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NATURAL EXTENSIONS AND PERIODIC APPROXIMATIONS OF SEMIGROUP ACTIONS

AUTHOR:
Miguel Donoso Echenique

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ADVISOR:
Raimundo Briceño

COMMITTEE:
María Isabel Cortez
Godofredo Iommi

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Introduction

Actions of groups over Polish spaces and their associated Borel measurable spaces are a widely studied subject in dynamical systems and ergodic theory. They arise from multiple situations, an example being the models from statistical physics, such as Ising and Potts models, which involve shift actions upon configurations of particles and their possible states on integer lattices. A natural direction is to consider actions over Polish spaces by transformations which are not necessarily invertible: actions of semigroups. The classical theory of dynamical systems is an example of this, as it studies both $\mathbb{Z}$-actions, such as irrational rotations on the unit circle, and $\mathbb{N}$-actions, such as the doubling map on the same space. In fact, these two transformations generate a semigroup within $\mathrm{Diffeo}(\mathbb{S}^1)$ with the presentation

$$\mathrm{BS}(1,2)^+ = \langle a, b \mid ab = b^2 a \rangle,$$

providing a natural example of an action of a (non-Abelian) semigroup upon a Polish space.

Studying semigroup actions can be fruitful for the theory of group actions. The worlds of $\mathbb{Z}^d$-actions and $\mathbb{N}^d$-actions are bonded by a construction called *the natural extension*, which associates to each non-invertible system an invertible one such that the former is a factor of the latter. We give a very brief introduction to this construction in §2.2, following [Sar09]. While this nexus enables to apply the theory of $\mathbb{Z}$-actions to study $\mathbb{N}$-actions, it can be used the other way around. For instance, a natural extension is implicitly utilized by R. Bowen in [Bow75] to introduce Gibbs measures on $\mathbb{N}$-shifts and then extend them to define Gibbs measures on $\mathbb{Z}$-shifts. It therefore becomes relevant to develop a theory of semigroup actions, not only on account of its intrinsic interest, but also for its implications towards group actions. Some recent results proven include an analogous of E. Lindenstrauss' [Lin01] generalization of Birkhoff's Ergodic Theorem to actions of amenable groups, proven in [Eas12] for left amenable bicancellative semigroups; an analogue of Ornstein-Weiss' Lemma for subadditive functions on the same class of semigroups, and its consequent applications to entropy theory, obtained in [CSCK14]; the introduction of a notion of soficity for semigroups, in [CSC14].

A special kind of actions that will receive particular attention during this thesis are symbolic actions. Symbolic dynamics originated as a technique for understanding more general dynamical systems via coding the position of elements in the phase space through a partition, but is of intrinsic interest and has found many applications in different areas. Introductions to these topics can be found in [LM21; Kit97] for $\mathbb{Z}$-shifts, [Sch90] for higher dimensional ($\mathbb{Z}^d$) shifts, and [CSC10] for general group shifts. In this context, the phenomenon of periodicity will be of major importance to us. Periodic points are the most fundamental examples of transitivity, and so are periodic measures of ergodicity and recurrence. The simpler nature of periodic structures makes desirable the ability to approximate other structures by periodic ones.

Let us put things in a more precise fashion now. Consider a semigroup S, a finite set $\mathcal{A}$, and define the space of configurations $\mathcal{A}^S$ consisting of all functions $x\colon S \to \mathcal{A}$, called configurations, and endowed with the product topology of the discrete topology on $\mathcal{A}$. This space comes with a natural action of S, called the shift action, which is given by $(s \cdot x)(t) = x(ts)$. The space of configurations together with the shift action will be referred to as the full S-shift on the alphabet $\mathcal{A}$, and this action might (or not) admit S-invariant measures, i.e., measures μ such that $\mu(s^{-1}A) = \mu(A)$ for every Borel subset. An important concern is the space $\mathcal{M}_S(\mathcal{A}^S)$ of S-invariant measures on $\mathcal{A}^S$.

When S is a group, there has been progress made. It is well known, for instance, that the space $\mathcal{M}_{\mathbb{Z}}(\mathcal{A}^{\mathbb{Z}})$ is a Poulsen simplex, meaning that the subset of ergodic $\mathbb{Z}$-invariant measures is dense in $\mathcal{M}_{\mathbb{Z}}(\mathcal{A}^{\mathbb{Z}})$ under the weak-* topology. This property has been further characterized in terms of Kazhdan's property (T) (see [GW97]). Continuing with the case $S = \mathbb{Z}$, there is a solid notion of periodicity for $\mathbb{Z}$-actions, and the ergodic periodic measures (i.e., those supported over a single finite orbit) are weak-* dense in $\mathcal{M}_{\mathbb{Z}}(\mathcal{A}^{\mathbb{Z}})$. This result is known to hold for $\mathbb{Z}^d$, and more generally for the class of residually finite amenable groups: we give here a detailed account of this. Outside the amenable case, the answer to the question about denseness of the set of periodic measures on the full S-shift is less clear. In [Bow03], L. Bowen shows that, for the free group $\mathbb{F}_n$ on n generators, the set of periodic measures is a dense subset of $\mathcal{M}_{\mathbb{F}_n}(\mathcal{A}^{\mathbb{F}_n})$. A. Kechris has also studied the topic in a different, equivalent formulation [Kec12; BK20]. Recently, C. Shriver related the denseness of periodic measures with free energy density of measures relative to sofic approximations [Shr23].

When we abandon the realm of groups to consider actions of semigroups, the territory remains fairly unexplored. Common restrictions upon the acting semigroup include being a monoid (i.e., having an identity element) and satisfying a two-sided cancellation law (that is, $a = b$ whenever $ca = cb$ or $ac = bc$ holds). An important class of semigroups satisfying the cancellation law is the one of semigroups that can be embedded into a group, a situation illustrated by the semigroup $\mathbb{N}$, which is a subsemigroup of the group $\mathbb{Z}$, and in this case there exists the mentioned construction of the natural extension.

This thesis has two main purposes. The first one is to define and study extensions of semigroup actions to actions of groups into which the semigroups embed, both from a topological and a measure-theoretical perspective. The second objective is to study which semigroups S have a weak-* dense set of periodic measures in the full S-shift. In order to achieve this, we apply the developed machinery of extensions to generalize the known results of denseness of periodic measures on group full shifts, with particular attention to the free semigroup on n generators, $\mathbb{F}_n^+$. This forces to ask simpler questions, such as: what is a good definition of periodicity for a semigroup action? What is a periodic measure in this context? What characteristics must a group containing S as a subsemigroup have in order to admit extensions of actions of S?

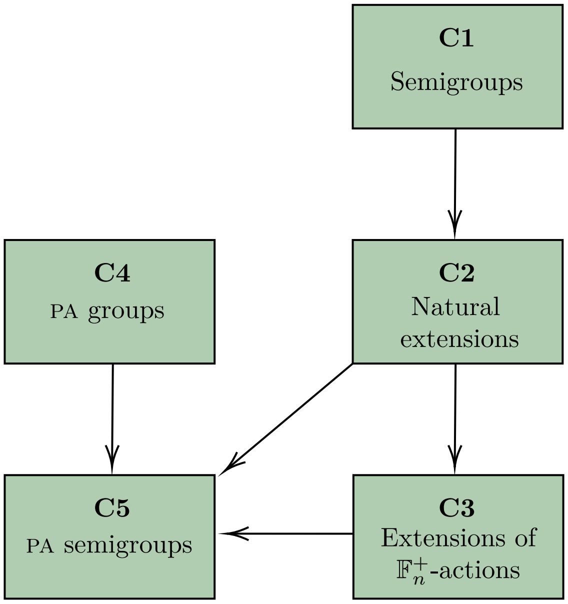
The document is structured in five chapters, which we detail now. We add links to the main original results.

- **Chapter 1**. We give a review of the theory of semigroups, with the basic notions from the algebraic theory and amenability. The reader may deepen into the algebraic theory in [CP61; CP67], and into the topic of amenability in [AW67; Nam64; FJ60; Day57; Don13; GK17; Mag21]. The focus of this chapter is to study the concepts of S-group, free S-group and group of fractions of S, relating them to the embeddability of S into a group. These notions play a fundamental role in this work and are carried on through the following chapters. We finish by giving a set of concrete examples of semigroups satisfying various combinations of properties.
- **Chapter 2**. The second chapter deals with actions of semigroups on Polish spaces, and the possibility of extending them to actions of the groups into which they can be embedded. Given a semigroup S and a group G containing S as a subsemigroup, we define the natural extension of an action of S upon a Polish space in both the topological and the measure-theoretical case. Then, we characterize the groups containing S that admit natural extensions in the topological case. More precisely, we prove that if G is the free S-group, then every surjective S-subshift is G-extensible (Theorem 2.4.4) and that every surjective S-action over a compact metric space is G-extensible (Corollary 2.5.4). On the other hand, we prove that for any S-group H other than the free S-group, there is an S-subshift (possibly in a countably infinite alphabet) which does not admit an extension to H, and that this can be achieved with a finite alphabet if G is residually finite (Proposition 2.5.2). Regarding measure-theoretical natural extensions, we prove that every p.m.p. action of a reversible bicancellative semigroup can be extended to an action of its group of fractions (Theorem 2.7.7).
- **Chapter 3**. The aim of the third chapter is to give account of measure-theoretical $\mathbb{F}_n$-extensions of symbolic actions of $\mathbb{F}_n^+$, a semigroup which is not reversible. The main tool developed here is the concept of a tree Markov shift both for the free semigroup $\mathbb{F}_n^+$ and the free group $\mathbb{F}_n$. We prove that every Markov measure on $\mathcal{A}^{\mathbb{F}_n^+}$ is $\mathbb{F}_n$-extensible (Corollary 3.2.4), and that every $\mathbb{F}_n^+$-invariant measure can be weak-* approximated by Markov measures (Proposition 3.3.1), thus concluding that every $\mathbb{F}_n^+$-invariant measure can be extended to $\mathbb{F}_n$ (Corollary 3.3.2). We end by giving an example of an $\mathbb{F}_2^+$-invariant measure which does not admit a natural extension to
$$\mathrm{BS}(1,2) = \langle a, b \mid ab = b^2 a \rangle,$$
which is an $\mathbb{F}_2^+$-group.

- **Chapter 4**. The fourth chapter is devoted to the study of periodic approximations of measures on full-shifts of groups. We start by introducing the main definitions and the classical results, as well as reviewing the class of residually finite groups. Then, we give a proof of the existence of a Følner sequence of fundamental domains for every residually finite amenable group, a result due to B. Weiss [Wei01], and which was proven in a stronger form in [CP14]. We take advantage of this fact to prove that every amenable residually finite group has periodic approximations. Finally, we follow L. Bowen in [Bow03] and prove that the free group $\mathbb{F}_n$ has periodic approximations as well.

- **Chapter 5**. The last chapter reunites all of the previous work in an application of the theory of natural extensions. We define an adequate notion of periodicity for semigroup actions, as well as introduce periodic measures. Then, we establish the relation between periodic and G-extensible measures when S embeds into its free S-group G (Corollary 5.3.3). Finally, we study residual finiteness conditions for semigroups and introduce the definition of a semigroup with periodic approximations, to then conclude (a) that free semigroups have periodic approximations (Proposition 5.3.4) and (b) that left amenable semigroups that are residually a finite group have the property as well (Theorem 5.3.5).

We include a diagram which summarizes the dependencies between the five chapters and the structural relation between our two main objectives.



It seems to make sense to apply the theory of natural extensions to a broad spectrum of problems, such as ergodic theorems, recurrence, entropy theory, amenability and soficity, to name a few. Nevertheless, there are some pending details within the theory that need to be taken care of as well, such as: how does the ergodicity/mixing of a semigroup action relate to ergodicity/mixing of its natural extension? Is it always possible, given any S-group H different to the free S-group, to construct a compact surjective action of S which cannot be extended to H? Can every surjective continuous action of a semigroup upon a Polish (not necessarily compact) space be extended to the free group on the semigroup? If S is not reversible, does every S-invariant measure admit an extension to the free S-group?

Chapter 1

Semigroups

The purpose of this chapter is to provide a review of some algebraic and geometric aspects of the semigroup theory. The most important goals are, on one side to study when and how exactly can a semigroup be embedded into a group, and on the other, provide a list of examples of semigroups satsifying various properties.

1.1 Basic concepts

We start out with some basic definitions. A **semigroup** is a set S together with an associative binary operation. We will always assume our semigroups to be countable. An element $1 \in S$ such that for every $a \in S$ we have $1a = a1 = a$ is called an **identity element** for S, while an element $0 \in S$ such that $0a = a0 = 0$ for all $a \in S$ is called a **zero element** for S. If an identity element or a zero element exists, it must be unique, so in that case we will write 1_S and 0_S, respectively.

A **monoid** is a semigroup which has an identity element. Although not every semigroup has identity, a monoid can be easily obtained from a semigroup by adjoining an extra element 1_S to S such that $1_S t = t 1_S = t$ for all $t \in S$ and $1_S 1_S = 1_S$. We denote the monoid $S \cup \{1_S\}$ by S^1. An analogous procedure can be applied to adjoint a zero element to a semigroup.

Definition 1.1.1. Let S be a semigroup, and $T \subseteq S$. We say T is a **subsemigroup** of S (denoted $T \leq S$) if it is closed under the binary operation, and we say it is a **left** (resp. **right**) **ideal** of S if $ST \subseteq T$ (resp. $TS \subseteq T$). The subset T will be called an **ideal** if it is both a right and left ideal.

Clearly, every left or right ideal is itself a subsemigroup. The left, right and two-sided ideals taking the respective forms Sa, aS and SaS for some $a \in S$, are called **principal ideals**.

Definition 1.1.2. Let S be a semigroup, and $K \subseteq S$. Define the **subsemigroup**

generated by K

$$\langle K\rangle = \bigcap_{K\subseteq T\leq S} T.$$

We say K **generates** S if $S=\langle K\rangle$.

Remark 1.1.3. If $K\subseteq S$, we can characterize the subsemigroup generated by K by

$$\langle K\rangle = \{k_1\dots k_n | n\in\mathbb{N}, k_1,\dots,k_n\in K\}.$$

Definition 1.1.4. Let S,T be two semigroups. A **semigroup homomorphism** is a function $\phi\colon S\to T$ such that $\phi(st)=\phi(s)\phi(t)$ for every $s,t\in S$. In the case S,T are monoids, we require ϕ to have the additional property that $\phi(1_S)=1_T$. Whenever we say $\phi\colon S\to T$ is an **embedding**, we mean an injective homomorphism.

A morphism $\phi\colon S\to T$ is an isomorphism if, and only if, it is bijective, since the inverse of a bijective semigroup morphism is itself a semigroup morphism. It is also easy to check that the homomorphic image of a semigroup is a subsemigroup of the codomain.

Remark 1.1.5. Given a semigroup S, there is a natural associated semigroup $(S,\star)$ defined by $a\star b=ba$ for all $a,b\in S$. We will denote this semigroup by S^-. This construction will be referred as the **opposite semigroup of** S, and it is anti-isomorphic to S, meaning that there is a bijective correspondence $S\to S^-$ (namely, the identity $\mathrm{id}\colon S\to S^-$) such that $\phi(ab)=\phi(b)\phi(a)$.

Any left (resp. right) property in the semigroup S will correspond with a right (resp. left) property in the opposite semigroup S^-. For instance, S has a left principal ideal if, and only if, S^- has a right principal ideal.

A semigroup such that for all $a,b\in S$, $ab=ba$, will be called **commutative** or **Abelian**. Equivalently, Abelian semigroups are the ones such that $\mathrm{id}\colon S\to S^-$ is an isomorphism.

We introduce now another fundamental class of semigroups, which will be of great importance throughout this work.

Definition 1.1.6. A semigroup S is said to be **left** (resp. **right**) **cancellative** if for every $a,b,c\in S$, $ab=ac$ (resp. $ba=ca$) implies that $b=c$. We say S is **bicancellative** if it is both right and left cancellative. Finally, when we refer to S as being **cancellative**, we mean it satisfies at least a one-sided cancellation law, without being specific about what side it is.

An alternative way to think of the definition of a bicancellative semigroup is to look at the morphisms $L_s\colon S\to S$ and $R_s\colon S\to S$ given by $L_s(t)=st$ and $R_s(t)=ts$ for $s,t\in S$. Bicancellative semigroups are those such that for all $s\in S$, L_s and R_s are injective functions. In particular, $|sA|=|A|$ for every $s\in S$ and $A\subseteq S$. Note that if a semigroup S has a zero element, then L_{0_S} is the constant function taking the value 0_S, and thus semigroups with zero cannot be bicancellative.

Remark 1.1.7. We point out some of the stability properties of bicancellative semigroups.

(i) Subsemigroups of bicancellative semigroups are bicancellative.

(ii) Direct products of bicancellative semigroups are bicancellative.

(iii) Homomorphic images of bicancellative semigroups *need not* be cancellative. As an example, let $k \geq 1$, and consider $(\mathbb{N}_k, \star)$ where $\mathbb{N}_k = \{0, \ldots, k\}$ and $m \star n = \min\{m+n, k\}$. This semigroup is an homomorphic image of the additive natural numbers: just consider $\phi_k(m) = \min\{m, k\}$. By checking cases, one finds that ϕ_k is a surjective morphism $(\mathbb{N}, +) \to (\mathbb{N}_k, \star)$. However, $k \star n = k$ for all $0 \leq n \leq k$, so $(\mathbb{N}_k, \star)$ has a zero element and thus cannot be cancellative.

In a sense, bicancellative semigroups stand close to groups. Obviously, every group is itself a bicancellative semigroup, but even though the converse is not true in general, we find that both notions coincide for finite semigroups.

Proposition 1.1.8. *Every finite bicancellative semigroup is a group.*

Proof. Let S be a finite bicancellative semigroup. By injectivity of L_a, for $a \in S$, we have that $aS = L_a(S) = S$, and thus there is an $1_a \in S$ satisfying $a1_a = a$. Pick now any $x \in S$, and observe that $ax = a1_a x$, which by left cancellativity implies $x = 1_a x$. Similarly, $x1_a x = x^2$ implies by right cancellativity that $x1_a = x$, so 1_a is a two-sided identity which does not depend on a. We will denote 1_a by 1_S.

Now, since $aS = S$, there is an $a^* \in S$ with $aa^* = 1_S$. Therefore, $aa^*a = a$, which implies (again, by left cancellativity) $a^*a = 1_S$, showing the existence of an inverse element for every $a \in S$. We conclude S is a group. ☐

An example of a left cancellative finite semigroup which is not a group is the right zero semigroup $R = \{a, b\}$, with $xy = y$ for all $x, y \in R$. Note that R is not right cancellative.

1.2 Congruences, quotients and free semigroups

We want to give a semigroup structure to the quotient of a semigroup by an equivalence relation. Given such a relation $\mathcal{R}$ in S, we denote by $\pi_{\mathcal{R}} \colon S \to S/\mathcal{R}$ the quotient map $\pi_{\mathcal{R}}(s) = [s]_{\mathcal{R}}$.

Definition 1.2.1. An equivalence relation $\mathcal{R}$ on a semigroup S is said to be **left** (resp. **right**) **compatible** if for every $a, b, c \in S$, $a\mathcal{R}b$ implies $ca\mathcal{R}cb$ (resp. $ac\mathcal{R}bc$). An equivalence relation both left and right compatible will be referred to as a **congruence**.

If $\mathcal{R}$ is a congruence on S and $a\mathcal{R}a'$, $b\mathcal{R}b'$, then $ab\mathcal{R}a'b$ and $a'b\mathcal{R}a'b'$, so by transitivity we conclude that $ab\mathcal{R}a'b'$. This motivates defining an operation $S/\mathcal{R} \times S/\mathcal{R} \longrightarrow S/\mathcal{R}$ given by $[a]_{\mathcal{R}} \cdot [b]_{\mathcal{R}} = [ab]_{\mathcal{R}}$. With this new structure, $S/\mathcal{R}$ becomes a semigroup and the quotient map $\pi\colon S \longrightarrow S/\mathcal{R}$ a semigroup homomorphism.

Just as in the group theoretic case, every homomorphic image of a semigroup S is isomorphic to a quotient of S.

Theorem 1.2.2. *Let $\theta\colon S \to T$ be a semigroup homomorphism, and define the congruence $\mathcal{R}$ on S by $a\mathcal{R}b$ if and only if $\theta(a) = \theta(b)$. Then, there is an injective homomorphism $\psi\colon S/\mathcal{R} \to T$ such that $\psi \circ \pi_{\mathcal{R}} = \theta$.*

Note that if $\mathcal{R}_0$ is any relation on a semigroup S, then there is a congruence containing $\mathcal{R}_0$, namely $S \times S$, thus making sense to introduce the **congruence generated** by $\mathcal{R}_0$ as the intersection of all congruences containing $\mathcal{R}_0$. In fact, there is a concrete description of the congruence $\mathcal{R}$ generated by $\mathcal{R}_0$. Let

$$\mathcal{R}_0^{-1} = \{(b,a) : (a,b) \in \mathcal{R}_0\} \text{ and } \Delta_S = \{(a,a) : a \in S\}.$$

We have the following description of $\mathcal{R}$.

Proposition 1.2.3. *Let $\mathcal{R}_0$ be any relation on a semigroup S, and $\mathcal{R}$ the congruence generated by $\mathcal{R}_0$. Define $\mathcal{R}_1 = \mathcal{R}_0 \cup \mathcal{R}_0^{-1} \cup \Delta_S$ and $\mathcal{R}_2$ by $a\mathcal{R}_2b$ if, and only if, $a = xcy$ and $b = xdy$ with $c\mathcal{R}_1d$. Then, $a\mathcal{R}b$ if, and only if, there exist $a_1, \dots, a_n \in S$ with $a\mathcal{R}_2a_1\mathcal{R}_2a_2\mathcal{R}_2 \cdots \mathcal{R}_2a_n = b$.*

Now let X be any set, and define $\mathbb{F}(X)^+$ the set of all finite words of elements from X. Together with the concatenation operation, $\mathbb{F}(X)^+$ becomes a semigroup, called the **free semigroup generated by** X. There is a canonical inclusion $\iota\colon X \hookrightarrow \mathbb{F}(X)^+$, which comes with a universal property, analogous to that of free groups.

Theorem 1.2.4. *Let S be a semigroup and $\varphi\colon X \to S$ be any function. Then, there is a unique semigroup morphism $\overline{\varphi}\colon \mathbb{F}(X)^+ \to S$ such that $\overline{\varphi} \circ \iota = \varphi$.*

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, hook, "\iota"'] & S \\
\mathbb{F}(X)^+ \arrow[ur, "\overline{\varphi}"'] &
\end{tikzcd}$$

In the particular case that X is finite, if $|X| = n$ we write $\mathbb{F}_n^+ := \mathbb{F}(X)^+$. This semigroup can be understood as the set X^* of all finite words that can be written with the elements of X, together with the binary operation of concatenation. Adjoining the empty word ε, the semigroups $\mathbb{F}_n^+$ become monoids, and will be of vital importance in the forthcoming chapters.

Given a set X, any subset $R = \{(u_\alpha, v_\alpha)\}_{\alpha \in A} \subseteq \mathbb{F}(X)^+ \times \mathbb{F}(X)^+$ generates a congruence $\mathcal{R}$ on $\mathbb{F}(X)^+$. We define the **semigroup generated by X with relations R**, by

$$\langle X \,|\, R \rangle = \mathbb{F}(X)^+/\mathcal{R}.$$

There is also a universal property regarding this construction.

Theorem 1.2.5. *Let X be a set, $R \subseteq \mathbb{F}(X)^+ \times \mathbb{F}(X)^+$ and $\mathcal{R}$ be the congruence generated by R. Then, if S is any semigroup and $\varphi\colon \mathbb{F}(X)^+ \to S$ a semigroup morphism constant in R-classes, then there is a unique semigroup morphism $\bar{\phi}\colon \mathbb{F}(X)^+/\mathcal{R} \to S$ such that $\bar{\phi} \circ \pi_{\mathcal{R}} = \phi$.*

1.3 Embedding semigroups into groups

Whenever a semigroup admits an embedding $\gamma\colon S \to G$ to a group G, we will say S is **embeddable**. It is clear that a necessary condition for a semigroup to be embedded in a group is to be bicancellative. A well known result states that in the Abelian case, bicancellativity is sufficient, so it becomes equivalent to being embeddable. However, in [Mal37] Mal'cev exhibited an example of a bicancellative semigroup which cannot be embedded into a group (we give account of the construction of Mal'cev in §1.5). Beyond the world of Abelian semigroups, Ore gave a famous sufficient condition for a bicancellative semigroup to be embeddable into a group: reversibility. For the original article[1], see [Ore31].

1.3.1 The reversible case

Definition 1.3.1. A semigroup S is said to be **left** (resp. **right**) **reversible** if for every $a, b \in S$ there are $x, y \in S$ such that $ax = by$ (resp. $xa = yb$), or equivalently if for all $a, b \in S$, $aS \cap bS \neq \varnothing$ (resp. $Sa \cap Sb \neq \varnothing$).

Remark 1.3.2. We point out some stability properties of reversible semigroups.

(i) Direct products of reversible semigroups are reversible.

