreducible sub-variety will be the generalized greatest common divisor of a point with respect to the sub-variety.
To motivate this, consider $a, b \in \mathbb{Z}$. The greatest common divisor of a and b is defined by

$$\gcd(a,b) = \prod_{p \text{ prime}} p^{\min\{\mathrm{ord}_p(a), \mathrm{ord}_p(b)\}},$$

then

$$\begin{aligned} \log\gcd(a,b) &= \sum_{p \text{ prime}} \min\{\mathrm{ord}_p(a), \mathrm{ord}_p(b)\} \log p \\ &= \sum_{v \in M_{\mathbb{Q}}^0} -\log\max\{|a|_v, |b|_v\} \\ &= \sum_{v \in M_{\mathbb{Q}}} \log^+ \frac{1}{\max\{|a|_v, |b|_v\}} \end{aligned}$$

From this we define the following:

Definition 2.2.1. Let α, β be elements of a number field K. The generalized (logarithmic) greatest common divisor of α, β is

$$h_{\mathrm{gcd}}(\alpha, \beta) = \prod_{v \in M_K} \log^+ \frac{1}{\max\{|\alpha|_v, |\beta|_v\}}.$$

Proposition 2.2.2. *Let K be a number field and let α, β be elements in K. Each α and β have associated points $[1:\alpha]$ and $[1:\beta]$ in $\mathbb{P}^1(K)$. Consider $X = (\mathbb{P}^1)^2$ and let $\pi : \tilde{X} \to X$ be the blow up of X at the point $([1:0],[1:0])$ with exceptional divisor E. Then*

$$h_{\mathrm{gcd}}(\alpha, \beta) = h_{\tilde{X},E}(\pi^{-1}([1:\alpha],[1:\beta])).$$

Proof. [Voj87, Lemma 2.5.2]. □

Definition 2.2.3. Let X be a smooth variety defined over K and $Y \subset X$ be an irreducible sub-variety of X also defined over K, of codimension $c \geq 2$. Let $\pi : \tilde{X} \to X$ be the blow up of X along Y and let E_Y be the exceptional divisor of the blow up. For $x \in X \setminus Y$ we let $\tilde{x} = \pi^{-1}(x)$.
The generalized (logarithmic) greatest common divisor of the point $x \in (X \setminus Y)(\overline{K})$ with respect to Y is

$$h_{\mathrm{gcd}}(x; Y) = h_{\tilde{X}, E_Y}(\tilde{x}).$$

Example 2.2.4. Let $X = \mathbb{P}^n$ and let $Y = [1:0:\ldots:0]$. For $x \in \mathbb{P}^n(\mathbb{Q})$, choose homogeneous coordinates $[x_0 : x_1 : \ldots : x_n]$ such that $\gcd(x_0, x_1, \ldots, x_n) = 1$. Then

$$h_{\mathrm{gcd}}(x; Y) = \log\gcd(x_1, \ldots, x_n) + O(1).$$

For more examples see [Sil05].

Remark. The heights from Definition 2.2.3 are also called global heights. For a more developed theory about this heights see [Sil87].

2.3 GCD bound

In this section we obtain a bound for the GCD defined in 2.2.3.

Throughout this section:

- K is a number field
- X is a smooth projective variety defined over K
- Y is an irreducible sub-variety of X of codimension $c \geq 2$ also defined over K
- $X' = X \setminus Y$

Proposition 2.3.1. *For all ample line sheaf $\mathcal{A}$ on X if there is an effective divisor D on X such that $\mathcal{O}(D) = \mathcal{A}^{\otimes m}$ and $r = m_Y(D)$ for some integers m, r, there is a properly contained Zariski closed set $Z \subset X$ such that for all $x \in (X' \setminus Z)(\overline{K})$:*

$$h_{\mathrm{gcd}}(x; Z) \leq \frac{m}{r} h(\mathcal{A}, x) + O(1)$$

Proof. Let E_Y be the exceptional divisor of the blow-up of X along Y and B be the base locus of $\tilde{D}$. By the assumption of D, given any $y \in (\tilde{X} \setminus B)(\overline{K})$ we have

$$\begin{aligned} r \cdot h_{\mathrm{gcd}}(x; Y) &= r \cdot h_{\tilde{X}, E_Y}(y) + O(1) \\ &\leq h_{\tilde{X}, \pi^*(D)}(y) + O(1) \\ &= h_{X,D}(x) + O(1)\ , \quad \pi(x) = y \\ &= m \cdot h_{X,\mathcal{A}}(x) + O(1) \end{aligned}$$