(ii) Subsemigroups of reversible semigroups *need not* be reversible: there are non-reversible semigroups which embed into groups, such as the free semigroup $\mathbb{F}_2^+$, which embeds into the free group $\mathbb{F}_2$, but all groups are clearly reversible, since their principal ideals are the whole group.

(iii) Homomorphic images of reversible semigroups are reversible.

Besides groups, all Abelian semigroups are reversible: if $a, b \in S$, then $ab \in aS$ and $ab = ba \in bS$, so $aS \cap bS \neq \varnothing$, and the same holds for right principal ideals. Therefore, the theorem due to Ore implies the embeddability of Abelian bicancellative semigroups.

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[1] Ore did this in the context of division rings, from where it follows the case of semigroups

Theorem 1.3.3 (Ore). *Let S be a left reversible bicancellative semigroup. Then, S can be embedded into a group.*

[Ore31, Theorem 1]

As the opposite semigroup of a group is the group itself (an isomorphism would be sending $g \in G$ to g^{-1}), this theorem tells us that a bicancellative right reversible semigroup S can be embedded in a group too. Indeed, S^- is left reversible and bicancellative, and so there is an embedding $\varphi \colon S^- \to G$ into a group, which means $\varphi \colon S \to G^- \simeq G$ is an embedding.

Later, in [Dub43], Dubreil noted that reversibility is a necessary and sufficient condition for a bicancellative semigroup to be embeddable in a group. Moreover, this embedding has a concrete simple manner.

Definition 1.3.4. Let S be a semigroup. A group G is of **right fractions** (resp. of **left fractions**) of S if there is an embedding $\gamma \colon S \to G$, and for every $g \in G$ there are $a, b \in S$ such that $g = \gamma(a)\gamma(b)^{-1}$ (resp. $g = \gamma(b)^{-1}\gamma(a)$).

Remark 1.3.5. If G and G' are groups of right fractions of S, then there is an isomorphism $G \longrightarrow G'$ that fixes S. In other words, the group of right fractions of S is unique modulo an S-fixing isomorphism, and may thus be denoted by $G_R(S)$. The same holds for the group of left fractions, which will be denoted by $G_L(S)$.

Theorem 1.3.6 (Ore-Dubreil). *Let S be a bicancellative semigroup. Then S is left (resp. right) reversible if and only if S can be embedded into its group of right (resp. left) fractions.*

A first direct consequence of these results is that all Abelian bicancellative semigroups can be embedded into their groups of fractions. Concretely, the additive semigroup $\mathbb{N}^n$ can be embedded into $\mathbb{Z}^n$, its group of fractions.

We finish this subsection with a lemma from [Don13].

Lemma 1.3.7. *Let T be a bicancellative left reversible semigroup and let $S \leq T$ be a left reversible subsemigroup. Then $G_R(S)$ is isomorphic to a subgroup of $G_R(T)$.*

Proof. Let $\eta \colon T \hookrightarrow G_R(T)$ be an injective morphism. Thus, $\eta|_S \colon S \to \langle \eta(S) \rangle \leq G_R(T)$ is an embedding. If $g \in \langle \eta(S) \rangle$, there exist $s_1, \dots s_n, \in S$ and $\epsilon_1, \dots, \epsilon_n \in \{1, -1\}$ such that

$$g = \eta(s_1)^{\epsilon_1} \dots \eta(s_n)^{\epsilon_n}.$$

We want to see that g can be written as a fraction $\eta(s')\eta(s)^{-1}$. If $n = 1$, fix $s_0 \in S$ and we have

$$g = \eta(s_1)^{\epsilon_1} = \begin{cases} \eta(s_1 s_0)\eta(s_0)^{-1} & \text{if } \epsilon_1 = 1, \\ \eta(s_0)\eta(s_1 s_0)^{-1} & \text{if } \epsilon_1 = -1. \end{cases}$$

Now, if $\eta(s_1)^{\epsilon_1} \dots \eta(s_{n-1})^{\epsilon_{n-1}} = \eta(s')\eta(s)^{-1}$ and $\epsilon_n = -1$ we are done. We just need to check that $\eta(s)^{-1}\eta(s_n)$ can be written as a right fraction. Since S is left reversible, there exist $a, b \in S$ with $sa = s_n b$, and thus $\eta(s)\eta(a) = \eta(s_n)\eta(b)$, which implies $\eta(s)^{-1}\eta(s_n) = \eta(b)\eta(a)^{-1}$ as desired.

Therefore, every element of $\langle \eta(S) \rangle$ can be written as a right fraction, so $\langle \eta(S) \rangle \simeq G_R(S)$. $\square$

1.3.2 The general case

The general case of embedding a semigroup into a group is delicate. Let S be a semigroup. A pair (H, η) will be called an S-**group** if $\eta \colon S \to H$ is a semigroup morphism with $\langle \eta(S) \rangle = H$. A morphism between two S-groups (H, η) and (H', η') will be a group morphism $\theta \colon H \to H'$ such that $\theta \circ \eta = \eta'$.

Definition 1.3.8. Let S be a semigroup. A **free group on the semigroup** S, or a **free** S-**group**, is an initial object in the category of S-groups, i.e., an S-group (G, γ) such that for every S-group (H, η) there is a unique morphism $\theta \colon G \to H$ with $\theta \circ \gamma = \eta$.

$$\begin{tikzcd}
S \arrow[r, "\gamma"] \arrow[rd, "\eta"'] & G \arrow[d, dashed, "\theta"] \\
& H
\end{tikzcd}$$

The free group on a semigroup S always exists, and as an initial object, it is unique up to isomorphism of S-groups. While in some cases the universal property of the free S-group will be really useful, the following concrete way of viewing this object will play a major role as well.

Proposition 1.3.9. *Let S be a semigroup and $S = \langle B | R \rangle$ a presentation for S, with $R = \{(u_\alpha, v_\alpha) : \alpha \in A\} \subseteq \mathbb{F}(B)^+ \times \mathbb{F}(B)^+$. Define $R' = \{u_\alpha v_\alpha^{-1} : \alpha \in A\} \subseteq \mathbb{F}(B)$. Then, the group $\langle B | R' \rangle$ is the free S-group.*

Proof. Let $G = \langle B | R' \rangle = \mathbb{F}(B)/\langle R' \rangle_\triangleleft$. First, we need to specify a semigroup morphism $\gamma \colon S \to G$ such that $\langle \gamma(S) \rangle = G$. Set $\gamma(b) = [b] = b\langle R' \rangle_\triangleleft$, and extend the function to all of S homomorphically. Clearly $\gamma(S)$ generates G. We need to check this function is well defined, i.e., that if $u, v \in \mathbb{F}(B)^+$ represent the same element of S, then $\gamma(u) = \gamma(v)$.

Let $\mathcal{R}$ be the congruence generated by R, and $\mathcal{R}' = \{uv^{-1} : (u, v) \in \mathcal{R}\}$. We will show $\mathcal{R}' \subseteq \langle R' \rangle_\triangleleft$. Let $N \trianglelefteq \mathbb{F}(B)$ is such that $R' \subseteq N$. If $u\mathcal{R}v$, there exist $u_1, \ldots, u_n \in S$ with $u\mathcal{R}_2 u_1 \mathcal{R}_2 \cdots \mathcal{R}_2 u_n = v$ (see §1.2 for notation). If $n = 1$, then $u = xay$, $v = xby$ and either aRb, bRa or $a = b$. Thus, either $x^{-1}uv^{-1}x = ab^{-1} \in R' \subseteq N$, $x^{-1}vu^{-1}x = ba^{-1} \in R' \subseteq N$ or $u = v$. In the first case, $uv^{-1} \in xNx^{-1} = N$, in the second we have $uv^{-1} = (vu^{-1})^{-1} \in (xNx^{-1})^{-1} = N$, and in the last scenario $uv^{-1} = 1_{\mathbb{F}(B)} \in N$. Therefore $uv^{-1} \in N$. Now assume $n > 1$ and $uu_{n-1}^{-1} \in N$. Again, we can write $u_{n-1} = xay$ and $v = xby$, and either aRb, bRa or $a = b$. In the same fashion as before, this will imply that $u_{n-1}v^{-1} \in N$, so $uv^{-1} = (uu_{n-1}^{-1})(u_{n-1}v^{-1}) \in N$. By induction we conclude $\mathcal{R}' \subseteq N$. Since N was an arbitrary normal subgroup containing R', this yields $\mathcal{R}' \subseteq \langle R' \rangle_{\triangleleft}$, so if $u\mathcal{R}v$ then $u\langle R' \rangle_{\triangleleft} = v\langle R' \rangle_{\triangleleft}$ and we get $\gamma(u) = \gamma(v)$. This shows G is an S-group.

To see G is the free S-group, let $\eta\colon S \to H$ be a semigroup morphism with $\eta(S)$ generating H, and consider the inclusion $\iota\colon B \to \mathbb{F}(B)$. The universal property of the free group allows us to construct a morphism $\theta\colon \mathbb{F}(B) \to H$ such that $\theta \circ \iota = \eta|_B$. Now, if $h \in \mathbb{F}(B)$ satisfies $\pi(h) = 1_G$, then

$$h = \prod_{j=1}^{n} w_j r_j w_j^{-1}$$

with $r_j \in R'$ for each $1 \leq j \leq n$. Fix j and note r_j writes as

$$r_j = \iota(a_1)^{i_1} \cdots \iota(a_m)^{i_m} \iota(b_k)^{-\ell_k} \cdots \iota(b_1)^{-\ell_1},$$

where $a_\alpha, b_\alpha \in B$, $a_1^{i_1} \cdots a_m^{i_m} = b_1^{\ell_1} \cdots b_k^{\ell_k}$ in S and $i_\alpha, \ell_\alpha \geq 1$. Hence,

$$\begin{aligned} \theta(r_j) &= \eta(a_1)^{i_1} \cdots \eta(a_m)^{i_m} \eta(b_k)^{-\ell_k} \cdots \eta(b_1)^{-\ell_1} \\ &= \eta(a_1^{i_1} \cdots a_m^{i_m}) \eta(b_1^{\ell_1} \cdots b_k^{\ell_k})^{-1} = 1_H. \end{aligned}$$

Therefore, $\theta(h) = 1_H$ and we have $\langle R' \rangle_{\triangleleft} \subseteq \ker(\theta)$. By the universal property of the quotient group, there is a unique group morphism $\theta'\colon G \to H$ such that $\theta' \circ \pi = \theta$. We need to show that $\theta' \circ \gamma = \eta$.

$$\begin{tikzcd}
B \arrow[r, "\iota"] \arrow[dr, "\eta|_B"'] & \mathbb{F}(B) \arrow[r, "\pi"] \arrow[d, dashed, "\theta"] & \mathbb{F}(B)/\langle R' \rangle_{\triangleleft} \arrow[dl, dashed, "\theta'"] \\
 & H &
\end{tikzcd}$$

Given $s \in S$, write $s = b_1^{n_1} \cdots b_k^{n_k}$ with $b_i \in B$ and $n_i \geq 1$. Since $\pi \circ \iota = \gamma|_B$, we get

$$\begin{aligned} \theta' \circ \gamma(s) &= \theta'(\gamma(b_1))^{n_1} \cdots \theta'(\gamma(b_k))^{n_k} \\ &= \theta(\iota(b_1))^{n_1} \cdots \theta(\iota(b_k))^{n_k} \\ &= \eta(b_1)^{n_1} \cdots \eta(b_k)^{n_k} = \eta(s). \end{aligned}$$

□

Remark 1.3.10 (Reversible case, revisited). An example of free group on a semigroup we have in hand is the group of right (or left) fractions in the reversible case. Indeed, if (H, η) is an S-group with η an embedding, applying Lemma 1.3.7 with $T = H$ we get that $G_R(S) \simeq G_R(\eta(S))$ is a subgroup of $G_R(G) = G$ such that $\langle \eta(S) \rangle = G_R(S)$. Since H was an S-group, this means $H = \langle \eta(S) \rangle = G_R(S)$. Thus, if S is either left or right reversible, there is only one S-group, namely $G_R(S)$ or $G_L(S)$, which hence equals the free S-group.

The really relevant thing about the free group on a semigroup is that it characterizes embeddability, meaning that a semigroup which can be embedded into a group can be embedded into its free group on the semigroup. A proof of this can be found in [CP61].

Theorem 1.3.11. *Let S be a semigroup, and (G, γ) be the free S-group. Then, the semigroup S can be embedded in a group if, and only if, $\gamma \colon S \to G$ is an embedding.*

1.3.3 A combinatorial criterion for embeddability

Sufficient conditions for embeddability of a semigroup into a group have been given in various forms. Famous conditions include Mal'cev's, Lambek's and Pták's: a detailed exposition of these can be found in [CP61, Vol. II, Chapter 12]. Here we review an interesting combinatorial criterion due to S.I. Adian ([Adi66]), who gives a sufficient (although not even close to necessary) condition for embeddability.

Let S be a semigroup presented as $\langle B \,|\, R\rangle$, where we write $B = \{a_1, \dots, a_n\}$ and $R = \{(u_\alpha, v_\alpha) : \alpha \in A\}$. For a given relation (u_α, v_β), where $u_\alpha = a$ we define its **left pair** $(\ell^1_\alpha, \ell^2_\alpha)$, where ℓ^1_α is the leftmost letter in the word u_α, and ℓ^2_α the leftmost letter in v_α. A **right pair** for (u_α, v_α) is defined analogously with rightmost letters. We call the graph $\Gamma = (V, E)$ with $V = B$ and $E = \{(\ell^1_\alpha, \ell^2_\alpha) : \alpha \in A\}$, the **left graph** of S. The **right graph** is the corresponding graph with right pairs as edges.

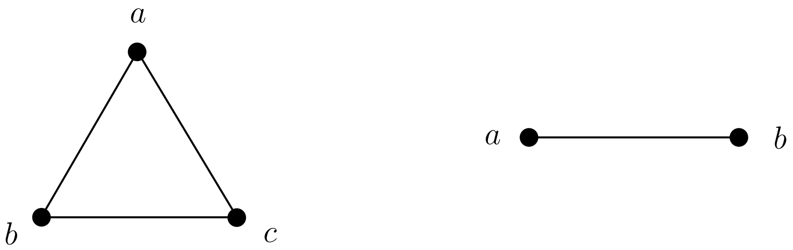


Figure 1.1: The left graphs of $\mathbb{N}^3 \simeq \langle a, b, c \mid ab = ba, bc = cb, ca = ac\rangle$ (on the left) and $\mathrm{BS}(m, n) = \langle a, b \mid ab^m = b^n a\rangle$, $n \geq 1$ (on the right).

A **cycle** in the left or right graph is a closed path $\{(v_1, v_2), (v_2, v_3), \dots, (v_n, v_{n+1})\}$ such that $v_{n+1} = v_1$, $v_k \neq v_{k+2}$ for all $k < n$, and $v_n \neq v_2$. We admit the possibility of a cycle consisting of a single edge.

The following theorem is proven in [Adi66].

Theorem 1.3.12. *Let S be a semigroup presented as $\langle B \,|\, R\rangle$ and suppose the left and right graphs of S associated with this presentation do not admit cycles. Then, S is bicancellative and isomorphic to the subsemigroup of the free S-group generated by B. In particular, it is an embeddable semigroup.*

A really useful application of the theorem is that all bicancellative finitely presented semigroups which admit a presentation with only one relation can be embedded into a group. Nonetheless, several bicancellative semigroups known to be reversible do not fall under the hypothesis of the criterion, an example being $\mathbb{N}^3$, which, as seen in Figure 1.1, contains a cycle in its left graph.

1.4 Amenable semigroups

The concept of amenability was introduced by J. von Neumann for groups in [Neu29] as an answer to the Banach-Tarski Paradox. The main purpose of this section is to review the amenability notion for semigroups and relate this notion to Følner-like properties and reversibility. Check [AW67; Nam64] for further details. Here, S will be a countable semigroup, and we define, for $s \in S$ and $f \in \ell^\infty(S)$, the functions sf and fs by setting

$$(fs)(t) = f(ts) \text{ and } (sf)(t) = f(st) \quad \forall t \in S$$

for all $t \in S$.

Definition 1.4.1. A **mean** on $\ell^\infty(S)$ is a positive, unital, linear functional $\sigma\colon \ell^\infty(S) \to \mathbb{C}$. A mean σ is called **left S-invariant** if $\sigma(sf) = \sigma(f)$ and **right S-invariant** if $\sigma(fs) = \sigma(f)$, for every $f \in \ell^\infty(S)$.

Note that a mean $\sigma \in \ell^\infty(S)$ is automatically bounded: if $f \in \ell^\infty(G)$, we may assume without loss of generality that $f \geq 0$. Note that $\|f\|_\infty - f \geq 0$, so $\sigma\Big(\|f\|_\infty \mathbf{1}\Big) \geq \sigma(f)$, yielding

$$|\sigma(f)| = \sigma(f) \leq \sigma\Big(\|f\|_\infty \mathbf{1}\Big) = \|f\|_\infty \sigma(\mathbf{1}).$$

Definition 1.4.2. The semigroup S is said to be **left** (resp. **right**) **amenable** if there is a left (resp. right) S-invariant mean on $\ell^\infty(S)$.

There is a parallel between means on $\ell^\infty(S)$ and finitely additive probability measures on $(S, \mathcal{P}(S))$, which we want to establish now. Given $A \subseteq S$ and $s \in S$, we introduce the following notation:

$$s^{-1}A = \{t \in S : st \in A\} \quad \text{and} \quad As^{-1} = \{t \in S : ts \in A\}.$$

Definition 1.4.3. A finitely additive probability measure μ on $(S, \mathcal{P}(S))$ is said to be **left S-invariant** if $\mu(s^{-1}A) = \mu(A)$ for every $s \in S$ and $A \in \mathcal{P}(S)$. Respectively, μ is called **right S-invariant** if $\mu(As^{-1}) = \mu(A)$ for every $s \in S$ and $A \in \mathcal{P}(S)$.

The parallel between left or right amenability and other correspondingly left or right properties will be a constant during this section. Since there will not be any essential difference in considering left or right properties, we will solely deal with the left-sided case from now on.

Proposition 1.4.4. *A semigroup S is left amenable if and only if it admits a left S-invariant finitely additive probabilty measure on $(S, \mathcal{P}(S))$.*

Proof. If S admits an invariant mean σ, then it admits an invariant finitely additive probability measure, which can be taken to be $\mu\colon \mathcal{P}(S) \to [0,1]$ given by $\mu(A) = \sigma(\mathbf{1}_A)$. That it is finitely additive is a consequence of the fact that σ is linear and $\mathbf{1}_{A\sqcup B} = \mathbf{1}_A + \mathbf{1}_B$. The invariance of μ follows from the fact that $\mathbf{1}_{s^{-1}A} = s\mathbf{1}_A$. Conversely, if S admits an invariant finitely additive probability measure μ, an invariant mean σ can be obtained setting

$$\sigma(f) = \int_S f\, d\mu$$

for every $f \in \ell^\infty(S)$. Indeed, such a function is clearly linear, unital and positive. To check invariance, note that if $f = \sum_k \alpha_k \mathbf{1}_{E_k}$ and $s \in S$, then

$$\sigma(sf) = \int_S f(st)\, d\mu(t) = \sum_k \alpha_k \int_S \mathbf{1}_{s^{-1}E_k}(t)\, d\mu(t) = \sum_k \alpha_k \mu(E_k) = \sigma(f).$$

By aproximation by simple functions, one can conclude for arbitrary $f \in \ell^\infty(S)$. ☐

1.4.1 Følner conditions for semigroups

One would like to connect amenability with Følner-like properties. While in the group-theoretic case we do not really need to distinguish between left and right amenability (for we already know that any left Følner sequence induces a right one, and vice versa, by taking inverses), in the semigroup case we have to. We will focus on left amenable semigroups here, although the right-amenable theory is completely analogous.

Definition 1.4.5. Let S be a semigroup.

(i) S satisfies the **left Følner condition** (LFC) if for every $\epsilon > 0$ and every finite subset $K \subseteq S$, there is a finite set $F \subseteq S$ such that for every $s \in K$ we have $|sF - F| < \epsilon|F|$.

(ii) S satisfies the **strong left Følner condition** (SLFC) if for every $\epsilon > 0$ and every finite subset $K \subseteq S$, there is a finite set $F \subseteq S$ such that for every $s \in K$ we have $|F - sF| < \epsilon|F|$.

(iii) A sequence $\{F_n\}_{n\in\mathbb{N}}$ of finite subsets of S is said to be a **left Følner sequence** if for every $s \in S$,

$$\lim_{n\to\infty} \frac{|sF_n \triangle F_n|}{|F_n|} = 0$$

Remark 1.4.6. For a finitely generated semigroup S, it suffices to check upon the generators to conclude that it has LFC. Indeed, if $\Omega \subseteq S$ is finite, $S = \langle \Omega \rangle$ and for all $\epsilon > 0$ there is a finite subset $F \subseteq S$ with $|\omega F - F| < \epsilon|F|$ for every $\omega \in \Omega$, then for any $s = \omega_1 \ldots \omega_n \in S$,

$$\begin{aligned}|sF - F| &= \left|(\omega_1 F - F) \cup \bigcup_{k=1}^{n-1} (\omega_1 \dots \omega_{k+1} F - \omega_1 \dots \omega_k F)\right| \\ &\leq |\omega_1 F - F| + \sum_{k=1}^{n-1} |\omega_1 \dots \omega_{k+1} F - \omega_1 \dots \omega_k F| \\ &\leq \sum_{k=1}^{n} |\omega_k F - F|,\end{aligned}$$

so taking a set F satisfying the property for ϵ/n with respect to Ω, we get $|sF - F| < \epsilon|F|$.

The weak and strong versions of the Følner condition relate to each other in the following manner.

Lemma 1.4.7. *Let S be a semigroup. Then, we have the following.*

(i) *If S satisfies* SLFC*, then it satisfies* LFC*.*

(ii) *If S is left cancellative and satisfies* LFC*, then it also satisfies* SLFC*.*

Proof. First, we prove (i). Let $\epsilon > 0$ and $K \subseteq S$ finite. Take a finite subset $F \subseteq S$ satisfying SLFC for ϵ and K. Note that, for every $s \in K$,

$$|sF - F| = |(sF \cup F) - F| = |sF \cup F| - |F|$$

Now, since F is finite, $|F| \geq |sF|$, which yields

$$|sF - F| \leq |sF \cup F| - |sF| = |(sF \cup F) - sF| = |F - sF| < \epsilon|F|.$$

To prove (ii), note that for a left cancellative semigroup, $|sF| = |F|$ for any $s \in S$ and finite $F \subseteq S$, so we obtain from our first equation that $|sF - F| = |F - sF|$. ☐

As a corollary of item (i) from the last lemma, we have that S satisfies SLFC if and only if S has a left Følner sequence. We now want to point out how the Følner conditions relate to amenability. The proofs, which require some work, will not be included, and can be found in [Nam64].

Proposition 1.4.8. *Let S be a semigroup. We have the following.*

(i) *If S satisfies* SLFC*, then it is left amenable.*

(ii) *If S is left amenable, then it satisfies* LFC*.*

Remark 1.4.9. Some further properties of amenability and Følner conditions for semigroups which can be found in the literature (check, for instance, [AW67]) are listed below.

- Every abelian semigroup is both left and right amenable.

- There are finite semigroups which are not left amenable. This shows that LFC does not imply left amenability in general, since every finite semigroup satisfies LFC ($|tS - S| = 0$ for all $t \in S$).

- A subsemigroup of a left amenable semigroup need not be left amenable, as noted by Frey in [FJ60]. Indeed, if S is the semigroup obtained by adjoining a zero element to $\mathbb{F}_2$, then $\sigma \colon f \mapsto f(0)$ is a left invariant mean. However, $\mathbb{F}_2$ is not amenable and embeds into S. Nevertheless, we have the following proposition.

Proposition 1.4.10. *Let S be a bicancellative semigroup and (H, η) be an S-group, with η an embedding. Then, if S is left amenable, the group H is amenable.*

Proof. Assume that S is left amenable. Since it is also bicancellative, it admits a Følner sequence $(F_n)_{n \geq 1}$. Now, any $g \in H$ can be written as $g = \eta(s_1)^{\epsilon_1} \cdots \eta(s_n)^{\epsilon_n}$, with $s_i \in S$ and $\epsilon_i \in \{1, -1\}$, since H is an S-group. In fact, we can take $\epsilon_i = (-1)^i$ without loss of generality. For the sake of readability, we will denote $\eta(F_n) \subseteq H$ as H_n through the main part of the computation.

For all $A, B, C \subseteq G$ we have $A \triangle B \subseteq (A \triangle C) \cup (B \triangle C)$, which implies $|A \triangle B| \leq |A \triangle C| + |B \triangle C|$ (in other words, $d(A, B) := |A \triangle B|$ defines a metric on $\mathcal{P}(G)$, this is a general set-theoretical truth). Therefore, we obtain

$$
\begin{aligned}
\frac{|gH_n \triangle H_n|}{|H_n|} &\leq \frac{|\eta(s_1)\eta(s_2)^{-1} \cdots \eta(s_n)^{\epsilon_n} H_n \triangle \eta(s_1) F_n|}{|H_n|} + \frac{|\eta(s_1) H_n \triangle H_n|}{|H_n|} \\
&= \frac{|\eta(s_2)^{-1} \cdots \eta(s_n)^{\epsilon_n} H_n \triangle H_n|}{|H_n|} + \frac{|\eta(s_1) H_n \triangle H_n|}{|H_n|} \\
&\leq \frac{|\eta(s_2)^{-1} \cdots \eta(s_n)^{\epsilon_n} H_n \triangle \eta(s_2)^{-1} H_n|}{|H_n|} + \frac{|\eta(s_2)^{-1} H_n \triangle H_n|}{|H_n|} + \frac{|\eta(s_1) H_n \triangle H_n|}{|H_n|} \\
&= \frac{|\eta(s_3) \cdots \eta(s_n)^{\epsilon_n} H_n \triangle H_n|}{|H_n|} + \frac{|\eta(s_2)^{-1}(H_n \triangle \eta(s_2) H_n)|}{|H_n|} + \frac{|\eta(s_1) H_n \triangle H_n|}{|H_n|} \\
&= \frac{|\eta(s_3) \cdots \eta(s_n)^{\epsilon_n} H_n \triangle H_n|}{|H_n|} + \frac{|\eta(s_2) H_n \triangle H_n|}{|H_n|} + \frac{|\eta(s_1) H_n \triangle H_n|}{|H_n|} \\
&\vdots \\
&= \sum_{i=1}^{n} \frac{|\eta(s_i)\eta(F_n) \triangle \eta(F_n)|}{|\eta(F_n)|} \\
&= \sum_{i=1}^{n} \frac{|s_i F_n \triangle F_n|}{|F_n|} \to 0,
\end{aligned}
$$

the last equality following by injectivity of η. Thus, G has a Følner sequence, and is therefore amenable. $\square$

1.4.2 Amenability and reversibility

Now that we have characterized amenability in the cancellative case, let us see how this notion relates to reversibility, and to the amenability of the corresponding group of fractions. The following result can be found in [Don13].

Proposition 1.4.11. *Let S be a semigroup which satisfies* SLFC*. Then, S is left reversible.*

Proof. Let $a, b \in S$. Since S satisfies SLFC, we can find a finite subset $F \subseteq S$ such that

$$\frac{|F - aF|}{|F|} < \frac{1}{4} \quad \text{and} \quad \frac{|F - bF|}{|F|} < \frac{1}{4},$$

which implies

$$\frac{|aF \cap F|}{|F|} = \frac{|F| - |F - aF|}{|F|} > 1 - \frac{1}{4} = \frac{3}{4},$$

and analogously for b. Hence,

$$\begin{aligned} |F| &\geq |F \cap (aF \cup bF)| \\ &= |F \cap aF| + |F \cap bF| - |F \cap aF \cap bF| \\ &> \frac{6}{4}|F| - |F \cap aF \cap bF|, \end{aligned}$$

so $|F \cap aF \cap bF| > |F|/2 > 0$. Clearly, this implies that $aS \cap bS \supseteq aF \cap bF \neq \varnothing$.

□

Corollary 1.4.12. *If S is left amenable and left cancellative, then S is left reversible.*

In order to have a converse for this result, we need an extra condition over the semigroup, as well as the following previous result.

Lemma 1.4.13. *Let S be a bicancellative reversible semigroup. Then, S is left amenable if and only if $G_R(S)$ is an amenable group.*

Proof. We already saw that S-groups are amenable whenever S is amenable. In particular, the group of right fractions must be amenable. Conversely, if $G = G_R(S)$ is amenable, let $(F_n)_{n\geq 1}$ be a left Følner sequence in G. For every $n \geq 1$ there is an element $g_n \in G$ with $g_n^{-1} \leq_S t$ for all $t \in F_n$, i.e., $F_n g_n \subseteq S$. Since for every $s \in S$,

$$\frac{|sF_n g_n \triangle F_n g_n|}{|F_n g_n|} = \frac{|sF_n \triangle F_n|}{|F_n|} \longrightarrow 0,$$

we obtain that $(F_n g_n)_{n\geq 1}$ is a left Følner sequence in S.

□

We state now the converse for Corollary 1.4.12. See [Don13, Theorem 5] for a more detailed result than the one we are going to exhibit.

Proposition 1.4.14. *If S is a semigroup which embeds into a bicancellative and left-amenable semigroup T, then the following statements are equivalent:*

(i) *S is left amenable,*

(ii) *S is left reversible.*

Proof. We have already seen in Corollary 1.4.12 that (i) implies (ii). Now assume S is left reversible. Since S is bicancellative, by Ore's Theorem it embeds into a group of fractions $G_R(S)$. The semigroup T is bicancellative and left amenable, so it embeds into its group of fractions, $G_R(T)$, as well, and by Lemma 1.3.7 we have that $G_R(S) \leq G_R(T)$. By the last lemma, we know $G_R(T)$ is amenable, and thus so is $G_R(S)$ as a subgroup of an amenable group. Finally, using again the last lemma we have that S is left amenable. ☐

Note that Remark 1.4.9 provides an example of a subsemigroup of a left amenable semigroup which is not left amenable. However, this subsemigroup does not satisfy the hypotheses required to inherit amenability. Indeed, even though $\mathbb{F}_2$ is left reversible (every group is), the ambient semigroup, which is $\mathbb{F}_2$ with a zero element adjoined, is left amenable, but obviously not bicancellative.

The following diagram summarizes some aspects about the relationship between LFC, SLFC, amenability and reversibility.

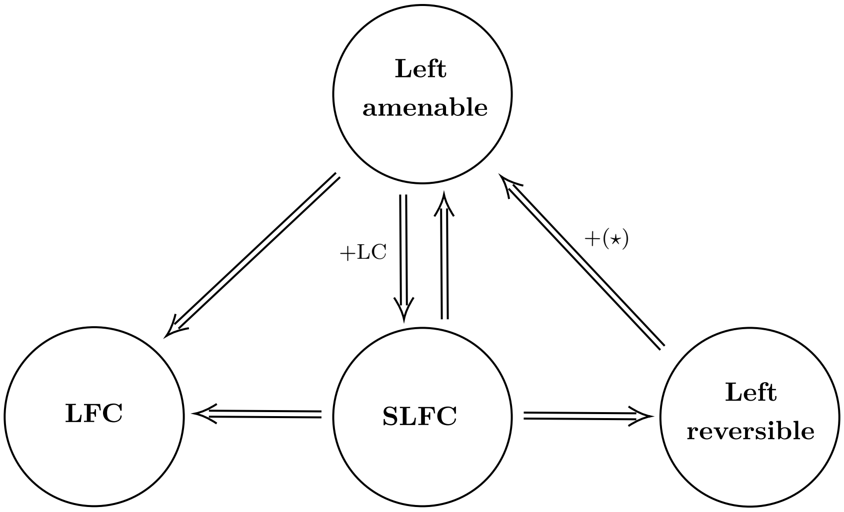


Figure 1.2: Summary of relations. Here, LC stands for "left cancellativity", and S satisfies the condition $(\star)$ if $S \leq T$ where T is a bicancellative left amenable semigroup.

We end this section with the following useful proposition, which was proven by Frey in [FJ60, Theorem 8.4].

Proposition 1.4.15. *Let S be a bicancellative semigroup. Then, S does not contain $\mathbb{F}_2^+$ as a subsemigroup if and only if every subsemigroup $T \leq S$ is left reversible.*

1.4.3 Graph amenability of semigroups

We want to relate the property LFC (which, recall, is equivalent to left amenability in the left cancellative case) with the amenability of the left Cayley graph of S. In order to do this, we need to consider different ways of speaking of amenability of graphs.

Definition 1.4.16. Let $\Gamma = (V, E)$ be a directed locally finite graph, and $F \subseteq V$ a finite set. We define

$$\partial^+ F = \{v \in V : (f, v) \in E \text{ for some } f \in F\},$$

$$\partial^- F = \{v \in V : (v, f) \in E \text{ for some } f \in F\}.$$

The set $\partial^+ F \cup \partial^- F$ will be denoted by ∂F.

We will distinguish between a directed and an undirected notion of amenability.

Definition 1.4.17. A directed locally finite graph $\Gamma = (V, E)$ will be called **amenable** if

$$\inf \left\{ \frac{|\partial F|}{|F|} : \varnothing \neq F \subseteq V \text{ is finite} \right\} = 0,$$

and we will say it is **out-amenable** (resp. **in-amenable**) if the same property holds for ∂^+ (resp. for ∂^-).

Since our definition of amenable graph coincides with the usual definition for undirected graphs, we will refer to it as the undirected version whenever we think it may be clarifying.

Remark 1.4.18. Since for every finite $F \subseteq V$ we have the inequality

$$\max\{|\partial^+ F|, |\partial^- F|\} \leq |\partial F| \leq |\partial^+ F| + |\partial^- F|,$$

if a graph Γ is amenable then it is both in-amenable and out-amenable.

Proposition 1.4.19. *Let S be a semigroup, and $\Omega \subseteq S$ a finite subset which generates S. If $\Gamma = \mathrm{Cay}_L(S, \Omega)$ is the left Cayley graph of S with respect to Ω, then*

(i) *S satisfies* LFC *if and only if Γ is out-amenable,*

(ii) *if S is left cancellative, then it satisfies* LFC *if and only if Γ is amenable.*

Proof. To prove (i), note that (here becomes important that Γ is the *left* Cayley graph)

$$\partial^+ F = \bigcup_{\omega\in\Omega}(\omega F - F).$$

Therefore, given any $\epsilon > 0$, if $|\omega F - F| < \epsilon|F|/|\Omega|$, we have $|\partial^+ F|/|F| < \epsilon$, so LFC implies Γ is out-amenable. Conversely, if Γ is out-amenable, taking F so that $|\partial^+ F| < \epsilon|F|$ for a given $\epsilon > 0$ ensures

$$|\omega_0 F - F| \leq \left|\bigcup_{\omega\in\Omega}(\omega F - F)\right| = |\partial^+ F| < \epsilon|F|$$

for every $\omega_0 \in \Omega$. Thus, S satisfies LFC with respect to Ω, and it suffices to show for the generators in the finitely generated case.

To prove (ii), since amenability of Γ implies out-amenability of Γ, it suffices to show that LFC implies Γ is amenable. Note that

$$\partial F = \bigcup_{\omega\in\Omega}(\omega F \cup \omega^{-1}F - F),$$

where $\omega^{-1}F = \{s \in S : \omega s \in F\}$. Since S is left cancellative,

$$\begin{aligned}|\omega F \cup \omega^{-1}F - F| &\leq |\omega F \cup F \cup \omega^{-1}F| - |F|\\ &\leq |\omega^2 F \cup \omega F \cup F| - |F|\\ &= |\omega^2 F \cup \omega F - F|\\ &\leq |\omega^2 F - F| + |\omega F - F|.\end{aligned}$$

Taking, for any $\epsilon > 0$, a finite set F such that $|sF - F| < \epsilon|F|/(2|\Omega|)$ for every $s \in \Omega \cup \Omega^2$, we get

$$|\partial F| \leq \sum_{\omega\in\Omega}|\omega F \cup \omega^{-1}F - F| \leq \sum_{\omega\in\Omega}(|\omega^2 F - F| + |\omega F - F|) < \epsilon|F|.$$

□

Remark 1.4.20. Not every graph that is out-amenable happens to be undirectedly amenable as well. One might just consider the binary tree with all edges directed towards the root.

1.5 Examples

This final section aims to build a nice set of examples of bicancellative semigroups with different properties, emphasizing in embeddable semigroups.

Example 1.5.1 (Free Abelian semigroups). We start out with the class of Free Abelian semigroups: the additive semigroup $(\mathbb{N}^n, +)$, with $n \geq 1$. These are Abelian semigroups which, as already mentioned, can be embedded into amenable groups ($\mathbb{Z}^n$, concretely). Thus they are embeddable, reversible and amenable.

Example 1.5.2 (Baumslag-Solitar semigroups). We want to give another example of an embeddable semigroup which is not reversible. Consider the Baumslag-Solitar semigroup $\mathrm{BS}(m,n)^+ = \langle a, b | ab^m = b^n a\rangle^+$, with $n, m \geq 1$, which is a subsemigroup of the corresponding Baumslag-Solitar group $\mathrm{BS}(m,n)$ as a consequence of Adian's Theorem 1.3.12. We will focus on the case $m = 1$. In this case the group $\mathrm{BS}(1,n)$ is solvable, thus amenable, and a dynamical realization is $\langle a, b\rangle \leq \mathrm{Homeo}(\mathbb{R})$, where $a(x) = nx$ and $b(x) = x + 1$ for every $x \in \mathbb{R}$.

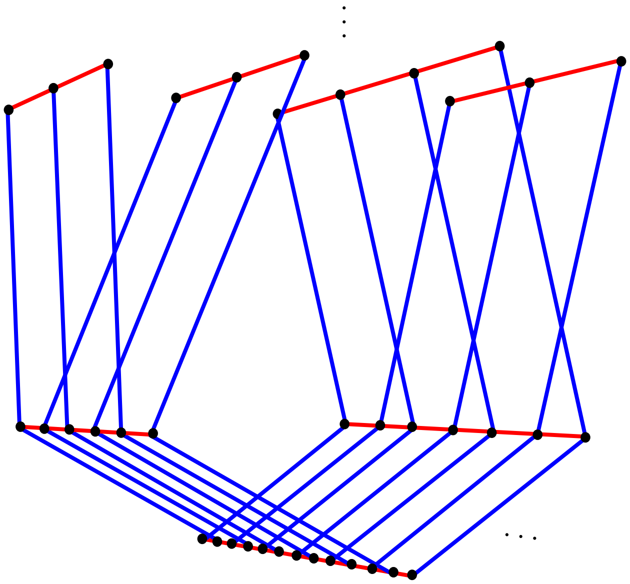


Figure 1.3: The left Cayley graph of $\mathrm{BS}(1,2)^+$. Blue edges correspond to a and red edges to b.

It can be inductively seen that, for all $k, j \in \mathbb{N}$, $a^j b^k = b^{kn^j} a^j$, so every element $s \in \mathrm{BS}(1,n)^+$ can be written as $s = b^k a^j$ with $j, k \in \mathbb{N}$. Assume $s = b^k a^j$ and $t = b^{k'} a^{j'}$ satisfy $as = bat$. Then $as = ab^k a^j = b^{kn} a^{j+1}$ and $bat = bab^{k'} a^{j'} = b^{k'n+1} a^{j'+1}$. Thus,

$$as = bat \implies b^{(k-k')n-1} a^{j'-j} = 1.$$

As a function in $\mathrm{Homeo}(\mathbb{R})$, this means $n^{j'-j} x + (k - k')n - 1 = x$, which implies $n(k - k') = 1$, and this cannot be the case. Therefore, $a\mathrm{BS}(1,n)^+ \cap (ba)\mathrm{BS}(1,n)^+ = \varnothing$, so $\mathrm{BS}(1,n)^+$ is not left reversible.

The fact that the semigroup $\mathrm{BS}(1,n)^+$ is not left reversible, as a consequence of Proposition 1.4.14, implies it cannot be left amenable. Nevertheless, they are right amenable. To show this, we exhibit a right Følner sequence (more details can be found in [SM20, Section 3.1]). Define for each $m \in \mathbb{N}$

$$F_m = \left\{ b^k a^j : 0 \leq k < n^m, 0 \leq j < m \right\}.$$

For $m \geq 1$, we have that $|F_m| = mn^m$, and

$$F_m a \triangle F_m = \left\{ b^j a^k : 0 \leq j < n^m, k = 0, m \right\},$$

so $|F_m a \triangle F_m| = 2n^m$. On the other hand,

$$\begin{aligned} F_m b \triangle F_m &= \left\{ b^{j+n^k} a^k : 0 \le j < n^m, 0 \le k < m \right\} \triangle \left\{ b^j a^k : 0 \le j < n^m, 0 \le k < m \right\} \\ &= \left\{ b^j a^k : n^k \le j < n^m + n^k, 0 \le k < m \right\} \triangle \left\{ b^j a^k : 0 \le j < n^m, 0 \le k < m \right\} \\ &= \left\{ b^j a^k : j \in [0, n^k) \cup [n^m, n^m + n^k), 0 \le k < m \right\}, \end{aligned}$$

so

$$|F_m b \triangle F_m| = \sum_{k=0}^{m-1} 2n^k = 2\frac{n^m - 1}{n - 1}.$$

Thus we get

$$\lim_{m \to \infty} \frac{|F_m a \triangle F_m|}{|F_m|} = 0 = \lim_{m \to \infty} \frac{|F_m b \triangle F_m|}{|F_m|},$$

showing right amenability of $\mathrm{BS}(1,n)^+$. Using again Proposition 1.4.14 we conclude $\mathrm{BS}(1,n)^+$ is right reversible.

In conclusion, the semigroups $\mathrm{BS}(1,n)^+$ are right reversible and amenable, but not left reversible nor amenable. This means there is only one $\mathrm{BS}(1,n)^+$-group containing $\mathrm{BS}(1,n)^+$ as a subsemigroup, namely, $\mathrm{BS}(1,n)$.

Example 1.5.3 (Free semigroups). The free semigroup in two generators $\mathbb{F}_2^+$ is an interesting example. It is bicancellative, since it can be embedded into the free group of rank 2, $\mathbb{F}_2$, but it is not reversible. Indeed, if we write $\mathbb{F}_2^+ = \langle a, b \rangle$, the elements a and b satisfy $a\mathbb{F}_2^+ \cap b\mathbb{F}_2^+ = \varnothing$, since there cannot be two equal words starting with different elements. This shows, by Ore-Dubreil's Theorem 1.3.6, that the group of right fractions of $\mathbb{F}_2^+$ does not exist: for instance, the word $b^{-1}a$ cannot be written as xy^{-1} with $x, y \in \mathbb{F}_2^+$. The same holds for the group of left fractions. Also, since bicancellative amenable semigroups are reversible, $\mathbb{F}_2^+$ cannot be neither left nor right amenable. All of these conclusions extend to $\mathbb{F}_n^+$, $n \ge 2$. Figure 2 shows the right Cayley graph of $\mathbb{F}_2^+$.

The fact that $\mathbb{F}_2^+$ is not reversible, and thus its group of fractions does not exist, can be complemented with the following observation. We already know $\mathbb{F}_2$ is an $\mathbb{F}_2^+$-group (it is the free $\mathbb{F}_2^+$-group actually). The group $\mathrm{BS}(1,2)$ is also a $\mathbb{F}_2^+$-group (for the details, see §2.5), and so $\mathbb{F}_2^+$ admits two non-isomorphic $\mathbb{F}_2^+$-groups, something that cannot happen in the reversible case, as already observed.

Example 1.5.4 (Heisenberg semigroup). The discrete Heisenberg semigroup

$$H(\mathbb{Z})^+ = \langle x, y, z | xz = zx, yz = zy, xy = zyx \rangle$$

is clearly not abelian, but it is reversible. Moreover, $H(\mathbb{Z})^+$ is a subsemigroup of the discrete Heisenberg group $H(\mathbb{Z})$, which has polynomial growth, and thus cannot contain $\mathbb{F}_2^+$ as a subsemigroup. Proposition 1.4.15 implies now that every subsemigroup of $H(\mathbb{Z})$ must be reversible. Since, as we already saw, for subsemigroups of bicancellative amenable semigroups reversibility and amenability become equivalent conditions, we conclude $H(\mathbb{Z})^+$ is both left and right amenable.

Example 1.5.5 (Mal'cev). Our last example of a bicancellative semigroup is one that cannot be embedded into a group, and was given by Mal'cev [Mal37]. Consider the semigroup

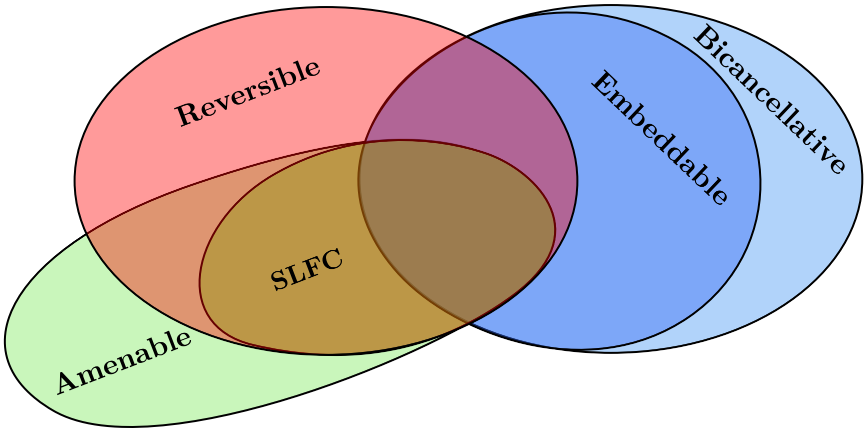
$$M = \langle a, b, c, d, x, y, u, v | ax = by, cx = dy, cu = dv \rangle.$$

Let $A = \{a, b, c, d, x, y, u, v\}$. Given any word $\alpha \in A^*$, we will say β is obtained by elementary transformations from α if it results of switching any subword of the form ax, cx, cu, by, dy, dv in α by its corresponding word following the defining relations.

The interesting thing about this presentation is that if you have a word α, then a subword $\alpha_{i-1}\alpha_i$ admits a switch if and only if $\alpha_i\alpha_{i+1}$ does not. Therefore, if $\alpha\beta = \alpha\gamma$, the switch must be ocurring between β and γ, showing $\beta = \gamma$. Thus, M is left cancellative. A symmetric argument shows right cancellativity.

Now, suppose that M embedds into a group. Directly from the defining relations, we would have $b^{-1}a = yx^{-1}, d^{-1}c = yx^{-1}$ and $d^{-1}c = vu^{-1}$. This would imply $b^{-1}a = vu^{-1}$, from where we get $au = vb$, which is not a valid relation in M. We conclude M is not embeddable into a group. In particular, as M is bicancellative, it cannot be reversible nor amenable on both its left and right sides.

We summarize the theoretical relationships between different semigroup properties through the following diagram.



The following table reunites all of the examples detailed above, giving a concrete monoid in each region of the bicancellative area of the last diagram. We add the group $\mathbb{F}_n$ as an example of a reversible, bicancellative, yet not left nor right amenable semigroup.

Semigroups	Embeddable	Reversible		Amenable		Abelian
		Left	Right	Left	Right	
$\mathbb{N}^n$	✓	✓	✓	✓	✓	✓
$H(\mathbb{Z})^+$	✓	✓	✓	✓	✓	✗
$\mathrm{BS}(1,n)^+$	✓	✗	✓	✗	✓	✗
$\mathbb{F}_n$	✓	✓	✓	✗	✗	✗
$\mathbb{F}_n^+$	✓	✗	✗	✗	✗	✗
Mal'cev	✗	✗	✗	✗	✗	✗

Finally, we include for completeness some examples of non-bicancellative groups.

Example 1.5.6. Consider the natural numbers $\mathbb{N}$ with the operation given by $n \star m = \min\{n, m\}$. It can be easily verified that $(\mathbb{N}, \star)$ is a semigroup, and $n \star n = n$ for every $n \in \mathbb{N}$. Hence, $(\mathbb{N}, \star)$ is an abelian semigroup which has no identity and has a zero element (therefore it is not cancellative).

Example 1.5.7. Consider $\mathbb{N}$ as the semigroup of additive natural numbers, and denote by $\mathbb{N}^\star$ the semigroup $(\mathbb{N}, \star)$ of our last example. The product semigroup $\mathbb{N} \oplus \mathbb{N}^\star$ is clearly not cancellative, although it has no zero element. It is a reversible semigroup, as a direct sum of reversible semigroups.

Example 1.5.8. We want to give an example of a semigroup that is neither cancellative nor reversible. An easy way is to take $\mathbb{F}_2^+$, which is not reversible, and $\mathbb{N}^\star$, which is not cancellative, and take their product, $S = \mathbb{F}_2^+ \oplus \mathbb{N}^\star$. Observe that $(a, 1)S \cap (b, 1)S = \varnothing$.

Chapter 2

Natural extensions of semigroup actions

2.1 General setting

Throughout this chapter S will be a countable monoid, and G will be an S-group containing S as a submonoid, so the morphism $S \to G$ is the inclusion. In fact, whenever we say H is an S-group, we will mean S is a submonoid of H such that S generates H as a group, despite the more general definition considered in Chapter 1.

We want to study extensions of actions of S to actions of G, in a sense which will be rigorously defined later. We will understand an **action** α of S over a set X, denoted as $S \overset{\alpha}{\curvearrowright} X$, by a function $\alpha\colon S \times X \to X$ such that $\alpha(1_S, x) = x$ and $\alpha(s, \alpha(t, x)) = \alpha(st, x)$ for all $x \in X$ and $s, t \in S$. Most of the time $\alpha(s, x)$ will be written simply as $s \cdot x$, and the function $\alpha(s, \cdot)\colon X \to X$ as simply α_s or s. An action $S \curvearrowright X$ is said to be **surjective** if for every $s \in S$ the function α_s is surjective. A morphism between two actions $S \curvearrowright X$ and $S \curvearrowright Y$, where X and Y are merely sets, will be an **S-equivariant** function $\varphi\colon X \to Y$, i.e., one such that $\varphi(s \cdot x) = s \cdot \varphi(x)$ for all $x \in X$ and $s \in S$. A function φ satisfying this last condition will be also called S-equivariant if G acts upon X instead of S. When we add extra structure, we will require that these morphisms preserve said structure. The two kinds of structure considered will be the following.