Thus

$$h_{\mathrm{gcd}}(x; Y) \leq \frac{m}{r} h_{X,\mathcal{A}}(x) + O(1).$$

□

Definition 2.3.2. Let X, Y and $\mathcal{A}$ be as above. We define

$$\eta(\mathcal{A}, Y) = \inf \left\{ \frac{m}{r} : \exists D \text{ an effective divisor on } X \text{ with } \mathcal{O}(D) = \mathcal{A}^{\otimes m} \ \& \ r = m_Y(D) \right\}.$$

Theorem 2.3.3. *For all $\varepsilon > 0$ there is a properly contained Zariski closed set $Z_\varepsilon \subset X$ such that for all $x \in (X' \setminus Z_\varepsilon)(\overline{K})$:*

$$h_{\mathrm{gcd}}(x; Y) \leq (\eta(\mathcal{A}, Y) + \varepsilon) h(\mathcal{A}, x) + O_\varepsilon(1).$$

Proof. It follows from Proposition 2.3.1 and characterization of infimum. □

Our goal then is to estimate the value of η. In this way we have the following result:

Theorem 2.3.4. *If $\mathcal{A}$ is very ample on X then:*

$$\eta(\mathcal{A}, Y) \leq \left(\frac{(\mathcal{A}^d \cdot Y)}{(\mathcal{A}^n)} \cdot \frac{n!}{c!} \right)^{\frac{1}{c}} + \varepsilon$$

Proof. We have to guarantee the existence of a divisor D in X such that $\mathcal{O}(D) = \mathcal{A}^{\otimes m}$ and $r = m_Y(D)$. The associated exact sequence for the ideal sheaf $\mathcal{I}_Y^r$ associated to Y is

$$0 \to \mathcal{I}_Y^r \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_Y^r \to 0$$

tensoring by $\mathcal{A}^{\otimes m}$ and passing to global sections we obtain

$$0 \to H^0(X, \mathcal{A}^{\otimes m} \otimes \mathcal{I}_Y^r) \to H^0(X, \mathcal{A}^{\otimes m}) \to H^0(X, \mathcal{A}^{\otimes m} \otimes \mathcal{O}_X/\mathcal{I}_Y^r)$$

If $h^0(X, \mathcal{A}^{\otimes m}) > h^0(X, \mathcal{A}^{\otimes m} \otimes \mathcal{O}_X/\mathcal{I}_Y^r)$ then $H^0(X, \mathcal{A}^{\otimes m} \otimes \mathcal{I}_Y^r)$ is non-trivial and we obtain the desired divisor (in fact we obtain $m_Y(D) \geq r$ but this is enough to bound η). Now, given any $\varepsilon > 0$ there is a sufficiently large choice for m, r such that

$$\frac{(\mathcal{A}^n)}{n!} m^n > \frac{r^c \cdot m^d}{c!} (\mathcal{A}^d \cdot Y) \quad \text{and} \quad \frac{m}{r} < \left(\frac{(\mathcal{A}^d \cdot Y)}{(\mathcal{A}^n)} \cdot \frac{n!}{c!} \right)^{\frac{1}{c}} + \varepsilon.$$

By Theorems 1.3.1 and 1.3.3 we obtain the result. □

Using the previous theorems, we obtain the main theorem of this work.