(i) **Polish spaces**, i.e., topological spaces which are separable and completely metrizable. An action $S \curvearrowright X$ over a Polish space is said to be **continuous** if α_s is continuous for each $s \in S$. We will denote by S-**Top** the category consisting of continuous actions of S over Polish topological spaces as objects, and S-equivariant continuous functions between the underlying topological spaces as morphisms.

(ii) **Standard probability spaces**, i.e., probability spaces such that the underlying measurable space is a Polish space with its Borel σ-algebra. For a given Polish space X, we will denote its Borel σ-algebra by $\mathscr{B}(X)$. A Borel-measurable map

$\varphi\colon (X,\mu) \to (Y,\nu)$ between standard probability spaces is **measure preserving** if $\mu(\varphi^{-1}A) = \nu(A)$ for all $A \in \mathscr{B}(Y)$. In the context where $X = Y$ and $\mu = \nu$, this is also referred to as μ being φ-invariant. An action $S \curvearrowright (X,\mu)$ is **probability measure preserving** (p.m.p.) if (X,μ) is a standard probability space, and for every $s \in S$, α_s is continuous and measure-preserving. Alternatively, such an action can be seen as an action $S \curvearrowright X$ in S-**Top** together with an S-invariant measure μ defined on $\mathscr{B}(X)$. A morphism between p.m.p. actions will be a measure-preserving continuous S-equivariant function between the underlying probability spaces.

Although the usage of terminology from Category Theory will be mild throughout this thesis, we refer the reader to [Awo10] for an introduction to the area.

Definition 2.1.1. We define the notion of a factor map in both contexts.

(i) Let $S \overset{\alpha}{\curvearrowright} X, \overset{\beta}{\curvearrowright} Y$ be two actions in S-**Top**. A surjective morphism $\varphi\colon X \to Y$ will be called a **topological S-factor**, or simply a **factor** if the structure is implicit. In this case, β is said to be a **factor** of α, and α to be an **extension** of β. If the factor φ is an homeomorphism, we will speak of a **topological S-conjugacy** and say both actions are **conjugate** to each other.

(ii) Let $S \overset{\alpha}{\curvearrowright} (X,\mu), \overset{\beta}{\curvearrowright} (Y,\nu)$ be two p.m.p. actions. Any morphism $\varphi\colon (X,\mu) \to (Y,\nu)$ will be considered a **measure-theoretical S-factor**. In this case, β will be called a **factor** of α, and the latter an **extension** of the former.

The reason to not request surjectivity in the measure-theoretical context is that measure preserving mappings are, essentially, surjective. Indeed, if $\varphi\colon (X,\mu) \to (Y,\nu)$ is measure preserving and $\varphi(X)$ is a measurable subset of Y (which is the case if (Y,ν) is complete, for instance), then

$$\nu(\varphi(X)) = \mu(\varphi^{-1}(\varphi(X))) = \mu(X) = 1,$$

so $\varphi(X)$ has full measure.

An S-**invariant subset** $A \subseteq X$ for an action $S \overset{\alpha}{\curvearrowright} X$ is a set such that $sA := \{s \cdot a : a \in A\} \subseteq A$ for all $s \in S$. In this case it makes sense to consider the action $S \curvearrowright A$ given by restricting $\alpha|_{S\times A}\colon S \times A \to A$, called the **subaction of S upon A**. If α is an action in S-**Top**, the additional requirement of A to be closed is needed to make sure the restricted action belongs to S-**Top** as well.

2.2 Natural extensions: the classical construction

Natural extensions are a fundamental construction in ergodic theory of $\mathbb{N}$-actions (see, e.g., [Pet89, §1.3, G]). We follow O. Sarig's lecture notes [Sar09] for a brief exposition of this construction. The reader may check [Lac95] for a generalization of this to actions of finitely generated Abelian groups.

Given a measure preserving $\mathbb{N}$-action $\mathbb{N} \overset{\alpha}{\curvearrowright} (X,\mu)$ determined by $T\colon X \to X$, its natural extension is the $\mathbb{Z}$-action $\mathbb{Z} \curvearrowright (\overline{X},\overline{\mu})$, where

(i) The set $\overline{X}$ is defined as

$$\overline{X} = \Big\{(x_i)_{i\in\mathbb{Z}} \in X^{\mathbb{Z}} : T(x_i) = x_{i+1} \text{ for all } i \in \mathbb{Z}\Big\}.$$

(ii) The σ-algebra $\overline{\mathscr{B}}$ is the one induced by $\overline{X}$ in $X^{\mathbb{Z}}$, i.e., the smallest containing the sets

$$[A;n] := \Big\{(x_i)_{i\in\mathbb{Z}} \in \overline{X} : x_n \in A\Big\}$$

for every $n \in \mathbb{Z}$ and $A \in \mathscr{B}$.

(iii) The measure $\overline{\mu}$ satisfies $\overline{\mu}\Big([A;n]\Big) = \mu(A)$ for every $n \in \mathbb{Z}$ and $A \in \mathscr{B}$.

(iv) The action of $\mathbb{Z}$ upon $\overline{X}$ is the shift action, i.e., $1 \cdot (x_i)_{i\in\mathbb{Z}} = (x_{i+1})_{i\in\mathbb{Z}}$.

In [Sar09] is proven that, if T is surjective, the measure $\overline{\mu}$ exists.

Theorem 2.2.1 ([Sar09, Theorem 1.6]). *Let $(X,\mathscr{B},\mu,T)$ be a measure preserving system with $T(X) = X$. Then, $\overline{\mu}$ exists and it is unique.*

Moreover, the natural extension is a $\mathbb{Z}$-factor of any other $\mathbb{Z}$-extension of $\mathbb{N} \overset{\alpha}{\curvearrowright} (X,\mu)$, and we have the following (for terminology, refer to Appendix C).

(i) α is ergodic if, and only if, $\mathbb{Z} \curvearrowright (\overline{X},\overline{\mu})$ is ergodic,

(ii) α is mixing if, and only if, $\mathbb{Z} \curvearrowright (\overline{X},\overline{\mu})$ is mixing.

2.3 Topological extensions

The generality in which the constructions of §2.2 are framed is far from being enough for our purposes, as we want to focus on non-Abelian embeddable semigroups, particularly on free semigroups of finite rank. The following definitions are fundamental for this chapter.

Definition 2.3.1. Let $S \overset{\alpha}{\curvearrowright} X$ be an action in S-**Top**. A pair (β,π), where β is an action $G \curvearrowright Y$ in G-**Top** and $\pi\colon Y \to X$ is a surjective S-equivariant continuous function will be called a G-**extension** of α. Given two G-extensions (β,π) and (β',π') of α, a morphism of actions $\varphi\colon \beta \to \beta'$ (i.e., a continuous G-equivariant function of the underlying spaces) is a **morphism of G-extensions** if $\pi' \circ \varphi = \pi$.

Definition 2.3.2. A **topological natural G-extension** of an action $S \overset{\alpha}{\curvearrowright} X$ in S-**Top** is a terminal object in the category of G-extensions. More concretely, it is a pair $(\widehat{\alpha}, \pi)$ where $\widehat{\alpha}$ is an action $G \curvearrowright \widehat{X}$ in G-**Top** and $\pi\colon \widehat{X} \to X$ a surjective S-equivariant continuous function, such that for every action $G \curvearrowright Y$ in G-**Top** and surjective S-equivariant continuous function $\tau\colon Y \to X$, there is a unique G-equivariant continuous function $\varphi\colon Y \to \widehat{X}$ satisfying $\pi \circ \varphi = \tau$.

$$\begin{tikzcd}
Y \arrow[r, dashed, "\varphi"] \arrow[dr, "\tau"'] & \widehat{X} \arrow[d, "\pi"] \\
& X
\end{tikzcd}$$

If such an object exists, it must be unique up to an isomorphism of G-extensions, since it is defined as a terminal object. Most of the time we will make an abuse of language and refer to the topological natural G-extension $(\widehat{\alpha}, \pi)$, with $\widehat{\alpha}$ being $G \curvearrowright \widehat{X}$, as $(\widehat{X}, \pi)$, $\pi\colon \widehat{X} \to X$ or simply as $\widehat{X}$, where in this last case the extension map π is implicit.

Finding a realization of the universal G-extension of an action $S \overset{\alpha}{\curvearrowright} X$ in S-**Top** is fairly straightforward, whenever it exists. Consider the set

$$X_G = \left\{ (x_g)_{g \in G} \in \prod_{g \in G} X : s \cdot x_g = x_{sg} \text{ for all } s \in S \text{ and } g \in G \right\},$$

endowed with the subspace topology of the product topology. For every $g \in G$, $\pi_g\colon X^G \to X$ will denote the projection on the coordinate corresponding to g. Note that these functions are continuous, and X_G can be seen to be closed in the following manner:

$$X_G = \bigcap_{g \in G, s \in S} \left\{ \bar{x} \in X^G : s \cdot \pi_g(\bar{x}) = \pi_{sg}(\bar{x}) \right\},$$

and $\{\bar{x} \in X^G : s \cdot \pi_g(\bar{x}) = \pi_{sg}(\bar{x})\}$ is the subset of X^G where the (continuous) functions $s \circ \pi_g$ and π_{sg} coincide, hence closed. Therefore, X_G is closed in X^G, and since G is countable we have X^G is Polish, so X_G is also Polish, i.e., $(X_G, \mathscr{B}(X_G))$ is a standard Borel space. To see it as an element in G-**Top** we let G act upon X_G by shifting on the right, that is, for every $h \in G$ and $(x_g)_{g \in G} \in X_G$,

$$h \cdot (x_g)_{g \in G} = (x_{gh})_{g \in G}.$$

This action is well defined, continuous, and will be denoted by α_G.

The function $\pi\colon X_G \to X$ will be the projection π_{1_G} on the coordinate of the identity 1_G, and will be of special importance for us. The function π is continuous, and for all $s \in S$ and $(x_g)_{g \in G} \in X_G$ we have

$$\pi(s \cdot (x_g)_{g \in G}) = x_s = s \cdot x_{1_G} = s \cdot \pi((x_g)_{g \in G}).$$

The following lemma will be really helpful throughout this chapter.

Lemma 2.3.3. *Let* $(X_\alpha)_{\alpha\in A}$ *be a family of metric spaces,* $\pi_\beta\colon \prod_{\alpha\in A} X_\alpha \to X_\beta$ *the coordinate projection for each* $\beta \in A$ *and* $F \subseteq \prod_{\alpha\in A} X_\alpha$ *any subset. If* Y *is a metric space, a function* $\varphi\colon Y \to F$ *is continuous if and only if* $(\pi_\alpha)|_F \circ \varphi$ *is continuous for every* $\alpha \in A$*. In particular, a function* $\varphi\colon Y \to X_G$ *is continuous if and only if* $(\pi_g)|_{X_G} \circ \varphi$ *is continuous for every* $g \in G$*, where* $\pi_g(\overline{x}) = \overline{x}(g) \in X$*.*

Proof. Define $\overline{\varphi}\colon Y \to \prod_{\alpha\in A} X_\alpha$ by $\overline{\varphi}(y) = \varphi(y)$. Since the topology in $\prod_{\alpha\in A} X_\alpha$ is the product topology, $\overline{\varphi}$ is continuous if and only if $\pi_\alpha \circ \overline{\varphi}$ is continuous for every $\alpha \in A$. Now, if φ is continuous, then so is $\overline{\varphi}$, hence if we have a convergent sequence in Y, $y_n \to y$, then

$$(\pi_\alpha)_F \circ \varphi(y_n) = \pi_\alpha \circ \overline{\varphi}(y_n) \to \pi_\alpha \circ \overline{\varphi}(y) = (\pi_\alpha)_F \circ \varphi(y).$$

Therefore $(\pi_\alpha)|_F \circ \varphi$ is continuous for every $\alpha \in A$. Conversely, if this last statement holds for every $\alpha \in A$ and $y_n \to y$, we have

$$\pi_\alpha \circ \overline{\varphi}(y_n) = (\pi_\alpha)|_F \circ \varphi(y_n) \to (\pi_\alpha)|_F \circ \varphi(y) = \pi_\alpha \circ \overline{\varphi}(y),$$

so $\pi_\alpha \circ \overline{\varphi}$ is continuous for all $\alpha \in A$, implying $\overline{\varphi}$ is continuous. Now let $C \subseteq F$ be closed, i.e., $C = F \cap C'$ with $C' \subseteq \prod_{\alpha\in A} X_\alpha$ closed. Then $\varphi^{-1}(C) = \overline{\varphi}^{-1}(C')$ is closed and we conclude φ is continuous. ☐

Thus far, we do not know whether X_G is non-empty or not.

Proposition 2.3.4. *Let* $S \overset{\alpha}{\curvearrowright} X$ *be an action in* S-**Top***. The following statements are equivalent.*

- (i) *The map* $\pi\colon X_G \to X$ *is surjective, and in particular* $X_G \neq \varnothing$*.*
- (ii) *The topological natural* G*-extension of* α *exists.*
- (iii) *There exists a surjective* S*-equivariant continuous function from an action in* G-**Top** *to* α*.*

Whenever one of these conditions holds, the topological natural G*-extension of* α *is isomorphic to* (X_G, π)*.*

Proof. Assume (i). We will prove that (X_G, π) is the topological natural G-extension of α, hence showing (ii) and the final statement of the proposition at once. Let $G \curvearrowright Y$ be an action in G-**Top** and $\pi'\colon Y \to X$ a surjective S-equivariant continuous function. We need to show there is a unique G-equivariant continuous function $\varphi\colon Y \to X_G$ with $\pi \circ \varphi = \pi'$. Note that, if the map φ exists, by G-equivariance and the fact that $\pi \circ \varphi = \pi'$ it must satisfy for each $y \in Y$ and $g \in G$ the following:

$$\varphi(y)(g) = (g \cdot \varphi(y))(1_G) = \pi(\varphi(g \cdot y)) = \pi'(g \cdot y).$$

Defining φ by this condition shows uniqueness, and that $\pi_g \circ \varphi = \pi' \circ \alpha_g$ is continuous for every $g \in G$, so φ is continuous by Lemma 2.3.3. Now, φ is G-equivariant: by definition of the shift action we have, for all $g \in G$,

$$\varphi(t \cdot y)(g) = \pi'(gt \cdot y) = \varphi(y)(gt) = (t \cdot \varphi(y))(g).$$

We conclude that (X_G, π) is the toplogical natural G-extension.

That (ii) implies (iii) is clear. Now let $G \curvearrowright Y$ be an action in G-**Top** and $\pi' \colon Y \to X$ a surjective S-equivariant continuous function. The same map φ constructed when proving that (i) implied (ii) allows us to conclude that for a given $x \in X$, $\varphi(y)$ is an element of X_G with $\pi(\varphi(y)) = x$, where $y \in Y$ is any element satisfying $\pi'(y) = x$. $\square$

By virtue of this last proposition, we will be henceforth interested in studying (X_G, π), which motivates the following definitions.

Definition 2.3.5. Let $S \overset{\alpha}{\curvearrowright} X$ be an action in S-**Top**.

(i) If $\pi \colon X_G \to X$ is surjective, α will be said to be **topologically G-extensible** and, as we proved, X_G will be the topological natural G-extension.

(ii) If $X_G \neq \varnothing$, we will say α is **topologically partially G-extensible**, and (X_G, π) will be called the **topological partial natural G-extension** of α.

Most of the time, if there is not risk of confusion with the (yet to introduce) measure-theoretical version of G-extensibility, a (partially) topologically G-extensible action will be referred to as simply (partially) G-extensible.

Note that, for a partially G-extensible action $S \overset{\alpha}{\curvearrowright} X$, the subset $\pi(X_G) \subseteq X$ is S-invariant, as π is S-equivariant, and so it makes sense to think of it as a subaction of $S \curvearrowright X$. Moreover, if $\pi(X_G)$ is closed (e.g., in the compact case), then $S \curvearrowright \pi(X_G)$ is an action in S-**Top**. Now, since $\pi(X_G) \subseteq X$, clearly the extension $\big(\pi(X_G)\big)_G$ is a subset of X_G. On the other hand, if $\overline{x} = (x_g)_{g \in G} \in X_G$, then for every $h \in G$, $h \cdot \overline{x} \in X_G$, so $x_h = \pi(h \cdot \overline{x}) \in \pi(X_G)$, which implies $X_G \subseteq \big(\pi(X_G)\big)_G$. Therefore, every topologically partially G-extensible action has a subaction which is topologically G-extensible.

Example 2.3.6. Some basic examples.

(i) **Invertible action**: if $S = G$, then (α_G, π) is isomorphic to (α, id) via π. Indeed, we just need to prove that π is bijective. Given $x \in X$, the image via π of $(g \cdot x)_{g \in G}$ is exactly x. On the other hand, if $\pi(x) = \pi(y)$ then, since $S = G$, $x_g = g \cdot x_1 = g \cdot y_1 = y_g$ for all $g \in G$, so $x = y$.

(ii) **Trivial action**: if S acts trivially upon X, i.e., $s \cdot x = x$ for all $x \in X$ and $s \in S$, then α_G is the trivial action of G upon X.

(iii) **Constant action**: if the action $S \curvearrowright X$ admits an element $x_0 \in X$ with $sx = x_0$ for all $s \in S$ and $x \in X$, then X_G is a singleton. Indeed, the only element with non-empty preimage would be x_0, which forces $X_G = \{(x_g)_{g\in G}\}$ where $x_g = x_0$ for all $g \in G$.

Remark 2.3.7 (G-extension functor). Any continuous S-equivariant map $\varphi\colon X \to Y$ of partially G-extensible S-actions induces a unique continuous G-equivariant map $\varphi_G\colon X_G \to Y_G$ such that $\pi' \circ \varphi_G = \varphi \circ \pi$. Said function is defined by $\varphi_G((x_g)_{g\in G}) = (\varphi(x_g))_{g\in G}$, which is continuous by Lemma 2.3.3.

$$\begin{tikzcd}
X_G \arrow[r, "\varphi_G"] \arrow[d, "\pi"'] & Y_G \arrow[d, "\pi'"] \\
X \arrow[r, "\varphi"'] & Y
\end{tikzcd}$$

The construction is functorial: if $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ are S-equivariant maps, and π_X, π_Y, π_Z are the respective G-extension maps, then $(\psi \circ \varphi)_G = \psi_G \circ \varphi_G$, since the latter is G-equivariant and satisfies $\pi_Z \circ (\psi_G \circ \varphi_G) = (\psi \circ \varphi) \circ \pi_X$.

In the case $\varphi\colon X \to Y$ is an S-factor, the function φ_G *need not* be a factor. Indeed, if $X = X' \sqcup \{x_0\}$, where X' is closed, S-invariant, and $S \curvearrowright X'$ satisfies $X'_G = \varnothing$ (i.e., X' is not partially G-extensible: these kind of actions exist, as we shall see in §2.5), and $s \cdot x_0 = x_0$ for all $s \in S$. Then, if $(x_g)_{g\in G} \in X_G$ and $x_t = x_0$ for some $t \in G$, we must have $x_{ts^{-1}} = x$ for all $s \in S$, which implies $x_g = x$ for all $g \in G$. Therefore, $|X_G| = 1$. Now, consider the action $S \curvearrowright Y$, where $Y = \{y\} \sqcup \{x_0\}$ and S acts trivially upon both elements. An S-factor map $\varphi\colon X \to Y$ can be defined by setting $\varphi(X') = \{y\}$ and $\varphi(x_0) = x_0$, but there cannot exist a surjective function $X_G \to Y_G$, since $|Y_G| = 2$.

Proposition 2.3.8. *Let $S \curvearrowright X$ and $S \curvearrowright Y$ be two actions in S-**Top**. If S is reversible, G is the group of right fractions of S and $\varphi\colon X \to Y$ is a factor with compact fibers (i.e., such that $\varphi^{-1}(y)$ is compact for every $y \in Y$), then φ_G is a factor.*

Proof. Fix $\overline{y} = (y_g)_{g\in G} \in Y_G$ and a sequence $(g_k)_{k\in\mathbb{N}}$ in G such that for every $g \in G$, $g \in Sg_k$ for some $k \in \mathbb{N}$ (this can be done by virtue of Lemma 2.7.1). Define $\overline{x}^k = (x^k_g)_{g\in G} \in X_G$ by choosing an arbitrary element $x^k_g \in \varphi^{-1}(y_g)$ for $g \in G - (S - \{1_S\})g_k$, and $x^k_g = s \cdot x^k_{g_k}$ if $g = sg_k$ with $s \in S - \{1_S\}$. Since φ has compact fibers, $(\overline{x}^k)_{k\geq 1}$ is a sequence in a compact set, namely,

$$\prod_{g\in G} \varphi^{-1}(y_g).$$

Letting $\overline{x} \in X^G$ be a limit point for this sequence, we see that

$$\varphi_G(\overline{x}) = \lim_{k\to\infty} \varphi_G(\overline{x}^k) = \overline{y},$$

and that $\overline{x} \in X_G$ by construction. $\square$

We point out now some stability properties of extensibility. An **inverse system** in S-$\textsf{Top}$ will be a pair $\big((X^i)_{i\in I}, (f_{ij})_{i\leq j}\big)$, where I is a directed set, each $S \curvearrowright X^i$ belongs to S-$\textsf{Top}$, each $f_{ij}\colon X^j \to X^i$ is continuous and S-equivariant function, $f_{ii} = \mathrm{id}_{X^i}$ and $f_{ik} = f_{ij} \circ f_{jk}$ whenever $i \leq j \leq k$.

Proposition 2.3.9 (Stability properties of extensions). *The following properties hold.*

(i) ***(Factors)*** *If $S \overset{\alpha}{\curvearrowright} X$ and $S \overset{\beta}{\curvearrowright} Y$ are elements in S-$\textsf{Top}$, α is G-extensible (resp. partially G-extensible) and $\varphi\colon X \to Y$ is a factor map (resp. S-equivariant map), then β is G-extensible (resp. partially G-extensible).*

(ii) ***(Inverse limits)*** *If $\big((X^i)_{i\in I}, (f_{ij})_{i\leq j}\big)$ is an inverse system in S-$\textsf{Top}$ and each X^i is G-extensible and compact, then so is its inverse limit, $\varprojlim X^i$, and*

$$\Big(\varprojlim X^i\Big)_G = \varprojlim (X^i)_G,$$

where the inverse limit of extensions is with respect to $((f_{ij})_G)_{i\leq j}$.

Proof. To prove (i), let α be a G-extensible action, so that $\pi_\alpha\colon X_G \to X$ is surjective. Then, by surjectivity of both π_α and φ, we know for all $y \in Y$ there is an element $\overline{x} \in \pi_\alpha^{-1}(\varphi^{-1}(y))$, and

$$\pi_\beta(\varphi_G(\overline{x})) = \varphi \circ \pi(\overline{x}) = y.$$

Therefore, π_β is surjective, and we conclude β is G-extensible. If α is only partially G-extensible, then X_G is non-empty, so there is a $\overline{x} \in X_G$. The element $\varphi_G(\overline{x})$ belongs to Y_G, which is hence non-empty. Therefore, β is partially G-extensible.

Now we prove (ii). Note that the spaces $((X^i)_G)_{i\in I}$ and the functions $\big((f_{ij})_G)_{i\leq j}\big)$ form an inverse system, since if $i \leq j \leq k$ in I, then $(f_{ij})_G \circ (f_{jk})_G = (f_{ik})_G$. Let p_k and $\overline{p}_k$ be the k-th coordinate projections on $\varprojlim X^k$ and $\varprojlim (X^k)_G$, respectively, and let $\pi_k\colon (X^k)_G \to X^k$ be the topological natural G-extension for each $k \in I$. By the definition of the maps $(f_{ij})_G$, and of the inverse limits, for each $i \leq j$ the following diagram commutes.

$$\begin{tikzcd}
 & (X^i)_G \arrow[r, "\pi_i"] & X^i & \\
\varprojlim (X^k)_G \arrow[ur, "\overline{p}_i"] \arrow[dr, "\overline{p}_j"'] & & & \varprojlim X^k \arrow[ul, "p_i"'] \arrow[dl, "p_j"] \\
 & (X^j)_G \arrow[uu, "(f_{ij})_G"] \arrow[r, "\pi_j"'] & X^j \arrow[uu, "f_{ij}"'] &
\end{tikzcd}$$

Step 1. We need to see $\varprojlim X^i$ as an action in G-$\textsf{Top}$ and specify a surjective S-equivariant continuous function $\pi\colon \varprojlim (X^i)_G \to \varprojlim X^i$. Define the actions $S \curvearrowright \varprojlim X^i$

and $G \curvearrowright \varprojlim(X^i)_G$ coordinate-wise. The extension map $\pi\colon \varprojlim(X^i)_G \to \varprojlim X^i$ will be

$$\pi\Big((\overline{x}_k)_{k\in I}\Big) = \Big(\pi_k(\overline{x}_k)\Big)_{k\in I} = (\overline{x}_k(1_G))_{k\in I}.$$

That π is well-defined follows from the fact that $f_{ij} \circ \pi_j = \pi_i \circ (f_{ij})_G$, and that $(f_{ij})_G(\overline{x}_j) = \overline{x}_i$ if $(\overline{x}_k)_{k\in I}$ belongs to $\varprojlim(X^k)_G$. The S-equivariance of π is straightforward, as each π_k is S-equivariant. The continuity of π follows from applying Lemma 2.3.3, since it is a restriction of the map

$$\begin{aligned} \overline{\pi}\colon \prod_{k\in I}(X^k)_G &\longrightarrow \prod_{k\in I} X^k \\ (\overline{x}_k)_{k\in I} &\longmapsto (\overline{x}_k(1_G))_{k\in I}, \end{aligned}$$

which is continuous and surjective as each π_k is continuous and surjective, to closed subsets of both its domain and codomain. We need to prove that π is surjective. For each $n \in I$ define the sets

$$Y^n = \left\{(x_k)_{k\in I} \in \prod_{k\in I} X^k : f_{ij}(x_j) = x_i \text{ for all } i \le j \le n\right\},$$

$$Y^n_G = \left\{(\overline{x}_k)_{k\in I} \in \prod_{k\in I} (X^k)_G : (f_{ij})_G(\overline{x}_j) = \overline{x}_i \text{ for all } i \le j \le n\right\}.$$

Clearly, the sets Y^n and Y^n_G are compact, and non-empty (as a consequence of the surjectivity of π_n, in the case of Y^n_G). Moreover, if $m \ge n$ are elements in I, then $Y^m \subseteq Y^n$ and $Y^m_G \subseteq Y^n_G$, so the sets

$$\varprojlim X^k = \bigcap_{n\in I} Y^n \quad \text{and} \quad \varprojlim(X^k)_G = \bigcap_{n\in I} Y^n_G$$

are non-empty. Note that $\overline{\pi}(Y^n_G) = Y^n$: given $(x_k)_{k\in I} \in Y^n$, a preimage is given by choosing $\overline{x}_k \in \pi_k^{-1}(x_k)$ for $k \ge n$, and $\overline{x}_k = (f_{nk})_G(\overline{x}_n)$ for $k < n$. Now, for any

$$x = (x_k)_{k\in I} \in \varprojlim X^k = \bigcap_{n\in I} \overline{\pi}(Y^n_G)$$

and $n \in I$, let $A = \overline{\pi}^{-1}(x)$ (which is non-empty, as $\overline{\pi}$ is surjective) and $A_n = A \cap Y^n_G$. Note that for every $n \in I$, $x \in \overline{\pi}(Y^n_G)$ and so there is an $y \in Y^n_G$ with $\overline{\pi}(y) = x$. In particular, $y \in A$, so $A \cap Y^n_G$ is non-empty. Since A is closed, $(A_n)_{n\in I}$ is a decreasing family of non-empty compact sets, hence has non-empty intersection. An element $y \in \bigcap_{n\in I} A \cap Y^n_G$ satisfies both $y \in \varprojlim(X^k)_G$ and $\pi(y) = \overline{\pi}(y) = x$, so $x \in \pi\left(\varprojlim(X^k)_G\right)$. Thus,

$$\pi\Big(\varprojlim(X^k)_G\Big) \supseteq \bigcap_{n\in I} \overline{\pi}(Y^n_G) = \bigcap_{n\in I} Y^n = \varprojlim X^k,$$

and we conclude π is surjective.

Step 2. We need to check the universal property of the natural G-extension. Let $G \curvearrowright Y$ be any action in G-**Top** and $\pi' \colon Y \to \varprojlim X^k$ an S-factor, and define a function $\varphi \colon Y \to \varprojlim (X^k)_G$ by

$$\overline{p}_i(\varphi(y))(g) = p_i(\pi'(g \cdot y)).$$

The map φ is well defined, since if $i < j$ are elements of I, $g \in G$ and $y \in Y$, we have $p_i(\pi'(g \cdot y)) = f_{ij}(p_j(\pi'(g \cdot y)))$, so we get

$$\overline{p}_i(\varphi(y))(g) = p_i(\pi'(g \cdot y)) = f_{ij}(p_j(\pi'(g \cdot y))) = f_{i,j}(\overline{p}_j(\varphi(y))(g)) = (f_{ij})_*(\overline{p}_j(\varphi(y)))(g).$$

Therefore, $\overline{p}_i(\varphi(y)) = (f_{ij})_*(\overline{p}_j(\varphi(y)))$ and we conclude $\varphi(y) \in \varprojlim (X^k)_G$.

The function φ is G-equivariant: if $t, g \in G$ and $y \in Y$ then

$$\overline{p}_k(g \cdot \varphi(y))(t) = (g \cdot \overline{p}_k(\varphi(y)))(t) = \overline{p}_k(\varphi(y))(tg) = p_k(\pi'(tg \cdot y)) = \overline{p}_k(\varphi(g \cdot y))(t).$$

That $\pi \circ \varphi = \pi'$ is by definition: $p_k(\pi \circ \varphi(y)) = \overline{p}_k(\varphi(y))(1_G) = p_k(\pi'(y))$. Also, the defining formula of φ is necessary from the requirements of G-invariance and $\pi \circ \varphi = \pi'$, showing uniqueness.

It just remains to prove continuity of φ. Let $y \in Y$ and $U \subseteq \varprojlim (X^k)_G$ be an open subset containing $\varphi(y)$. It suffices to consider U of the form

$$U = \bigcap_{j \in J, g \in F_j} g \cdot \overline{p}_j^{-1}(\pi_j^{-1}(U_{j,g})),$$

where $J \subseteq I$ and $F_j \subseteq G$ are finite, and the sets $U_{j,g} \subseteq X^j$ are open. If $j \in J$ and $g \in F_j$, since $\varphi(g^{-1} \cdot y) \in \overline{p}_j^{-1}(\pi_j^{-1}(U_{j,g}))$, we have

$$\pi'(g^{-1} \cdot y) = \pi \circ \varphi(g^{-1} \cdot y) \in U_{j,g}.$$

Thus, by continuity of π' there is an open neighborhood $V_{j,g}$ of $g^{-1} \cdot y$ such that $\pi'(V_{j,g}) \subseteq U_{j,g}$, which means $\pi_j \circ \overline{p}_j \circ \varphi(V_{j,g}) = \pi \circ \varphi(V_{j,g}) = \pi'(V_{j,g}) \subseteq U_{j,g}$. Therefore,

$$\varphi(g \cdot V_{j,g}) = g \cdot \varphi(V_{j,g}) \subseteq g \cdot \overline{p}_j^{-1}(\pi_j^{-1}(U_{j,g})).$$

Each $g \cdot V_{j,g}$ contains y, so taking the intersection of all these sets yields an open neighborhood V_J of y with $\varphi(V_J) \subseteq U$.

□

2.4 Extensions of symbolic actions

Symbolic actions will be of a crucial importance to us throughout this work. Given a semigroup S and any set $\mathcal{A}$, which will be called the **alphabet** and assumed finite of cardinality $|\mathcal{A}| \geq 2$ unless explicitly told otherwise, we define the **space of configurations** $\mathcal{A}^S$ of all functions $x \colon S \to \mathcal{A}$. This space will be endowed with the **prodiscrete**

topology, i.e., we consider the discrete topology upon $\mathcal{A}$ and then the product topology on $\mathcal{A}^S$, which is the smallest topology making all the coordinate projections $\pi_s\colon \mathcal{A}^S \to \mathcal{A}$ continuous. Note that in this space the convergence is characterized by

$$x_n \to x \iff x_n(s) \to x(s) \text{ for all } s \in S.$$

There is a natural action $S \curvearrowright \mathcal{A}^S$ given by $(s \cdot x)(t) = x(ts)$ for all $s, t \in S$. It is standard that $\mathcal{A}^S$ is a Cantor space, i.e., it is metrizable, compact, totally disconnected and without isolated points, and that the action $S \curvearrowright \mathcal{A}^S$ is continuous. The space of configurations $\mathcal{A}^S$ together with the action $S \curvearrowright \mathcal{A}^S$ is called the **full** S**-shift**, and a closed S-invariant subset $X \subseteq \mathcal{A}^S$ will be called a **subshift**.

We will now introduce some notation that will be carried on through this and the following chapters. If F, K are disjoint subsets of S, $x \in \mathcal{A}^F$ and $y \in \mathcal{A}^K$, we denote by $x \wedge y$ the configuration in $\mathcal{A}^{F \sqcup K}$ such that $(x \wedge y)|_F = x$ and $(x \wedge y)|_K = y$. Given a configuration $x \in \mathcal{A}^S$ and a finite subset $F \subseteq S$, we will write

$$[x\,; F] = \left\{ x' \in \mathcal{A}^S : x'|_F = x|_F \right\}.$$

Such a set will be called a **cylinder**, and the collection of all cylinders form a basis of clopen sets that generate the topology of $\mathcal{A}^S$. If $y \in \mathcal{A}^F$, i.e., y is a function $F \to \mathcal{A}$, the set $\{x' \in \mathcal{A}^S : x'|_F = y\}$ will be denoted by $[y\,; F]$. This will not cause any confusion, as we will always be specifying whether we are dealing with a finite or a full configuration. Finally, if we just want to explicitly list the finite configuration, we will write

$$[k_1, \ldots, k_n; \{t_1, \ldots, t_n\}] = \left\{ x \in \mathcal{A}^S : x(t_i) = k_i \text{ for all } 1 \leq i \leq n \right\}.$$

We have the following useful properties of translations of cylinders.

Proposition 2.4.1. *Let $s \in S$, $F \subseteq S$ a finite subset and $x \in \mathcal{A}^S$. Then,*

(i) *$s[x\,; F] = [s \cdot x; Fs^{-1}]$, where $Fs^{-1} = \{t \in S : ts \in F\}$, and*

(ii) *$s^{-1}[x\,; F] = [z\,; Fs]$, where $z \in \mathcal{A}^S$ is any element satisfying $s \cdot z = x$.*

Proof. For the first statement, let $y = s \cdot z$, with $z \in [x\,; F]$. Then, if $ts \in F$, we have

$$(s \cdot x)(t) = x(ts) = z(ts) = (s \cdot z)(t) = y(t),$$

so $y \in [s \cdot x; Fs^{-1}]$. For the other inclusion, if $y|_{Fs^{-1}} = s \cdot x$, define z as follows. If $t \in Ss$, there is a unique $t' \in S$ with $t's = t$, so define $z(t) = y(t')$. If $t \in S - Ss$, define $z(t) = x(t)$. First note that $(s \cdot z)(t) = z(ts) = y(t)$ by definition, so $s \cdot z = y$. Now, to see z belongs to $[x\,; F]$, it just remains to show $z(t) = x(t)$ for any $t \in F \cap Ss$. Indeed, if $t = t's \in F$, then $t' \in Fs^{-1}$ and

$$z(t) = z(t's) = y(t') = (s \cdot x)(t') = x(t).$$

To prove the second statement, let z be such that $s \cdot z = x$ and $y \in s^{-1}[x\,;F]$. If $t \in F$, then $y(ts) = (s \cdot y)(t) = x(t) = (s \cdot z)(t) = z(ts)$, showing the first inclusion. For the reverse inclusion, if $y \in [z\,;Fs]$ then $(s \cdot y)(t) = y(ts) = z(ts) = (s \cdot z)(t) = x(t)$, thus $s \cdot y \in [x\,;F]$. ☐

Remark 2.4.2. Let us briefly discuss the G-extensibility of $\mathcal{A}^S$ and what $(\mathcal{A}^S)_G$ is. Let $\rho\colon \mathcal{A}^G \to \mathcal{A}^S$ be the restriction to S, so that $(\mathcal{A}^G, \rho)$ is a G-extension of $\mathcal{A}^S$. We will identify this G-extension with the topological natural G-extension by exhibiting an isomorphism of extensions, which will be useful later.

Define $\varphi\colon (\mathcal{A}^S)_G \to \mathcal{A}^G$ as the function sending an element $\overline{x} = (\overline{x}_g)_{g \in G} \in (\mathcal{A}^S)_G$ to the configuration given by $\varphi(\overline{x})(g) = \overline{x}_g(1_S)$ for all $g \in G$. The inverse of φ is easily found to be given, on each coordinate $g \in G$, by

$$\begin{aligned} (\varphi^{-1}(x))_g \colon S &\longrightarrow \mathcal{A} \\ s &\longmapsto x(sg). \end{aligned}$$

This function is well defined, as for all $x \in \mathcal{A}^G$, $s, t \in S$ and $g \in G$,

$$(s \cdot (\varphi^{-1}(x))_g)(t) = x(tsg) = (\varphi^{-1}(x))_{sg}(t),$$

yielding $s \cdot (\varphi^{-1}(x))_g = (\varphi^{-1}(x))_{sg}$, so $\varphi^{-1}(x) \in (\mathcal{A}^S)_G$. Note that $\rho \circ \varphi = \pi$ and $\pi \circ \varphi^{-1} = \rho$. Also,

$$\varphi(h \cdot \overline{x})(g) = (h \cdot \overline{x})_g(1_S) = \overline{x}_{gh}(1_S) = \varphi(\overline{x})(gh) = h \cdot \varphi(\overline{x})(g),$$

so φ is G-equivariant.

It remains to see that φ is an homeomorphism, in order to conclude that it is an isomorphism of G-extensions. Since for each $i \in \mathcal{A}$ and $g \in G$,

$$\varphi^{-1}([i; g]) = \{\overline{x} \in (\mathcal{A}^S)_G : \varphi(\overline{x})(g) = \overline{x}_g(1_S) = \pi(g \cdot \overline{x})(1_S) = i\} = g^{-1}\pi^{-1}([i; 1_S]),$$

the preimage of every cylinder via φ is open by continuity of the action $G \curvearrowright (\mathcal{A}^S)_G$ and of π. Thus φ is continuous. On the other hand, the topology of X_G is generated by the sets of the form

$$\bigcap_{t \in F} t^{-1}\pi^{-1}(A_t),$$

where $F \subseteq G$ is finite and each $A_t \subseteq \mathcal{A}^S$ is open. Now,

$$\varphi\left(\bigcap_{t \in F} t^{-1}\pi^{-1}(A_t)\right) = \bigcap_{t \in F} t^{-1}\varphi(\pi^{-1}(A_t)) = \bigcap_{t \in F} t^{-1}\rho^{-1}(A_t).$$

Since ρ is continuous, we have that φ is open, hence an homeomorphism.

Proposition 2.4.3. *Let $X \subseteq \mathcal{A}^S$ be a subshift, and define $X' \subseteq \mathcal{A}^G$ by*

$$X' = \{x' \in \mathcal{A}^G : (g \cdot x')|_S \in X \text{ for all } g \in G\}.$$

Then, (X_G, π) is isomorphic as a G-extension to $(X', \rho|_{X'})$.

Proof. Since X_G is a G-invariant closed subset of $(\mathcal{A}^S)_G$ and $\varphi\colon (\mathcal{A}^S)_G \to \mathcal{A}^G$ is an isomorphism of G-extensions of $\mathcal{A}^S$, it suffices to show that $\varphi(X_G) = X'$ to conclude $\varphi|_{X_G}\colon X_G \to X'$ is a G-equivariant homeomorphism with $\rho|_{X'} \circ \varphi|_{X_G} = \pi|_{X_G}$. Let $x' \in X'$ and define $\bar{x}(g) = (g \cdot x')|_S \in X$. It is clear that for all $s, t \in S$ and $g \in G$ we have

$$[s \cdot (\bar{x}(g))](t) = (\bar{x}(g))(ts) = (g \cdot x')(ts) = (sg \cdot x')(t) = [x(sg)](t),$$

so $\bar{x} \in X_G$. Thus, since $\varphi(\bar{x}) = x'$, we have proven the first statement. From the fact that $\rho \circ \varphi = \pi$ and φ is bijective, we obtain the second statement. ☐

Theorem 2.4.4. *Let $X \subseteq \mathcal{A}^S$ be a subshift (with $\mathcal{A}$ possibly infinite) such that the action $S \overset{\alpha}{\curvearrowright} X$ is surjective. Then, if G is the free S-group, α is G-extensible.*

Proof. Let $S = \langle B \mid R \rangle$ be a presentation of S, let $\rho\colon \mathbb{F}(B) \to G$ be a surjective morphism with $\rho(B) = B$, and $\rho^+ = \rho|_{\mathbb{F}(B)^+}\colon \mathbb{F}(B)^+ \to S$. Here, $\mathbb{F}(B)$ and $\mathbb{F}(B)^+$ denote the free group and the free semigroup on B, respectively. Note that both ρ and ρ^+ are surjective.

Define $\widehat{X} \subseteq \mathcal{A}^{\mathbb{F}(B)^+}$ as $\widehat{X} := \{x \circ \rho^+ : x \in X\}$. It is a closed set: if $x_n \circ \rho^+ \to \hat{x} \in \mathcal{A}^{\mathbb{F}(B)^+}$, for any $s \in S$ and $\sigma \in \mathbb{F}(B)^+$ with $\rho^+(\sigma) = s$ we have $x_n(s) \to \hat{x}(\sigma)$, so $(x_n)_n$ is Cauchy. By completeness of X, $x_n \to y \in X$, so for any $\sigma \in \mathbb{F}(B)^+$ and $n \geq 1$ large enough we have $\hat{x}(\sigma) = x_n(\rho^+(\sigma)) = y(\rho^+(\sigma))$, implying $\hat{x} = y \circ \rho^+ \in \widehat{X}$. The set $\widehat{X}$ is $\mathbb{F}(B)^+$-invariant: if $\sigma \in \mathbb{F}(B)^+$ and $\hat{x} \in \widehat{X}$ then

$$(\sigma \cdot \hat{x})(\tau) = \hat{x}(\tau\sigma) = x(\rho^+(\tau)\rho^+(\sigma)) = (\rho^+(\sigma) \cdot x)(\rho^+(\tau))$$

for some $x \in X$. Thus, $\sigma \cdot \hat{x} = (\rho^+(\sigma) \cdot x) \circ \rho^+ \in \widehat{X}$. Finally, $\mathbb{F}(B)^+ \curvearrowright \widehat{X}$ is surjective. Indeed, if $\hat{x} = x \circ \rho^+ \in \widehat{X}$ and $\sigma \in \mathbb{F}(B)^+$, by surjectivity of $S \curvearrowright X$ there is an $y \in X$ with $\rho^+(\sigma) \cdot y = x$, yielding

$$(\sigma \cdot (y \circ \rho^+))(\tau) = y \circ \rho^+(\tau\sigma) = (\rho^+(\sigma) \cdot y)(\rho^+(\tau)) = x \circ \rho^+(\tau).$$

Therefore, by choosing a function $\sigma^{-1}\colon \widehat{X} \to \widehat{X}$ such that $\sigma \circ \sigma^{-1} = \mathrm{id}_{\widehat{X}}$ for every $\sigma \in B \subseteq \mathbb{F}(B)^+$, we may define, given any $\hat{x} = x \circ \rho^+ \in \widehat{X}$, a configuration $\bar{x} \in \mathcal{A}^{\mathbb{F}(B)}$ by

$$\bar{x}(\sigma_1^{\epsilon_1} \cdots \sigma_n^{\epsilon_n}) = \Big(\sigma_1^{\epsilon_1} \circ \cdots \circ \sigma_n^{\epsilon_n}(\hat{x})\Big)\Big(1_{\mathbb{F}(B)^+}\Big),$$

where $\epsilon_i \in \{1, -1\}$ for $1 \leq i \leq n$. Clearly, for $\gamma \in \mathbb{F}(B)$ and $\sigma \in \mathbb{F}(B)^+$,

$$(\gamma \cdot \bar{x})(s) = \bar{x}(s\gamma) = (s\gamma(\hat{x}))(1_{\mathbb{F}(B)^+}) = (s \cdot \gamma(\hat{x}))(1_{\mathbb{F}_n^+}) = (\gamma(\hat{x}))(s),$$

so $(\gamma \cdot \bar{x})|_{\mathbb{F}(B)^+} = \gamma(\hat{x}) \in \widehat{X}$ for all $\gamma \in \mathbb{F}(B)$ and we conclude $\bar{x}$ is an element of the natural $\mathbb{F}(B)$-extension of $\widehat{X}$ with $\bar{x}|_{\mathbb{F}(B)^+} = \hat{x}$, so $\widehat{X}$ is $\mathbb{F}(B)$-extensible.

We now want to see that every element of $\widehat{X}_{\mathbb{F}(B)}$ defines an element in X_G. Note that, for every $\gamma \in \mathbb{F}(B)$ and $\bar{x} \in \widehat{X}_{\mathbb{F}(B)}$, we have $(\gamma \cdot \bar{x})|_{\mathbb{F}(B)^+} = x^\gamma \circ \rho^+$ for some $x^\gamma \in X$. This implies, for any $\sigma, \tau \in \mathbb{F}(B)^+$ such that $\rho^+(\sigma) = \rho^+(\tau)$, the following:

$$\bar{x}(\sigma\gamma) = (\gamma \cdot \bar{x})(\sigma) = x^\gamma \circ \rho^+(\sigma) = x^\gamma \circ \rho^+(\tau) = \bar{x}(\tau\gamma).$$

Therefore, if $\rho^+(\sigma) = \rho^+(\tau)$ and $\bar{x} \in \widehat{X}_{\mathbb{F}(B)}$,

$$\bar{x}(\gamma\sigma\tau^{-1}\gamma^{-1}) = (\tau^{-1}\gamma^{-1} \cdot \bar{x})(\gamma\sigma) = (\tau^{-1}\gamma^{-1} \cdot \bar{x})(\gamma\tau) = \bar{x}(1_{\mathbb{F}(B)}).$$

Now, if $\gamma \in \mathbb{F}(B)$ satisfies $\rho(\gamma) = 1_G$, as a consequence of G being the free S-group it can be written as

$$\gamma = \prod_{j=1}^{n} \omega_j \sigma_j \tau_j^{-1} \omega_j^{-1},$$

where $\rho^+(\sigma_j) = \rho^+(\tau_j)$, i.e., $(\sigma_j, \tau_j) \in R$, and $\omega_j \in \mathbb{F}(B)$, for $1 \leq j \leq n$. Therefore,

$$\begin{aligned}
\bar{x}(\gamma) &= \bar{x}\left(\prod_{j=1}^{n} \omega_j \sigma_j \tau_j^{-1} \omega_j^{-1}\right) = \left(\prod_{j=2}^{n} \omega_j \sigma_j \tau_j^{-1} \omega_j^{-1} \cdot \bar{x}\right)(\omega_1 \sigma_1 \tau_1^{-1} \omega_1^{-1}) \\
&= \bar{x}\left(\prod_{j=2}^{n} \omega_j \sigma_j \tau_j^{-1} \omega_j^{-1}\right) = \cdots = \bar{x}(1_{\mathbb{F}(B)}).
\end{aligned}$$

Hence, any $\bar{x} \in \widehat{X}_{\mathbb{F}(B)}$ can be viewed as an element of $\mathcal{A}^G$ via $\bar{x}(\rho(\gamma)) = \bar{x}(\gamma)$. We want to see they are elements of the natural G-extension of X as well, i.e., that for all $g \in G$, $(g \cdot \bar{x})|_S \in X$. We already know for any $\bar{x} \in \widehat{X}_{\mathbb{F}(B)}$ and $\gamma \in \mathbb{F}(B)$ there is a $x^\gamma \in X$ with $(\gamma \cdot \bar{x})|_{\mathbb{F}(B)^+} = x^\gamma \circ \rho^+$. If $s = \rho^+(\sigma) \in S$ and $g = \rho(\gamma) \in G$,

$$(g \cdot \bar{x})(s) = \bar{x}(\rho(\sigma\gamma)) = (\gamma \cdot \bar{x})(\sigma) = x^\gamma \circ \rho^+(\sigma) = x^\gamma(s),$$

so $(g \cdot \bar{x})|_S = x^\gamma \in X$, concluding $\bar{x} \in X_G$. Finally, $\bar{x}|_S$ can be chosen to be any $x \in X$ by the $\mathbb{F}(B)$-extensibility of $\widehat{X}$, so X is G-extensible. $\square$

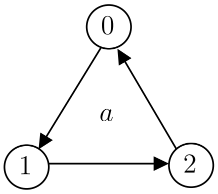
2.5 Non-extensibility and the free S-group

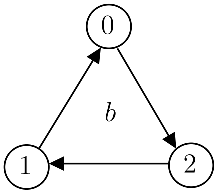
It is clear that a necessary condition for an action $S \overset{\alpha}{\curvearrowright} X$ to be G-extensible is that α is a surjective action. Similarly, a necessary condition for α to be partially G-extensible is the existence of an element $x \in X$ with $s^{-1}(x) \neq \varnothing$ for all $s \in S$. Note, however, that there exist surjective actions (even over compact spaces) with $X_G = \varnothing$. An example will be provided now with the free semigroup on two generators, $\mathbb{F}_2^+ = \langle a, b\rangle^+$, and the $\mathbb{F}_2^+$-group $\mathrm{BS}(1,2) = \langle \alpha, \beta | \alpha\beta = \beta^2\alpha\rangle$. It is known that α and $\beta\alpha$ generate $\mathbb{F}_2^+$ within $\mathrm{BS}(1,2)$. Defining $a = \alpha$ and $b = \beta\alpha$ we have that $\alpha\beta = \beta^2\alpha$ if and only if $\alpha(\beta\alpha) = (\beta\alpha)\alpha^{-1}(\beta\alpha)\alpha$, which is equivalent to $b^{-1}ab = a^{-1}ba$. Hence,

$$\mathrm{BS}(1,2) = \langle a, b \mid b^{-1}ab = a^{-1}ba\rangle \quad \text{and} \quad \langle a, b\rangle^+ \simeq \mathbb{F}_2^+.$$

We will work with this presentation from now on.