Theorem 2.3.5. *Given any ample line sheaf $\mathcal{A}$ on X and $\varepsilon > 0$ there is a properly contained Zariski closed set $Z_\varepsilon \subset X$ such that for all $x \in (X' \setminus Z_\varepsilon)(\overline{K})$:*

$$h_{\mathrm{gcd}}(x; Y) \leq \left(\left(\frac{(\mathcal{A}^d \cdot Y)}{(\mathcal{A}^n)} \cdot \frac{n!}{c!} \right)^{\frac{1}{c}} + \varepsilon \right) h_{X,\mathcal{A}}(x) + O_\varepsilon(1).$$

Proof. In the very ample case the result follows from Theorems 2.3.3 and 2.3.4.
In the general case, consider an integer k such that $\mathcal{A}^{\otimes k}$ is very ample. Applying the result to $\mathcal{A}^{\otimes k}$ we obtain that

$$\begin{aligned} h_{\mathrm{gcd}}(x; Y) &\leq \left(\left(\frac{((\mathcal{A}^{\otimes k})^d \cdot Y)}{((\mathcal{A}^{\otimes k})^n)} \cdot \frac{n!}{c!} \right)^{\frac{1}{c}} + \varepsilon \right) h_{X,\mathcal{A}^{\otimes k}}(x) + O_\varepsilon(1) \\ &\leq \left(\left(\frac{k^d (\mathcal{A}^d \cdot Y)}{k^n (\mathcal{A}^n)} \cdot \frac{n!}{c!} \right)^{\frac{1}{c}} + \varepsilon \right) k \cdot h_{X,\mathcal{A}}(x) + O_\varepsilon(1) \\ &\leq \left(\frac{1}{k} \left(\frac{(\mathcal{A}^d \cdot Y)}{(\mathcal{A}^n)} \cdot \frac{n!}{c!} \right)^{\frac{1}{c}} + \varepsilon \right) k \cdot h_{X,\mathcal{A}}(x) + O_\varepsilon(1). \end{aligned}$$

From this, we can conclude. □

2.4 Applications

In this section we give some interesting applications of the GCD bound.

Theorem 2.4.1. *Let* $x_1, \dots, x_n$ *be rational numbers. Then for all* $\varepsilon > 0$ *we have:*

$$\log \gcd(x_1, \dots, x_n) \le (1+\varepsilon) h(x) + O(1)$$

where $x = [1 : x_1 : \dots : x_n] \in \mathbb{P}^n$.

Proof. It follows from Example 2.2.4 and 2.3.5 with $X = \mathbb{P}^n$, $Y = [1 : 0 : \dots : 0]$, $\mathcal{A} = \mathcal{O}(1)$ and $x = [1 : x_1 : \dots : x_n]$. $\square$

Theorem 2.4.2. *Let* K *be a number field and* $\alpha, \beta \in K$. *Then for all* $\varepsilon >$ *hold the inequality*

$$h_{\text{gcd}}(\alpha, \beta) \le \left(\sqrt{\frac{1}{2}} + \varepsilon \right) [h(\alpha) + h(\beta)] + O(1).$$

Proof. By Proposition 2.2.2 we have

$$h_{\text{gcd}}(\alpha, \beta) = h_{\text{gcd}}(([1:\alpha],[1:\beta]); ([1:0],[1:0])).$$

Then the result follows from Theorem 2.3.5 with $X = (\mathbb{P}^1)^2$, $Y = ([1 : 0], [1 : 0])$, $\mathcal{A} = \pi_1^* \mathcal{O}(1) \otimes \pi_2^* \mathcal{O}(1)$ where π_i are the projections and $x = ([1 : \alpha], [1 : \beta])$ $\square$

Theorem 2.4.3. *Let* $\varepsilon > 0$ *and* $Y \subset \mathbb{P}^n$ *be a sub-variety of degree* d. *Then there is a properly contained Zariski closed set* $Z_\varepsilon \subset X$ *such that*

$$h_{\text{gcd}}(x; Y) \le (d^{\frac{1}{c}} + \varepsilon) h(x) + O(1)$$

for all $x \in (X \setminus (Y \cup Z_\varepsilon))(\overline{K})$, *where* K *is a number field and* $c = \operatorname{codim}(Y, X)$.

Proof. Consider $X = \mathbb{P}^n$, $Y = Y$ and $\mathcal{A} = \mathcal{O}(1)$ in Theorem 2.3.5. $\square$

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