Example 2.5.1. Let $G = \mathrm{BS}(1,2)$. Now consider the subshift $X \subseteq 3^{\mathbb{F}_2^+}$ determined by the following graphs.





Recall that the natural G-extension of X is

$$X_G = \left\{ x' \in 3^G : (g \cdot x')|_{\mathbb{F}_n^+} \in X \text{ for all } g \in G \right\}.$$

Thus, any configuration $\overline{x} \in X_G$ must be determined by the same rules (graphs) which define X, following the directions of a and b. More precisely, for any $g \in G$ and $s \in \{a, b\}$, the value of $\overline{x}(sg)$ is determined from $\overline{x}(g)$ by the same graph associated to s in the definition of X. Indeed, $(g \cdot \overline{x})|_{\mathbb{F}_2^+}$ belongs to X, $\overline{x}(g) = (g \cdot \overline{x})(1_G)$ and $\overline{x}(sg) = (g \cdot \overline{x})(s)$.

Now, the path $\mathfrak{c} = \{1_G, b, ab, b^{-1}ab = a^{-1}ba, ba, a, 1_G\}$ defines a cycle in G, but its directed structure does not allow to follow the defining rules of X while colouring the vertices, as shown by the diagram below. Therefore, $X_G = \varnothing$. Note, however, that the action $\mathbb{F}_2^+ \curvearrowright X$ is surjective.

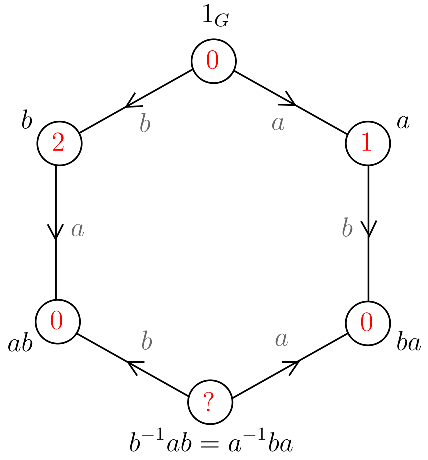


Figure 2.1: The directed structure of the cycle $\mathfrak{c}$ with respect to a and b. In red, it is shown how this cycle cannot be coloured with the symbols 0, 1 and 2, starting with 0 at 1_G, following the defining rules of X. The same happens if we start with 1 or 2 at 1_G.

Last example can be viewed in the following manner. Considering the symbols $0, 1$ and 2 as elements from $\mathbb{Z}_3$, the subshift X is defined by adding up 1 mod 3 in the a direction, and 2 mod 3 in the b direction. The group $\mathbb{Z}_3$ is a $\mathbb{F}_2^+$-group via the morphism η sending a to 1 and b to 2, and since the relation $b^{-1}a^{-1}ba^{-1}ba$ does not hold for this pair of elements, there is no morphism $\theta\colon \mathrm{BS}(1,2) \to \mathbb{Z}_3$ such that $\theta \circ \iota = \eta$, with ι

the inclusion $\mathbb{F}_2^+ \hookrightarrow \mathrm{BS}(1,2)$. This gives us a hint on how to generate examples of non extensible surjective actions for any S-group which is not the free S-group.

To make the exposition of the following result more clear, we will briefly resume the usage of embeddings of semigroups, rather than assuming S is a subsemigroup of the involved groups.

Proposition 2.5.2. *For any S-group (H,η) with η injective which is not the free S-group (G,γ), there is a non H-extensible surjective action α in S-**Top**. Moreover, if G is residually finite, α can be chosen to be an action over a compact space.*

Proof. Since there is a surjective morphism $\theta\colon G \twoheadrightarrow H$ and H is not isomorphic to G, $\ker(\theta)$ is non-trivial and so there is an element $1_G \neq w \in \ker(\theta)$. Writing $w = \gamma(s_1)^{\epsilon_1}\cdots\gamma(s_n)^{\epsilon_n}$, we have

$$1_H = \theta(w) = \eta(s_1)^{\epsilon_1}\cdots\eta(s_n)^{\epsilon_n}.$$

Define $X \subseteq G^S$, where $x \in X$ if and only if $x(s) = \gamma(s)x(1_S)$ for all $s \in S$. In other words, $X = \{\gamma g : g \in G\}$, with $(\gamma g)(s) = \gamma(s)g$ for all $s \in S$. The subset X is S-invariant, since if $x \in X$ and $s \in S$ then $(s\cdot x)(t) = x(ts) = \gamma(ts)x(1_S) = \gamma(t)x(s) = \gamma(t)(s\cdot x)(1_S)$. It is also a closed subset: if $x \in \overline{X}$ and $s \in S$ then there is an element $y \in X$ which coincides with x at both 1_S and s, implying $x(s) = y(s) = \gamma(s)y(1_S) = \gamma(s)x(1_S)$. Thus, X is a subshift.

Observe now that, if $\overline{x} \in X_H$, for all $s \in S$ and $h \in H$ we have

$$\overline{x}(\eta(s)h) = (h\cdot\overline{x})(\eta(s)) = \gamma(s)(h\cdot\overline{x})(1_H) = \gamma(s)\overline{x}(h),$$

and

$$\gamma(s)\overline{x}(\eta(s)^{-1}h) = \gamma(s)(\eta(s)^{-1}h\cdot\overline{x})(1_H) = (\eta(s)^{-1}h\cdot\overline{x})(\eta(s)) = \overline{x}(h).$$

Thus, $\overline{x}(\eta(s)h) = \gamma(s)\overline{x}(h)$ and $\overline{x}(\eta(s)^{-1}h) = \gamma(s)^{-1}\overline{x}(h)$. Now let $g \in G$ be any element and assume $\overline{x} \in X_H$ satisfies $\overline{x}(1_H) = g$. The local rules that we have just proven imply that

$$g = \overline{x}(\eta(s_1)^{\epsilon_1}\cdots\eta(s_n)^{\epsilon_n}) = \gamma(s_1)^{\epsilon_1}\cdots\gamma(s_n)^{\epsilon_n}\overline{x}(1_H) = wg,$$

but we chose $w \neq 1_G$, so we obtain a contradiction. Thus $X_H = \varnothing$. However, if $x \in X$ then there is a $g \in G$ with $x(s) = \gamma(s)g$ for all $s \in S$. Given $t \in S$, let $y \in X$ be defined by $y(s) = \gamma(s)\gamma(t)^{-1}g$. We have that

$$(t\cdot y)(s) = y(st) = \gamma(st)\gamma(t)^{-1}g = \gamma(s)g = x(s),$$

so $s\cdot y = x$ and we conclude $S \curvearrowright X$ is surjective.

In the case G is residually finite, there is a finite group F and a morphism $\varphi\colon G \to F$ such that $\varphi(w) \neq 1_F$. Defining $X \subseteq F^S$ to be $X = \{(\varphi\circ\gamma)f : f \in F\}$ and replicating the argument shown above we obtain that for every $f \in F$ and $\overline{x} \in X_H$ such that $\overline{x}(1_H) = f$,

$$f = \overline{x}(\eta(s_1)^{\epsilon_1} \cdots \eta(s_n)^{\epsilon_n}) = \varphi(\gamma(s_1))^{\epsilon_1} \cdots \varphi(\gamma(s_n))^{\epsilon_n} \overline{x}(1_H) = \varphi(w) f,$$

a contradiction, so $X_H = \varnothing$. Note that X is compact as F is finite, and given $x \in X$, $s \in S$, the configuration defined by $y(t) = (\varphi \circ \gamma(t))(\varphi \circ \gamma(s))^{-1} x(1_S)$ is a preimage of x under s, as $(s \cdot y)(t) = y(ts) = (\varphi \circ \gamma(t))(\varphi \circ \gamma(s))(\varphi \circ \gamma(s))^{-1} x(1_S) = x(t)$. Hence $S \curvearrowright X$ is a surjective action over a compact metric space which is not partially H-extensible. ☐

This rises the question whether any surjective S-action over a compact metric space is G-extensible when G is the free S-group. The answer is yes, and to prove it we need the following lemma. The first part is a generalization to actions of semigroups of a result due to Giordano and de la Harpe (see [GDLH97]), which is itself a group-equivariant version of a theorem due to Alexandroff and Hausdorff (namely, that compact spaces are continuous images of the Cantor space).

Lemma 2.5.3. *Every action $\alpha \in S$-**Top** over a compact space is a factor of an inverse limit of subshifts. More precisely, the following statements hold.*

- (i) *Let $S \overset{\alpha}{\curvearrowright} X$ be an action in S-**Top** with X compact, and $\mathcal{C} = 2^{\mathbb{N}}$ be the Cantor space. Then, α is a factor of a closed S-invariant subset $Y \subseteq \mathcal{C}^S$, where the action $S \curvearrowright \mathcal{C}^S$ is the shift action: $(s \cdot y)(t) = y(ts)$ for all $s, t \in S$ and $y \in \mathcal{C}^S$.*
- (ii) *The action $S \curvearrowright Y$ is S-conjugate to an inverse limit of subshifts $X_k \subseteq (2^k)^S$.*
- (iii) *If α is surjective, then $S \curvearrowright Y$ is surjective and each $S \curvearrowright X_k$ is surjective as well.*

Proof. Proof of (i). Since every compact metrizable space is a continuous image of $\mathcal{C}$, we can choose a continuous surjective function $\theta : \mathcal{C} \longrightarrow X$. Let

$$Y = \{y \in \mathcal{C}^S : s \cdot \theta(y(t)) = \theta(y(st)) \text{ for all } s, t \in S\}.$$

It is easy to show that Y is S-invariant and closed. The closedness follows from the fact that θ and α are continuous, and the metric structure of $\mathcal{C}^S$: if $(y_n)_n \subseteq Y$, then $y_n \to y$ if and only if $y_n(t) \to y(t)$ for every $t \in S$), so

$$s \cdot \theta(y(t)) = \lim_{n\to\infty} s \cdot \theta(y_n(t)) = \lim_{n\to\infty} \theta(y_n(st)) = \theta(y(st)).$$

Define the continuous surjective function $\varphi := \theta \circ \pi_{1_S} : \mathcal{C}^S \longrightarrow X$, where π_{1_S} is the projection on the coordinate of $1_S \in S$. Let us see that $\varphi|_Y$ is surjective. Given $x \in X$, define $y_x \in \mathcal{C}^S$ by choosing for every $t \in S$ a $y_x(t) \in \theta^{-1}(t \cdot x)$. Note that

$$s \cdot \theta(y_x(t)) = s \cdot (t \cdot x) = st \cdot x = \theta(y_x(st)),$$

so $y_x \in Y$, and $\varphi(y_x) = \theta(y_x(1_S)) = x$, proving surjectivity. Also, if $y \in Y$ and $s \in S$, then

$$s \cdot \varphi|_Y(y) = s \cdot \theta(y(1_S)) = \theta(y(s)) = \theta((s \cdot y)(1_S))) = \varphi|_Y(s \cdot y).$$

Thus, $\varphi|_Y$ is a continuous, surjective and S-equivariant function $Y \to X$.

Proof of (ii). Let $\pi_k \colon \mathcal{C}^S \to (2^k)^S$ be defined as $\pi_k(\omega)(s) = \omega(s)|_{[0,\ldots,k]}$. This map is S-equivariant and Y is S-invariant, so $\pi_k(Y)$ is S-invariant as well. Also, $\pi_k(Y)$ is compact as a continuous image of a compact set, thus it is closed. For each $i \leq j$, define the map $\pi_{ij} \colon \pi_j(Y) \to \pi_i(Y)$ by $\pi_{ij}(\omega)(s) = \omega(s)|_{[0,\ldots,i]}$, which is S-equivariant. Clearly, if $i \leq j$ then $\pi_{ij} \circ \pi_j = \pi_i$, so for any $\omega \in \pi_i(Y)$, $\omega = \pi_i(y)$, the element $\pi_j(y)$ is a preimage of ω, showing π_{ij} is surjective.

Let $\varphi \colon Y \to \varprojlim \pi_k(Y)$ be given by $y \mapsto (\pi_k(y))_{k \in \mathbb{N}}$. This map can be seen as a continuous function to the product of the spaces $\pi_k(Y)$, restricted to $\varprojlim \pi_k(Y)$, a closed subset, and therefore it is continuous. It is also injective: if $y \neq y'$ are elements from Y, then there exist $s \in S$ and $k \in \mathbb{N}$ with $y(s)(k) \neq y'(s)(k)$, implying $\pi_k(y) \neq \pi_k(y')$. Now let $(x_k)_{k \in \mathbb{N}} \in \varprojlim \pi_k(Y)$. Then, for each $k \in \mathbb{N}$ there is a $y_k \in Y$ with $x_k = \pi_k(y_k)$. Since Y is compact, we can assume that $y_k \to y \in Y$, and by the continuity of φ we have $\varphi(y) = (x_k)_{k \in \mathbb{N}}$, so φ is surjective. Finally, since the functions π_k are S-equivariant and the action $S \curvearrowright \varprojlim \pi_k(Y)$ is defined to be coordinate-wise, φ is S-equivariant. Thus, it is an S-equivariant continuous bijection from a compact space to a Hausdorff space, i.e., an S-equivariant homeomorphism.

Proof of (iii). First note that if $S \curvearrowright Y$ were surjective, since $\pi_k \colon Y \to \pi_k(Y)$ is S-equivariant and surjective, for every $x \in \pi_k(Y)$ and $s \in S$ we can choose a $y \in s^{-1}\pi_k^{-1}(x)$, which immediately yields $s \cdot \pi_k(y) = \pi_k(s \cdot y) = x$, so $S \curvearrowright \pi_k(Y)$ is surjective. It thus suffices to prove surjectivity of $S \curvearrowright Y$. Let $s \in S$ and $y \in Y$. We will define a preimage $\tau \in s^{-1}(y)$. For each $t = t's \in Ss$, define $\tau(t) = y(t')$ (note that this is well defined, as an element t can be uniquely written in this fashion by bicancellativity of S). Now we distinguish to cases. First case, if $1_S = s^* s$ for some $s^* \in S$, then $ss^* = 1_S$ by left cancellativity, and

$$s \cdot \theta(\tau(1_S)) = s \cdot \theta(y(s^*)) = \theta(y(ss^*)) = \theta(y(1_S)).$$

Second case, $1_S \notin Ss$, and we define $\tau(1_S)$ by choosing any element in $\theta^{-1}(s^{-1}(\theta(y(1_S))))$. This can be done by the surjectivity of both θ and $S \curvearrowright X$. In either situation, $\tau(1_S)$ is defined and $s \cdot \theta(\tau(1_S)) = \theta(y(1_S))$. Now we define τ in the remaining of S: if $t \in S - Ss$, define $\tau(t)$ by choosing any element of $\theta^{-1}(t \cdot \theta(\tau(1_S)))$.

Observe that if $t = t's$, then $\theta(\tau(t)) = \theta(y(t'))$ by definition, while

$$t \cdot \theta(\tau(1_S)) = t's \cdot \theta(\tau(1_S)) = t' \cdot \theta(y(1_S)) = \theta(y(t')),$$

thus yielding $\tau(t) \in \theta^{-1}(t \cdot \theta(\tau(1_S)))$. Therefore, if $t, t' \in S$, we get

$$t \cdot \theta(\tau(t')) = t \cdot (t' \cdot \theta(\tau(1_S))) = (tt') \cdot \theta(\tau(1_S)) = \theta(\tau(tt')),$$

so $\tau \in Y$. Finally, by definition we have $(s \cdot \tau)(t) = \tau(ts) = y(t)$, so $s \cdot \tau = y$.

$\square$

As a direct consequence of this lemma, the fact that surjective subshifts are G-extensible if G is the free S-group, and that G-extensibility is preserved under taking factors and inverse limits (in the compact case), we obtain the following corollary.

Corollary 2.5.4. *Let G be the free S-group, and $S \overset{\alpha}{\curvearrowright} X$ be a continuous, surjective action over a compact metric space. Then, α is G-extensible.*

Putting all together, we have the following characterization of the free S-group of a semigroup S.

Theorem 2.5.5. *Let G be an S-group.*

(i) *The group G is the free S-group if and only if every surjective S-subshift is G-extensible.*

(ii) *Assume G is residually finite. Then G is the free S-group if and only if every surjective action $\alpha \in S$-$\textbf{Top}$ over a compact space is G-extensible.*

2.6 Measure extensions

For a given Polish space X, we denote by $\mathcal{M}(X)$ the set of probability measures defined on the Borel σ-algebra $\mathscr{B}(X)$. Recall that this set can be viewed as a subset of the dual vector space of the continuous bounded functions $\mathscr{C}_b(X)$ due to Riesz-Markov-Kakutani's Theorem. Therefore, we can endow $\mathcal{M}(X)$ with the (metrizable) weak-* topology with respect to $\mathscr{C}_b(X)$, i.e., the one in which the convergence is characterized by what is known as the Portmanteau Theorem: $\mu_n \to \mu$ if and only if for all $f \in \mathscr{C}_b(X)$

$$\int_X f \, d\mu_n \to \int_X f \, d\mu.$$

A basis for the topology consists of the sets

$$V(\mu, \mathcal{F}, \epsilon) = \left\{ \mu' \in \mathcal{M}(X) : \left| \int f \, d\mu' - \int f \, d\mu \right| < \epsilon, \forall f \in \mathcal{F} \right\},$$

where $\epsilon > 0$, $\mathcal{F} \subseteq \mathscr{C}_n(X)$ is finite, and $\mu \in \mathcal{M}(X)$. When X is compact, the space $\mathcal{M}(X)$ is weak-* compact as a consequence of the theorem of Banach-Alaoglu. When a semigroup S acts continuously upon X, the space of S-invariant measures $\mathcal{M}_S(X)$ is closed in $\mathcal{M}(X)$. Indeed, if $(\mu_n)_{n \in \mathbb{N}}$ is a sequence in $\mathcal{M}_S(X)$ converging to an element $\mu \in \mathcal{M}(X)$ and $s \in S$, then

$$\int_X f \, d(s_* \mu) = \int_X f \circ s \, d\mu = \lim_{n \to \infty} f \circ s \, d\mu_n = \lim_{n \to \infty} \int_X f \, d\mu_n = \int_X f \, d\mu$$

for all $f \in \mathscr{C}_b(X)$. This implies that $s_* \mu = \mu$, so $\mu \in \mathcal{M}_S(X)$. In particular, when X is compact, both $\mathcal{M}(X)$ and $\mathcal{M}_S(X)$ are weak-* compact.

Returning to the case we are dealing with, observe that the projection map $\pi\colon X_G \to X$ induces a push-forward map $\pi_*\colon \mathcal{M}(X_G) \to \mathcal{M}(X)$ given by $\pi_*\hat{\mu}(A) = \mu(\pi^{-1}(A))$, for all $A \in \mathscr{B}(X)$. Moreover, since π is S-equivariant, the image of a G-invariant measure on X_G via π_* is an S-invariant measure on X, so the operator $\pi_*\colon \mathcal{M}_G(X_G) \to \mathcal{M}_S(X)$ is well defined. From now on, π_* will denote the push-forward at invariant level, i.e., $\pi_*\colon \mathcal{M}_G(X_G) \to \mathcal{M}_S(X)$.

Definition 2.6.1. Let $S \overset{\alpha}{\curvearrowright} X$ be a partially G-extensible action in S-**Top**.

(i) If $\mu \in \mathcal{M}_S(X)$ belongs to $\operatorname{im}(\pi_*)$, then μ will be called **G-extensible**, and the p.m.p. action $S \curvearrowright (X, \mu)$ will be called **measure-theoretically G-extensible**. The subset $\operatorname{im}(\pi_*)$ of G-extensible measures of $\mathcal{M}_S(X)$ will be denoted by $\operatorname{Ext}_G(X, S)$.

(ii) If $\mu \in \mathcal{M}_S(X)$ and $\mu_G \in (\pi_*)^{-1}(\mu)$, then (X_G, μ_G) together with the map $\pi\colon X_G \to X$ will be called a **measure-theoretical natural G-extension** of α.

Note that the requirement that $S \curvearrowright X$ is partially G-extensible in order to define measure-theoretical G-extensibility is redundant, as there cannot be a probability measure defined over an empty set.

Remark 2.6.2. Via the identification of the topological G-extension of $\mathcal{A}^S$ with $(\mathcal{A}^G, \rho)$, where $\rho\colon \mathcal{A}^G \to \mathcal{A}^S$ is the restriction $\overline{x} \mapsto \overline{x}|_S$, a measure $\mu \in \mathcal{M}_S(\mathcal{A}^S)$ is G-extensible if there is a measure $\mu_G \in \mathcal{M}_G(\mathcal{A}^G)$ such that $\rho_*\mu_G = \mu$.

Proposition 2.6.3. *Let $S \overset{\alpha}{\curvearrowright} X$ be a partially G-extensible action in S-**Top**. The operator $\pi_*\colon \mathcal{M}_G(X_G) \to \mathcal{M}_S(X)$ is continuous with respect to the weak-* topologies.*

Proof. Let $\mu \in \mathcal{M}_G(X_G)$ and set $\nu = \pi_*\mu$. Let $\epsilon > 0$, $\mathcal{F} \subseteq \mathscr{C}_b(X)$ be a finite subset, and consider

$$V(\nu, \mathcal{F}, \epsilon) = \left\{\nu' \in \mathcal{M}(X) : \left|\int f\,d\nu' - \int f\,d\nu\right| < \epsilon, \forall f \in \mathcal{F}\right\},$$

i.e., an arbitrary basic open neighborhood of ν. Define $\mathcal{F}' = \{f \circ \pi : f \in \mathcal{F}\} \subseteq \mathscr{C}_b(X_G)$, which is finite, and take any $\mu' \in V(\mu, \mathcal{F}', \epsilon)$. Then, given any $f \in \mathcal{F}$,

$$\left|\int f\,d\pi_*\mu' - \int f\,d\nu\right| = \left|\int f \circ \pi\,d\mu' - \int f \circ \pi\,d\mu\right| < \epsilon,$$

so $\pi_*\mu' \in V(\nu, \mathcal{F}, \epsilon)$. Therefore, $\pi_*(V(\mu, \mathcal{F}', \epsilon)) \subseteq V(\nu, \mathcal{F}, \epsilon)$, and we conclude the desired continuity of π_* at μ. □

Corollary 2.6.4. *Assume X is compact. Then, the subset $\operatorname{Ext}_G(X, S) \subseteq \mathcal{M}_S(X)$ is weak-* closed. In particular, it is weak-* compact and if a family $\mathcal{F} \subseteq \operatorname{Ext}_G(X, S)$ is weak-* dense in $\mathcal{M}_S(X)$, then $\operatorname{Ext}_G(X, S) = \mathcal{M}_S(X)$.*

Proof. Let $(\mu_n)_{n\geq 1}$ be a sequence in $\mathrm{Ext}_G(X,S)$ converging to $\mu \in \mathcal{M}_S(X)$. Consider, for every $n \in \mathbb{N}$, a measure $\overline{\mu}_n \in \mathcal{M}_G(X_G, \pi)$ such that $\pi_*\bar{\mu}_n = \mu_n$. Since X_G is a closed subset of X^G and the latter is compact, so is X_G. Thus, $\mathcal{M}_G(X_G)$ is weak-* compact and we can find a subsequence $\bar{\mu}_{n_k} \to \bar{\mu} \in \mathcal{M}_G(X_G)$. By continuity of π_*, we get $\mu_{n_k} = \pi_*\bar{\mu}_{n_k} \to \pi_*\bar{\mu}$, so $\mu = \pi_*\bar{\mu}$. Therefore $\mu \in \mathrm{Ext}_G(X,S)$. $\square$

Proposition 2.6.5. *Let $S \overset{\alpha}{\curvearrowright} (X,\mu), S \overset{\beta}{\curvearrowright} (Y,\nu)$ be two p.m.p. actions such that there are measurable S-invariant subsets $X' \subseteq X$ and $Y' \subseteq Y$ with $\mu(X') = \nu(Y') = 1$ and an S-equivariant measure-preserving function $\varphi\colon X' \to Y'$. Then, if α is measure-theoretically G-extensible, so is β.*

Proof. Let (X_G, π_α) and (Y_G, π_β) be the corresponding topological G-extensions to α and β, and $\mu_G \in \mathcal{M}_G(X_G)$ such that $(\pi_\alpha)_*\mu_G = \mu$. The first thing we need to check is that Y_G is non-empty (and hence β is topologically partially G-extensible). Indeed, since μ_G is G-invariant, $\mu_G(g\pi_\alpha^{-1}(X')) = \mu(X') = 1$ for all $g \in G$, so the fact that G is countable implies

$$\mu_G\left(\bigcap_{g\in G} g\pi_\alpha^{-1}(X')\right) = 1.$$

Thus, there is an element $(x_g)_{g\in G} \in X_G$ with $x_g \in X'$ for all $g \in G$. The element $(\varphi(x_g))_{g\in G}$ is an element of Y_G.

We want to construct a measure $\nu_G \in \mathcal{M}_G(Y_G)$ satisfying $(\pi_\beta)_*\nu_G = \nu$. Consider the function

$$\begin{aligned} \varphi_*\colon \bigcap_{g\in G} g\pi_\alpha^{-1}(X') &\longrightarrow \bigcap_{g\in G} g\pi_\beta^{-1}(Y') \\ (x_g)_{g\in G} &\longmapsto (\varphi(x_g))_{g\in G}, \end{aligned}$$

which is well defined, as $\bigcap_{g\in G} g\pi_\alpha^{-1}(X') \subseteq (X')^G$ and φ is S-equivariant. As we already mentioned, φ_* is defined upon a full-measure subset of X_G. Also, φ_* is G-equivariant as a consequence of the G-equivariance of φ, and the fact that it is defined upon elements of X_G. Define, for $A \in \mathscr{B}(Y_G)$,

$$\nu_G(A) = \mu_G\left(\varphi_*^{-1}\left(A \cap \bigcap_{g\in G} g\pi_\beta^{-1}(Y')\right)\right).$$

The set function ν_G is a probability measure on Y_G. The G-invariance comes as a consequence of G-equivariance of φ_* and G-invariance of $\bigcap_{g\in G} g\pi_\beta^{-1}(Y')$. To see that π_β becomes a measure-preserving mapping note that $\pi_\beta \circ \varphi_* = \varphi \circ \pi_\alpha$, so for every

$A \in \mathscr{B}(X)$,

$$\begin{aligned}
\nu_G(\pi_\beta^{-1}(A)) &= \mu_G\left(\varphi_*^{-1}\left(\pi_\beta^{-1}(A) \cap \bigcap_{g\in G} g\pi_\beta^{-1}(Y')\right)\right)\\
&= \mu_G\left(\varphi_*^{-1}(\pi_\beta^{-1}(A)) \cap \bigcap_{g\in G} g\varphi_*^{-1}(\pi_\beta^{-1}(Y'))\right)\\
&= \mu_G\left(\pi_\alpha^{-1}\left(\varphi^{-1}(A)\right) \cap \bigcap_{g\in G} g\pi_\alpha^{-1}(X')\right)\\
&= \mu(\varphi^{-1}(A)) = \nu(A).
\end{aligned}$$

□

2.7 Measure extensions: the reversible case

The case where S is left reversible and G is the group of right fractions deserves special attention. In this situation, G presents a certain directed structure which allows to construct G-invariant measures on X_G such that the extension map π is measure-preserving. Define the preorder $\leq_S$ on G by

$$g \leq_S h \iff h \in Sg \iff hg^{-1} \in S.$$

Observe that this preorder has a dynamical interpretation: an element is smaller than other if the latter is in the "dynamical future" of the former, i.e., it can be obtained from the first one via left multiplication by an element of S. The preorder $(G, \leq_S)$ being downward directed, in the sense that finite sets have common lower bounds, will be a fundamental feature for the construction of π-preserving measures in $\mathcal{M}_G(X_G)$.

Lemma 2.7.1. *Assume S is left reversible. Then, G is the group of right fractions of S if and only if the pre-order $(G, \leq_S)$ is downward directed, i.e., if and only if for every finite subset $F \subseteq G$ there is an $m_F \in G$ such that $m_F \leq_S t$ for all $t \in F$.*

Proof. If $G = G_R(S)$ and $g, h \in G$, then we can find $a, b, c, d \in S$ such that $g = ab^{-1}$ and $h = cd^{-1}$. Since S is left reversible, there are $x, y \in S$ such that $bx = dy$, which means $x^{-1}b^{-1} = y^{-1}d^{-1}$ in G. Thus,

$$x^{-1}a^{-1}g = x^{-1}b^{-1} = y^{-1}d^{-1} = y^{-1}c^{-1}h.$$

Define $m = x^{-1}a^{-1}g = y^{-1}c^{-1}h \in G$. Then, by definition, $g = axm$ and $h = cym$, so m is a common lower bound for g and h. Iterating the argument, we obtain a lower bound for any finite subset $F \subseteq G$.

Conversely, assume $(G, \leq_S)$ is directed. Let $g \in G$, and take a common lower bound $m \in G$ for g and 1_G, that is, an element m such that there are $s, t \in S$ satisfying $g = sm$ and $1_G = tm$. Therefore, $s^{-1}g = t^{-1}$ and we have $g = st^{-1}$. Therefore, every element of G can be written as a right quotient, which means $G = G_R(S)$. □

Let now $S \overset{\alpha}{\curvearrowright} (X, \mu)$ be a p.m.p. action, and G be the group of right fractions of S. denote, as usual, the topological natural G-extension of α by (X_G, π). Given a finite set $F \subseteq G$, fix any lower bound m_F for F (i.e., such that $m_F \leq_S t$ for all $t \in F$). Note that, by definition, this means $tm_F^{-1} \in S$ for each $t \in F$. Define the following collection of subsets of $\mathscr{B}(X^F)$:

$$\mathscr{C}_F = \left\{ \prod_{t \in F} A_t : A_t \in \mathscr{B}(X) \text{ for every } t \in F \right\},$$

and the set function $\mu_F \colon \mathscr{C}_F \to [0,1]$ by

$$\mu_F \left(\prod_{t \in F} A_t \right) = \mu \left(\bigcap_{t \in F} (tm_F^{-1})^{-1}(A_t) \right).$$

Observe that, due to the S-invariance of μ, the value of the function μ_F does not depend on the choice of m_F. Indeed, if m_1 and m_2 are two distinct lower bounds of F, take a lower bound m for both m_1 and m_2. Then $s_i := m_i m^{-1}$ belongs to S for $i = 1, 2$, so

$$\begin{aligned} \mu \left(\bigcap_{t \in F} (tm^{-1})^{-1} A_t \right) &= \mu \left(\bigcap_{t \in F} (tm_i^{-1} m_i m^{-1})^{-1} A_t \right) \\ &= \mu \left(\bigcap_{t \in F} s_i^{-1} \left[(tm_i^{-1})^{-1}(A_t) \right] \right) \\ &= \mu \left(\bigcap_{t \in F} (tm_i^{-1})^{-1}(A_t) \right). \end{aligned}$$

Lemma 2.7.2. *The function μ_F extends to a finitely additive probability measure on the algebra of sets $\mathscr{A}_F$ generated by $\mathscr{C}_F$.*

Proof. Since $\mathscr{A}_F$ consists of finite disjoint unions of elements of $\mathscr{C}_F$, we just need to check that, for any $C \in \mathscr{A}_F$ and finite partitions $\mathcal{P}, \mathcal{Q}$ of C by elements of $\mathscr{C}_F$,

$$\sum_{A \in \mathcal{P}} \mu_F(A) = \sum_{B \in \mathcal{Q}} \mu_F(B).$$

First, if $\mathcal{Q}$ is a refinement of $\mathcal{P}$, take any $A \in \mathcal{P}$ and write it as a union of a collection $\{B^{(i)}\}_{i=1}^n \subseteq \mathcal{Q}$:

$$A = \prod_{t \in F} A_t = \bigsqcup_{i=1}^n B^{(i)}, \quad B^{(i)} = \prod_{t \in F} B_t^{(i)}.$$

Since this union is disjoint, for $1 \leq i < j \leq n$ there must be a $t_{ij} \in F$ such that $B_{t_{ij}}^{(i)} \cap B_{t_{ij}}^{(j)} = \varnothing$, which implies

$$(t_{ij} m_F^{-1})^{-1} \left(B_{t_{ij}}^{(i)} \right) \cap (t_{ij} m_F^{-1})^{-1} \left(B_{t_{ij}}^{(j)} \right) = \varnothing.$$

In particular, this means

$$\left[\bigcap_{t \in F} (tm_F^{-1})^{-1} \left(B_t^{(i)} \right) \right] \cap \left[\bigcap_{t \in F} (tm_F^{-1})^{-1} \left(B_t^{(j)} \right) \right] = \varnothing,$$

therefore the collection

$$\left\{\bigcap_{t\in F}(tm_F^{-1})^{-1}\left(B_t^{(i)}\right) : 1\leq i\leq n\right\}$$

is pairwise disjoint. Hence,

$$\mu\left(\bigsqcup_{i=1}^{n}\bigcap_{t\in F}(tm_F^{-1})^{-1}\left(B_t^{(i)}\right)\right)=\sum_{i=1}^{n}\mu_F\left(B^{(i)}\right).$$

Now, for every $1\leq i\leq n$ and $t\in F$, $B_t^{(i)}\subseteq A_t$, from where we obtain

$$\bigsqcup_{i=1}^{n}\bigcap_{t\in F}(tm_F^{-1})^{-1}\left(B_t^{(i)}\right)\subseteq\bigcap_{t\in F}(tm_F^{-1})^{-1}(A_t).$$

We want to see this last inclusion is an equality. Take any $x\in\bigcap_{t\in F}(tm_F^{-1})^{-1}(A_t)$. Then, the tuple $(tm_F^{-1}\cdot x)_{t\in F}$ belongs to A, which means it belongs to some $B^{(i)}$. Thus,

$$x\in\bigcap_{t\in F}(tm_F^{-1})^{-1}\left(B_t^{(i)}\right),$$

and we get the desired opposite inclusion. Putting all together yields

$$\mu_F(A)=\mu\left(\bigcap_{t\in F}(tm_F^{-1})^{-1}(A_t)\right)=\mu\left(\bigsqcup_{i=1}^{n}\bigcap_{t\in F}(tm_F^{-1})^{-1}\left(B_t^{(i)}\right)\right)=\sum_{i=1}^{n}\mu_F\Big(B^{(i)}\Big).$$

Finally, summing over $\mathcal{P}$:

$$\sum_{A\in\mathcal{P}}\mu_F(A)=\sum_{A\in\mathcal{P}}\sum_{\substack{B\in\mathcal{Q}\\ B\subseteq A}}\mu_F(B)=\sum_{B\in\mathcal{Q}}\mu_F(B).$$

The remaining case, where $\mathcal{Q}$ need not be a refinement of $\mathcal{P}$, follows by considering a refinement $\mathcal{P}\vee\mathcal{Q}$ of both $\mathcal{P}$ and $\mathcal{Q}$. $\square$

Now that we have a finitely additive probability measure on $\mathscr{A}_F$, we would like to prove that it is, in fact, σ-additive on $\mathscr{A}_F$. There are multiple ways to approach this. Here we recall a nice result which will help us at this step.

Lemma 2.7.3. *Let ν be a finite measure on an algebra $\mathscr{A}$ which is finitely additive and continuous at $\varnothing$, meaning that whenever $\bigcap_{n\geq 1}A_n=\varnothing$, with $A_1\supseteq A_2\supseteq\cdots$ in $\mathscr{A}$ (this is sometimes denoted by $A_n\downarrow\varnothing$), we have*

$$\lim_{n\to\infty}\nu(A_n)=0.$$

Then, ν is σ-additive in $\mathscr{A}$.

Proof. Let $(E_i)_{i\geq 1} \subseteq \mathscr{A}$ be a pairwise disjoint sequence such that $E := \bigsqcup_{i\geq 1} E_i \in \mathscr{A}$. Define $B_n = E \setminus \bigsqcup_{i=1}^n E_i$. It is clear that $B_n \downarrow \varnothing$. Also, since $E \in \mathscr{A}$ and $\mathscr{A}$ is an algebra, we have that $B_n \in \mathscr{A}$ for all $n \in \mathbb{N}$, and so

$$\nu(E) = \nu\left(B_n \sqcup \bigsqcup_{i=1}^n E_i\right) = \nu(B_n) + \sum_{i=1}^n \nu(E_i),$$

Therefore,

$$\nu(E) = \lim_{n\to\infty}\left[\nu(B_n) + \sum_{i=1}^n \nu(E_i)\right] = \sum_{i=1}^\infty \nu(E_i)$$

The last equality follows by continuity of ν at $\varnothing$. □

Lemma 2.7.4. *The measure $\mu_F \colon \mathscr{A}_F \to [0,1]$ is continuous at $\varnothing$, hence σ-additive.*

Proof. Let $A_n \downarrow \varnothing$ in $\mathscr{A}_F$. For each $n \in \mathbb{N}$ write

$$A_n = \bigsqcup_{i=1}^{k_n} \prod_{t\in F} A_t^{(i,n)}.$$

Observe that

$$\begin{aligned}\mu_F(A_n) &= \sum_{i=1}^{k_n} \mu_F\left(\prod_{t\in F} A_t^{(i,n)}\right) = \sum_{i=1}^{k_n} \mu\left(\bigcap_{t\in F} (tm_F^{-1})^{-1}\left(A_t^{(i,n)}\right)\right) \\ &= \mu\left(\bigsqcup_{i=1}^{k_n}\bigcap_{t\in F} (tm_F^{-1})^{-1}\left(A_t^{(i,n)}\right)\right).\end{aligned}$$

These last sets are decreasing in n. Suppose there exists an element

$$x \in \bigcap_{n\geq 1}\left[\bigsqcup_{i=1}^{k_n}\bigcap_{t\in F} (tm_F^{-1})^{-1}\left(A_t^{(i,n)}\right)\right].$$

Then, for each $t \in F$ define $x_t := tm_F^{-1}\cdot x$. We would have that for all $n \geq 1$ there is a $1 \leq i_n \leq k_n$ such that $x_t \in A_t^{(i_n,n)}$ for all $t \in F$, meaning

$$(x_t)_{t\in F} \in \prod_{t\in F} A_t^{(i_n,n)} \subseteq \bigsqcup_{i=1}^{k_n}\prod_{t\in F} A_t^{(i,n)}$$

for all $n \geq 1$, which contradicts the fact that $A_n \downarrow \varnothing$. Thus, the intersection was empty, and by continuity of μ we conclude

$$\lim_{n\to\infty} \mu_F(A_n) = \lim_{n\to\infty} \mu\left(\bigsqcup_{i=1}^{k_n}\bigcap_{t\in F} (tm_F^{-1})^{-1}\left(A_t^{(i,n)}\right)\right) = 0.$$

□

By applying Carathéodory's Extension Theorem, we obtain a unique extension of μ_F to the σ-algebra $\sigma(\mathscr{C}_F) = \mathscr{B}(X^F)$ generated by $\mathscr{C}_F$.

Corollary 2.7.5. *Let $F \subseteq G$ be a finite subset and $m_F \leq_S t$ for all $t \in F$. Then, there is a unique probability measure $\mu_F \colon \mathscr{B}(X^F) \to [0,1]$ such that*

$$\mu_F\left(\prod_{t\in F} A_t\right) = \mu\left(\bigcap_{t\in F}(tm_F^{-1})^{-1}(A_t)\right)$$

for every $\prod_{t\in F} A_t \in \mathscr{B}(X^F)$.

We now want to extend this collection of measures to a measure on X^G, via Kolmogorov's Extension Theorem. Denote by $\mathcal{F}(G)$ the collection of all finite subsets of G. If $F \subseteq K \subseteq G$, we define $\pi_F^K \colon X^K \to X^F$ as the canonical projection, and we omit the super-index if $K = G$. Recall that a family of measures $\{\nu_F \colon \mathscr{B}(X^F) \to [0,1] \mid F \in \mathcal{F}(G)\}$ is called **consistent** if whenever $F \subseteq K \in \mathcal{F}(G)$, we have $(\pi_F^K)_*\mu_K = \mu_F$.

Lemma 2.7.6. *The family of probability measures $\{\mu_F : F \in \mathcal{F}(G)\}$ is consistent.*

Proof. Let $F \subseteq K \in \mathcal{F}(G)$. Then, a lower bound m_K for K is a lower bound for F as well. Let $\prod_{t\in F} A_t$ be an arbitrary element of $\mathscr{C}_F$, and define, for $g \in K - F$, $A_g = X$. Then,

$$\begin{aligned}\left(\pi_F^K\right)_* \mu_K\left(\prod_{t\in F} A_t\right) &= \mu_K\left(\prod_{g\in K} A_g\right) = \mu\left(\bigcap_{g\in K}(gm_K^{-1})^{-1}(A_g)\right)\\ &= \mu\left(\bigcap_{t\in F}(tm_K^{-1})^{-1}(A_t)\right) = \mu_F\left(\prod_{t\in F} A_t\right).\end{aligned}$$

Now, the sets in $\mathscr{B}(X^F)$ which satisfy the formula $\left(\pi_F^K\right)_* \mu_K = \mu_F$ form a σ-algebra, which implies the result for all sets in $\mathscr{B}(X^F)$. $\square$

By Kolmogorov's Extension Theorem, we obtain a unique probability measure $\overline{\mu}_G$ on $\mathscr{B}(X^G)$ satisfying the condition $\mu_F = \left(\pi_F^G\right)_* \overline{\mu}_G$ for every finite subset $F \subseteq G$. This allows to establish the following result.

Theorem 2.7.7. *The measure $\overline{\mu}_G$ is G-invariant, and we have $\overline{\mu}_G(X_G) = 1$. Therefore, $\mu_G := \overline{\mu}_G|_{\mathscr{B}(X_G)}$ is a G-invariant probability measure, and $\pi_*\mu_G = \mu$.*

Proof. To show G-invariance of $\overline{\mu}_G$, let $g \in G$ and define $\nu = g_*\overline{\mu}_G$. It suffices to check that, for a finite subset $F \subseteq G$, $\mu_F = \left(\pi_F^G\right)_* \nu$ on cylinders, so that by the uniqueness granted by Kolmogorov's Extension Theorem we get $\nu = \overline{\mu}_G$.

It can be easily verified that, if m_F is a lower bound for F, then so is $m_F g$ for Fg. Now let $\prod_{t\in F} A_t$ be any element of $\mathscr{C}_F$, and define $A_g = X$ for $g \in G - F$. We obtain the following.

$$\begin{aligned}\left(\pi^G_F\right)_* \nu\left(\prod_{t\in F} A_t\right) &= \overline{\mu}_G\left(g^{-1}\prod_{t\in G} A_t\right) = \overline{\mu}_G\left(\prod_{t\in Fg} A_{tg^{-1}}\right)\\ &= \mu_{Fg}\left(\prod_{t\in Fg} A_{tg^{-1}}\right) = \mu\left(\bigcap_{t\in Fg}(tg^{-1}m_F^{-1})^{-1}A_{tg^{-1}}\right)\\ &= \mu\left(\bigcap_{t\in F}(tm_F{}^{-1})^{-1}A_t\right) = \mu_F\left(\prod_{t\in F} A_t\right).\end{aligned}$$

Now we prove X_G has full measure. We already know X_G is closed, thus measurable. Now, as it can be written as

$$X_G = \bigcap_{t\in G}\bigcap_{s\in S}\Big\{(x_t)_{t\in G} \in X^G : s\cdot x_t = x_{st}\Big\},$$

defining, for each $s \in S$ and $t \in G$, $A_{s,t} = \{(x_t)_{t\in G} \in X^G : s\cdot x_t = x_{st}\}$ (which is also closed, by continuity of $S \curvearrowright X$ and of the coordinate projections), it suffices to check that each $A_{s,t}$ has full measure. To see this, fix $s \in S$, $t \in G$. The set $X^G - A_{s,t}$ is open, and can hence be written as a countable union of cylinders of X^G (X^G is second countable):

$$X^G - A_{s,t} = \bigcup_{n\geq 1}\prod_{g\in G} C^{(n)}_g,$$

where for each n, $C^{(n)}_g = X$ if $g \in G - F_n$. We must have that $C^{(n)}_t \cap s^{-1}\Big(C^{(n)}_{st}\Big) = \varnothing$ (in particular $t, st \in F_n$ for every $n \geq 1$), and therefore, for all $n \geq 1$,

$$\left(tm_{F_n}^{-1}\right)^{-1}\left[C^{(n)}_t \cap s^{-1}\Big(C^{(n)}_{st}\Big)\right] = \left(tm_{F_n}^{-1}\right)^{-1}\left(C^{(n)}_t\right) \cap \left(stm_{F_n}^{-1}\right)^{-1}\Big(C^{(n)}_{st}\Big) = \varnothing.$$

This last fact directly implies that

$$\overline{\mu}_G\left(X_G - A_{s,t}\right) \leq \sum_{n\geq 1}\mu_{F_n}\left(\prod_{h\in F_n} C^{(n)}_h\right) = \sum_{n\geq 1}\mu\left(\bigcap_{h\in F_n}(hm_F^{-1})^{-1}\left(C^{(n)}_h\right)\right) = 0,$$

obtaining $\overline{\mu}_G(A_{s,t}) = 1$, as desired.

Finally, let $\mu_G := \overline{\mu}_G|_{\mathscr{B}(X_G)}$. This measure is G-invariant, as $\mu_G(A) = \overline{\mu}_G(A)$ for every $A \in \mathscr{B}(X_G)$. Since $(\pi^G_F)_*\overline{\mu}_G = \mu_F$ and $\pi^G_{1_G}(\overline{x}) = \pi(\overline{x})$ for every $\overline{x} \in X_G$, we have

$$\mu_G(\pi^{-1}(A)) = \mu_G\Big([\pi^G_{1_G}]^{-1}(A) \cap X_G\Big) = \overline{\mu}_G\Big([\pi^G_{1_G}]^{-1}(A)\Big) = \mu(A)$$

for all $A \in \mathscr{B}(X_G)$. □

Chapter 3

Measure extensions of $\mathbb{F}_n^+$-shifts

The matter of extending p.m.p. actions of reversible semigroups to their groups of right fractions is settled. Nevertheless, there are interesting actions which do not fall under this category, such as $\mathbb{F}_n^+$-full shifts. The aim of this chapter is to show that every $\mathbb{F}_n^+$-invariant measure on the $\mathbb{F}_n^+$-full shift is $\mathbb{F}_n$-extensible.

We introduce some notation which will be used throughout the chapter. Let $\mathcal{A}$ be a finite alphabet, and $n \geq 2$. Consider the free semigroup $\mathbb{F}_n^+$ on the generators $a_1, \dots, a_n$, and the shift action $\mathbb{F}_n^+ \curvearrowright \mathcal{A}^{\mathbb{F}_n^+}$. When we refer to the Cayley graph of $\mathbb{F}_n^+$, we mean the right Cayley graph, i.e., edges correspond to elements $(t, a_i t)$ for some $t \in \mathbb{F}_n^+$ and $1 \leq i \leq n$. For notational economy, the identity $1_{\mathbb{F}_n^+}$ will be denoted by ε, and the ball of radius r with center on the identity ε (corresponding to the word metric associated to $a_1, \dots, a_n$) will be denoted by B_r. We introduce the following definitions as well.

- A finite subset $T \subseteq \mathbb{F}_n^+$ will be called a **finite subtree** of $\mathbb{F}_n^+$ (sometimes referred to as finite tree as well) if the induced subgraph $(T, E(T))$ of $\mathrm{Cay}(\mathbb{F}_n^+) = (\mathbb{F}_n^+, E)$, where $E(T) = \{(t, t') \in E : t, t' \in T\}$, is a finite tree.
- Let T be a finite subtree. An element $t \in T$ will be called a **leaf** if $a_i t \notin T$ for every $1 \leq i \leq n$. The set of leaves of T will be denoted by $L(T)$.
- The **root** of a finite subtree T will be the unique element $r(T) \in T$ of minimal distance to ε with respect to the word metric, and the depth of an element $t \in T$ will be the distance to $r(T)$ in the same metric.
- For each element $t \in T$, there is a unique path

  $$\mathfrak{p}(t) = \{r(T), a_{i_1} r(T), \dots, a_{i_k} \cdots a_{i_1} r(T) = t\}$$

  joining $r(T)$ and t. For $m \geq 0$, the subpath of $\mathfrak{p}(t)$ whose vertices lie within the ball $B(t, m)$ will be denoted by $\mathfrak{p}_m(t)$.

3.1 Markov shifts on $\mathbb{F}_n^+$

A really important class of $\mathbb{Z}$-shifts are Markov shifts. A (p, P)-Markov shift is a p.m.p. action $\mathbb{Z} \curvearrowright (\mathcal{A}^{\mathbb{Z}}, \mu)$, where μ is given on cylinders by

$$\mu([x\,;k,\dots,k+n]) = \mathbf{p}_{x(k)}\mathbf{P}_{x(k),x(k+1)}\cdots\mathbf{P}_{x(k+n-1),x(k+n)},$$

where $\mathbf{p}$ and $\mathbf{P}$ are a probability vector of length $|\mathcal{A}|$ and a stochastic $|\mathcal{A}|\times|\mathcal{A}|$ matrix, respectively, such that $\mathbf{pP} = \mathbf{p}$. These notions will be pertinently defined later. A more detailed account of this class of shifts can be found, for instance, in [Wal81]. One can also define a generalized notion of Markov shift, which has transition probabilities depending on the last m steps (see [Kit97, pp. 163-164], or [VO16, Exercise 7.2.4]). We now want to extend these definitions to free semigroups.

Definition 3.1.1. If $m \geq 0$, a measure $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{F}_n^+}, \mathbb{F}_n^+)$ is **Markov of memory** m (or m-**Markov**) if for every $x \in \mathcal{A}^{\mathbb{F}_n^+}$, every finite subtree $T \subseteq \mathbb{F}_n^+$ and every leaf $t \in L(T)$ of depth $\ell \geq m$ with $\mu([x\,;T-\{t\}]) > 0$, we have

$$\mu\Big([x\,;t]\,\Big|\,[x\,;T-\{t\}]\Big) = \mu\Big([x\,;t]\,\Big|\,[x\,;\mathfrak{p}_m(t)-\{t\}]\Big),$$

where

$$\mu(A\,|\,B) = \frac{\mu(A\cap B)}{\mu(B)}$$

for any pair of sets $A, B \in \mathscr{B}\Big(\mathcal{A}^{\mathbb{F}_n^+}\Big)$ with $\mu(B) > 0$.

Our first goal is to relate general m-Markov measures to 1-Markov measures via a suitable change of the alphabet $\mathcal{A}$. To accomplish this, we start by establishing a useful characterization of 1-Markov measures. A matrix $\mathbf{P} \in \mathrm{Mat}_{|\mathcal{A}|}(\mathbb{R})$ is a **stochastic matrix** if $\mathbf{P} \geq \mathbf{0}$, and for all $k \in \mathcal{A}$,

$$\sum_{\ell\in\mathcal{A}} \mathbf{P}_{k,\ell} = 1.$$

A vector $\mathbf{p} \in \mathbb{R}^{|\mathcal{A}|}$ is a **probability vector** if $\mathbf{p} \geq \mathbf{0}$ and

$$\sum_{\ell\in\mathcal{A}} \mathbf{p}_\ell = 1.$$

Proposition 3.1.2. *A measure $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{F}_n^+}, \mathbb{F}_n^+)$ is 1-Markov if, and only if, there exist a probability vector $\mathbf{p}$ of length $|\mathcal{A}|$ and a family $\{\mathbf{P}^{(i)} : 1 \leq i \leq n\}$ of stochastic $|\mathcal{A}|\times|\mathcal{A}|$ matrices such that*

(i) *$\mathbf{pP}^{(i)} = \mathbf{p}$ for every $1 \leq i \leq n$, i.e., $\mathbf{p}$ is a left eigenvector of each $\mathbf{P}^{(i)}$, and*

(ii) *for every $x \in \mathcal{A}^{\mathbb{F}_n^+}$ and every finite tree T with root $r(T) = \varepsilon$ we have*

$$\mu([x\,;T]) = \mathbf{p}_{x(\varepsilon)} \cdot \prod_{(t,a_it)\in E(T)} \mathbf{P}^{(i)}_{x(t),x(a_it)}.$$

Proof. If μ is a 1-Markov measure, let $\mathbf{p}_\ell = \mu([\ell;\varepsilon])$ for each $\ell \in \mathcal{A}$, and choose any $\boldsymbol{\lambda} \in [0,1]^{|\mathcal{A}|}$ with $\sum_{\ell\in\mathcal{A}} \boldsymbol{\lambda}_\ell = 1$. Define, for $1 \le i \le n$ and $k, \ell \in \mathcal{A}$,

$$(\mathbf{P}^{(i)})_{k,\ell} = \begin{cases} \mu\Big([\ell; a_i] \Big| [k;\varepsilon]\Big) & \text{if } \mu([k;\varepsilon]) \neq 0 \\ \boldsymbol{\lambda}_\ell & \text{if } \mu([k;\varepsilon]) = 0. \end{cases}$$

Nonnegativity of $\mathbf{p}$ and $\mathbf{P}^{(i)}$ for each i follow directly, as well as the fact that $\mathbf{p}$ is a probability vector. To check that each $\mathbf{P}^{(i)}$ is stochastic, let J be the set of all $k \in \mathcal{A}$ such that $\mathbf{p}_k = 0$, and note that if $k \in J$ then

$$\sum_{\ell\in\mathcal{A}} \mathbf{P}^{(i)}_{k,\ell} = \sum_{\ell\in\mathcal{A}} \boldsymbol{\lambda}_\ell = 1,$$

while if $k \notin J$ we get

$$\sum_{\ell\in\mathcal{A}} \mathbf{P}^{(i)}_{k,\ell} = \sum_{\ell\in\mathcal{A}} \frac{\mu([k,\ell\,;\{\varepsilon, a_i\}])}{\mu([k\,;\varepsilon])} = \frac{1}{\mu([k\,;\varepsilon])}\mu([k\,;\varepsilon]) = 1.$$

On the other hand, to see $\mathbf{p}\mathbf{P}^{(i)} = \mathbf{p}$, fix $\ell \in \mathcal{A}$. Then

$$\sum_{k\in\mathcal{A}} \mathbf{p}_k \mathbf{P}^{(i)}_{k,\ell} = \sum_{k\in J^c} \mu([k,\ell\,;\{\varepsilon, a_i\}]) = \sum_{k\in\mathcal{A}} \mu([k,\ell\,;\{\varepsilon, a_i\}]) = \mu([\ell\,;a_i]) = \mathbf{p}_\ell.$$

Let us check property (ii) now. If $x \in \mathcal{A}^{\mathbb{F}_n^+}$, $T \subseteq \mathbb{F}_n^+$ is a finite subtree with root ε, $t \in L(T)$ is not the root and $s \in T$ is such that $t = a_i s$, we have two options. If $\mathbf{p}_{x(s)} = \mu([x\,;s]) = 0$, then $\mu([x\,;T]) = 0$ and the statement holds true. Otherwise, by definition of 1-Markov measure we have

$$\begin{aligned} \mu([x\,;T]) &= \mu([x\,;T - \{t\}) \cdot \frac{\mu([x\,;\{s, a_i s\}])}{\mu([x\,;s])} \\ &= \mu([x\,;T - \{t\}) \cdot \frac{\mu(s^{-1}[s\cdot x\,;\{\varepsilon, a_i\}])}{\mu(s^{-1}[s\cdot x\,;\varepsilon])} \\ &= \mu([x\,;T - \{t\}]) \cdot \mathbf{P}^{(i)}_{x(s),x(a_i s)}. \end{aligned}$$

The statement is thus proved by iterating this argument.

Conversely, let $\mathbf{p}$ and $\{\mathbf{P}^{(i)} : 1 \le i \le n\}$ be such as in the statement of the proposition, and let μ be defined on cylinders of the form $[x\,;T]$ (with T a finite tree) as in (ii). We proceed in several steps.

Step 1. We want to extend μ to the whole σ-algebra. Define the measure of an arbitrary finitely-supported cylinder $[x\,;F]$ as

$$\mu([x\,;F]) = \sum_{y\in\mathcal{A}^{T-F}} \mu([y \wedge x\,;T]) = \sum_{y\in\mathcal{A}^{T-F}} \mu([y\,;T-F] \cap [x\,;F]),$$

where T is any finite tree with root ε containing F. Note that this is well defined: if T is such a tree and $t \in T$ is a leaf, then, writing $T' = T \cup \{a_j t\}$ we have

$$\begin{aligned}
\sum_{y\in\mathcal{A}^{T'-F}} \mu([y\wedge x\,;T']) &= \sum_{y\in\mathcal{A}^{T-F}}\sum_{\ell\in\mathcal{A}} \mathbf{p}_{y(\varepsilon)}\cdot \prod_{(g,a_ig)\in E(T)} \mathbf{P}^{(i)}_{(y\wedge x)(g),(y\wedge x)(a_ig)}\mathbf{P}^{(j)}_{(y\wedge x)(t),\ell} \\
&= \sum_{y\in\mathcal{A}^{T-F}} \mathbf{p}_{y(\varepsilon)}\cdot \prod_{(g,a_ig)\in E(T)} \mathbf{P}^{(i)}_{(y\wedge x)(g),(y\wedge x)(a_ig)}\left(\sum_{\ell\in\mathcal{A}}\mathbf{P}^{(j)}_{(y\wedge x)(t),\ell}\right) \\
&= \sum_{y\in\mathcal{A}^{T-F}} \mu([y\wedge x\,;T]),
\end{aligned}$$

so recursively (it can be thought of as if we were able to "prune" the tree to get a minimal tree containing F) we find that $\mu([x\,;F])$ is independent of the choice of T. Now, we want to see that μ is finitely additive on cylinders. If $[x\,;F]$ writes as a finite union

$$[x\,;F] = \bigsqcup_{i=1}^{N}[x\wedge x_i\,;F_i]$$

with $F \subsetneq F_i$ for each i, then fixing a tree T with root ε such that $F_i \subseteq T$ for all i, we have each $[x\wedge x_i\,;F_i]$ can be decomposed as $\bigsqcup_{j=1}^{N_i}[x\wedge x_i\wedge y^i_j\,;T]$, where $\{y^i_j : 1 \leq j \leq N_i\} = \mathcal{A}^{T-F_i}$. Now, since all of these unions are disjoint, if $x\wedge x_i\wedge y^i_j = x\wedge x_k\wedge y^k_\ell$ then $x_i\wedge y^i_j = x_k\wedge y^k_\ell$, which implies $F_i = F_k$ and $x_i = x_k$, thus $i = k$, and $y^i_j = y^k_\ell = y^i_\ell$, so $j = \ell$. This is because $F_i\cap F_k\cap F^{\mathrm{c}} \neq \varnothing$ (otherwise the first union would not be disjoint) and x_i must differ from x_k on this last set if $i = k$ (by the same reason). In other words, if $(i,j) \neq (k,\ell)$ then $x_i\wedge y^i_j$ and $x_k\wedge y^k_\ell$ are different $\mathcal{A}$-colorings of $T-F$. Also, if y is any $\mathcal{A}$-coloring of $T-F$ and $\overline{y} \in [x\,;F]\cap[y\,;T-F]$, there are i,j such that $\overline{y} \in [x\wedge x_i\wedge y^i_j\,;T]$. Thus, $y = x_i\wedge y^i_j$, so there is a bijection $\mathcal{A}^{T-F} \leftrightarrow \{x_i\wedge y^i_j : 1 \leq i \leq N, 1 \leq j \leq N_i\}$. This implies

$$\begin{aligned}
\mu([x\,;F]) &= \sum_{y\in\mathcal{A}^{T-F}}\mu([x\wedge y\,;T]) = \sum_{i,j}\mu([x\wedge x_i\wedge y^i_j\,;T]) \\
&= \sum_{i=1}^{n}\sum_{y\in\mathcal{A}^{T-F_i}}\mu([x\wedge x_i\wedge y\,;T]) = \sum_{i=1}^{n}\mu([x\wedge x_i\,;F_i]).
\end{aligned}$$

What we have proven so far is that μ defines a finitely additive probability measure upon the semi-algebra of cylinder sets of $\mathcal{A}^{\mathbb{F}_n^+}$, and so it extends to a finitely additive measure upon the algebra $\mathscr{A}$ generated by the cylinders. Moreover, a cylinder set cannot be expressed as a disjoint union of infinitely many cylinder sets, as cylinders are both open and compact. Thus, μ is σ-additive on cylinders, hence upon $\mathscr{A}$, so it extends to a unique σ-additive measure in the Borel σ-algebra.

Step 2. To show μ is $\mathbb{F}_n^+$-invariant, first we prove that any finite tree T (i.e., not necessarily one with the identity as its root) satisfies

$$\mu([x\,;T]) = \mathbf{p}_{x(r(T))}\cdot\prod_{(t,a_it)\in E(T)}\mathbf{P}^{(i)}_{x(t),x(a_it)}.$$

This can be seen in the following way. If r is the root of T and

$$\mathfrak{p} = \{\varepsilon, a_{i_1}, a_{i_2}a_{i_1}, \ldots, a_{i_\ell}\cdots a_{i_1} = r\}$$

is the path joining ε with r, since $\mathbf{p}\mathbf{P}^{(i)} = \mathbf{p}$ for each i, we get

$$\begin{aligned}
\mu([x\,;T]) &= \sum_{\alpha_0\in\mathcal{A}}\cdots\sum_{\alpha_{\ell-1}\in\mathcal{A}} \mu([\alpha_0,\ldots,\alpha_{\ell-1}\,;\mathfrak{p}(r)-\{r\}]\cap[x\,;T]) \\
&= \left(\sum_{\alpha_0\in\mathcal{A}}\cdots\sum_{\alpha_{\ell-1}\in\mathcal{A}} \mathbf{p}_{\alpha_0}\prod_{j=1}^{\ell-1}\mathbf{P}^{(i_j)}_{\alpha_{j-1},\alpha_j}\mathbf{P}^{(i_\ell)}_{\alpha_{\ell-1},x(r)}\right)\prod_{(t,a_it)\in E(T)}\mathbf{P}^{(i)}_{x(t),x(a_it)} \\
&= \left(\sum_{\alpha_1=0}^{k-1}\cdots\sum_{\alpha_{\ell-1}=0}^{k-1}\left(\sum_{\alpha_0=0}^{k-1}\mathbf{p}_{\alpha_0}\mathbf{P}^{(i_1)}_{\alpha_0,\alpha_1}\right)\prod_{j=2}^{\ell-1}\mathbf{P}^{(i_j)}_{\alpha_{j-1},\alpha_j}\mathbf{P}^{(i_\ell)}_{\alpha_{\ell-1},x(r)}\right)\prod_{(t,a_it)\in E(T)}\mathbf{P}^{(i)}_{x(t),x(a_it)} \\
&= \left(\sum_{\alpha_1=0}^{k-1}\cdots\sum_{\alpha_{\ell-1}=0}^{k-1}\mathbf{p}_{\alpha_1}\prod_{j=2}^{\ell-1}\mathbf{P}^{(i_j)}_{\alpha_{j-1},\alpha_j}\mathbf{P}^{(i_\ell)}_{\alpha_{\ell-1},x(r)}\right)\prod_{(t,a_it)\in E(T)}\mathbf{P}^{(i)}_{x(t),x(a_it)} \\
&\;\vdots \\
&= \mathbf{p}_{x(r)}\cdot\prod_{(g,a_ig)\in E(T)}\mathbf{P}^{(i)}_{x(g),x(a_ig)}.
\end{aligned}$$

Now recall that the preimage of a cylinder satisfies, for every $s \in \mathbb{F}_n^+$, $x \in \mathcal{A}^{\mathbb{F}_n^+}$ and finite $F \subseteq \mathbb{F}_n^+$,

$$s^{-1}[x\,;F] = [y\,;Fs],$$

where $y \in \mathcal{A}^{\mathbb{F}_n^+}$ is any element satisfying $s\cdot y = x$. Hence, we have for a tree T with root r that

$$\begin{aligned}
\mu(s^{-1}[x\,;T]) &= \mathbf{p}_{y(rs)}\cdot\prod_{(ts,a_its)\in E(Ts)}\mathbf{P}^{(i)}_{y(ts),y(a_its)} \\
&= \mathbf{p}_{(s\cdot y)(r)}\cdot\prod_{(t,a_it)\in E(T)}\mathbf{P}^{(i)}_{(s\cdot y)(t),(s\cdot y)(a_it)} = \mu([x\,;T]).
\end{aligned}$$

Since it suffices to show invariance upon the generators of a σ-algebra and cylinder sets over finite trees generate the Borel σ-algebra, we conclude μ is $\mathbb{F}_n^+$-invariant.

Step 3. Let us see that μ is a 1-Markov measure. Let $x \in \mathcal{A}^{\mathbb{F}_n^+}$, let T be a finite tree and t a leaf of depth $\ell \geq 1$. Write $\mathfrak{p}_m(t) = \{p_0,\ldots,p_\ell = t\}$. Clearly, if

$$0 < \mu([x\,;T-\{t\}]) = \mathbf{p}_{x(r)}\cdot\prod_{(t',a_it')\in E(T-\{t\})}\mathbf{P}^{(i)}_{x(t'),x(a_it')},$$

then all of the coefficients in the above product are positive, yielding

$$\frac{\mu([x\,;T])}{\mu([x\,;T-\{t\}])} = \mathbf{P}^{(i)}_{x(p_{\ell-1}),x(t)} = \frac{\mu([x\,;\mathfrak{p}_m(t)])}{\mu([x\,;\mathfrak{p}_m(t)-\{t\}])}.$$

This shows by definition that μ is 1-Markov. □

We now want to relate m-Markov measures to 1-Markov measures via a suitable change of alphabet. Let $m \geq 1$ and $\mathcal{B}_m = \mathcal{A}^{B_{m-1}}$. Define $\varphi_m \colon \mathcal{A}^{\mathbb{F}_n^+} \to (\mathcal{B}_m)^{\mathbb{F}_n^+}$ by $\varphi_m(x)(s) = (s \cdot x)|_{B_{m-1}}$ for every $x \in \mathcal{A}^{\mathbb{F}_n^+}$. This map is injective: if $x, y \in \mathcal{A}^{\mathbb{F}_n^+}$ differ on a coordinate $s \in \mathbb{F}_n^+$, then $s \cdot x$ and $s \cdot y$ differ at $\varepsilon \in B_{m-1}$, so $\varphi_m(x) \neq \varphi_m(y)$. Also, if $s, t \in \mathbb{F}_n^+$ and $x \in \mathcal{A}^{\mathbb{F}_n^+}$ then

$$\varphi_m(t \cdot x)(s) = (st \cdot x)|_{B_{m-1}} = \varphi_m(x)(st) = (t \cdot \varphi_m(x))(s),$$

so $t \cdot \varphi_m(x) = \varphi_m(t \cdot x)$ and hence φ_m is an injective $\mathbb{F}_n^+$-equivariant function. Finally, φ_m is clearly continuous: if $x_i \to x$ in $\mathcal{A}^{\mathbb{F}_n^+}$ and $F \subseteq \mathbb{F}_n^+$ is finite, then for sufficiently large i we have $x_i(s) = x(s)$ for all $s \in B_{m-1}F$, implying $\varphi_m(x_i)(s) = \varphi(x)(s)$ for all $s \in F$. Thus $\varphi_m(x_i) \to \varphi(x)$. In particular, φ_m is Borel measurable and closed. Define

$$\Lambda_m = \left\{ \nu \in \mathcal{M}_{\mathbb{F}_n^+}\left((\mathcal{B}_m)^{\mathbb{F}_n^+} \right) : \operatorname{supp}(\nu) \subseteq \operatorname{im}(\varphi_m) \right\}.$$

Proposition 3.1.3. *There exists an $\mathbb{F}_n^+$-equivariant Borel map $\psi_m \colon (\mathcal{B}_m)^{\mathbb{F}_n^+} \to \mathcal{A}^{\mathbb{F}_n^+}$ such that $\psi_m \circ \varphi_m(x) = x$ for all $x \in \mathcal{A}^{\mathbb{F}_n^+}$. Furthermore, we have the following:*

(i) *for every $\mu \in \mathcal{M}_{\mathbb{F}_n^+}\left(\mathcal{A}^{\mathbb{F}_n^+}\right)$, we have $(\psi_m)_* \circ (\varphi_m)_* \mu = \mu$,*

(ii) *if, moreover, μ is m-Markov, then $(\varphi_m)_*\mu$ is 1-Markov and belongs to Λ_m,*

(iii) *for every $\nu \in \Lambda_m$ and $A \in \mathscr{B}\left(\mathcal{A}^{\mathbb{F}_n^+}\right)$, $(\psi_m)_*\nu(A) = \nu(\varphi_m(A))$.*

Proof. Define the local function $\Psi_m \colon \mathcal{B}_m \to \mathcal{A}$ by $\Psi_m(u) = u(1_S)$; this defines a sliding block code $\psi_m \colon (\mathcal{B}_m)^{\mathbb{F}_n^+} \to \mathcal{A}^{\mathbb{F}_n^+}$ by $\psi_m(x) = (\Psi_m(x_s))_{s \in S}$. This implies that $\psi_m(\varphi_m(x)) = x$ for every $x \in \mathcal{A}^{\mathbb{F}_n^+}$, and the map ψ_m is equivariant and continuous (hence measurable) as it is defined as a sliding block code. By construction, we immediately get that ψ_m satisfies (i). Note that, for every $A \in \mathcal{B}(\mathcal{A}^{\mathbb{F}_n^+})$, $\varphi_m(A) = \psi_m^{-1}(A) \cap \operatorname{im}(\varphi_m)$. Therefore, if $\nu \in \Lambda_m$ we get

$$\nu(\varphi_m(A)) = \nu(\psi_m^{-1}(A)) = (\psi_m)_*\nu(A),$$

and (iii) is proven.

It remains to prove (ii). We want to see that $\nu = \varphi_*\mu$ is a 1-Markov measure on $\mathcal{A}^{\mathbb{F}_n^+}$ for every m-Markov measure μ on $\mathcal{A}^{\mathbb{F}_n^+}$. Note that

$$\varphi_m^{-1}([x\,;F]) = \bigcap_{t \in F} [x(t)\,; B_{m-1}t]$$

for every $x \in \mathcal{A}^{\mathbb{F}_n^+}$ and finite subset $F \subseteq \mathbb{F}_n^+$. If $T \subseteq \mathbb{F}_n^+$ is a finite subtree, then $\bigcup_{t \in T} B_{m-1}t$ is a finite subtree as well, and if

$$\bigcap_{t \in F} [x(t)\,; B_{m-1}t] \neq \varnothing,$$

then there is a sequence $y \in \mathcal{A}^{\mathbb{F}_n^+}$ such that

$$\bigcap_{t\in F}[x(t)\,;B_{m-1}t] = \left[y\,;\bigcup_{t\in T} B_{m-1}t\right].$$

Thus, if $g \in L(T)$ has depth 1 or greater and $\nu([x\,;T-\{g\}]) > 0$,

$$\frac{\nu([x\,;T])}{\nu([x\,;T-\{g\}])} = \frac{\mu\left(\left[y\,;\bigcup_{t\in T} B_{m-1}t\right]\right)}{\mu\left(\left[y\,;\bigcup_{t\in T-\{g\}} B_{m-1}t\right]\right)} = \prod_{t\in L(B_{m-1}g)} \frac{\mu([y\,;\mathfrak{p}(t)])}{\mu([y\,;\mathfrak{p}(t)-\{t\}])},$$

while,

$$\frac{\nu([x\,;\mathfrak{p}_1(g)])}{\nu([x\,;\mathfrak{p}_1(g)-\{g\}])} = \frac{\mu\left(\left[y\,;\bigcup_{t\in\mathfrak{p}(g)} B_{m-1}t\right]\right)}{\mu\left(\left[y\,;\bigcup_{t\in\mathfrak{p}(g)-\{g\}} B_{m-1}t\right]\right)} = \prod_{t\in L(B_{m-1}g)} \frac{\mu([y\,;\mathfrak{p}(t)])}{\mu([y\,;\mathfrak{p}(t)-\{t\}])}.$$

Finally, we check that $\nu = (\varphi_m)_*\mu$ satisfies $\operatorname{supp}(\nu) \subseteq \operatorname{im}(\varphi_m)$. Indeed, φ_m is continuous, so $\operatorname{im}(\varphi_m)$ is a closed subset of $\mathcal{A}^{\mathbb{F}_n^+}$. Take any $y \in \mathcal{A}^{\mathbb{F}_n^+} - \operatorname{im}(\varphi_m)$. Then, there is an open neighborhood U of y contained in $\mathcal{A}^{\mathbb{F}_n^+} - \operatorname{im}(\varphi_m)$, which implies $\nu(U) = \mu(\varphi_m^{-1}(U)) = \mu(\varnothing) = 0$. Thus, $\operatorname{supp}(\nu) \subseteq \operatorname{im}(\varphi_m)$.

□

Now we have proven that every m-Markov measure on $\mathcal{A}^{\mathbb{F}_n^+}$ is a factor of a 1-Markov measure on $(\mathcal{B}_m)^{\mathbb{F}_n^+}$. Therefore, as $\mathbb{F}_n$-extensibility is preserved under factors of p.m.p. actions (a particular case of Proposition 2.6.5), we obtain the following result.

Corollary 3.1.4. *Markov shifts are $\mathbb{F}_n$-extensible if, and only if, 1-Markov shifts are $\mathbb{F}_n$-extensible.*

3.2 Extensions of Markov shifts

In order to extend 1-Markov $\mathbb{F}_n^+$-shifts to $\mathbb{F}_n$, we need to speak of 1-Markov shifts on $\mathbb{F}_n$ on the first place. However, unlike the case of $\mathbb{F}_n^+$, a difficulty arises as finite subtrees of $\mathbb{F}_n$ may have multiple "dynamical pasts", i.e., multiple minimal elements in the partial order $\leq_{\mathbb{F}_n^+}$, which, recall, is given by

$$g \leq_{\mathbb{F}_n^+} h \iff hg^{-1} \in \mathbb{F}_n^+.$$

This leads to the question on how to assign a measure to cylinders supported on such subtrees. To give us an idea on how to do this, let us explore these notions in $\mathbb{N}$ and $\mathbb{Z}$.

Consider a $(\mathbf{p}, \mathbf{P})$-Markov measure μ on $\mathcal{A}^{\mathbb{Z}}$ with $\mathbf{p} > 0$, and define a set function ν on the cylinders of $\mathcal{A}^{\mathbb{Z}}$ as a 1-Markov measure, by thinking of $\mathbb{Z}$ as a rooted tree (see Figure 3.1), with transition probabilities determined by $\mathbf{P}^{(1)} = \mathbf{P}$ and

$$\mathbf{P}^{(-1)}_{k,\ell} = \frac{\mathbf{p}_\ell}{\mathbf{p}_k}\mathbf{P}_{\ell,k}$$

for $k, \ell \in \mathcal{A}$. In other words, for all $k \leq 0$ and $\ell \geq 0$,

$$\nu([i_k, \ldots, i_0, \ldots i_\ell\,; k, \ldots, \ell]) = \mathbf{p}_{i_0} \left(\mathbf{P}^{(1)}_{i_0,i_{-1}} \ldots \mathbf{P}^{(1)}_{i_{\ell-1},i_\ell}\right) \left(\mathbf{P}^{(-1)}_{i_0,i_1} \ldots \mathbf{P}^{(-1)}_{i_{(k+1)},i_k}\right).$$

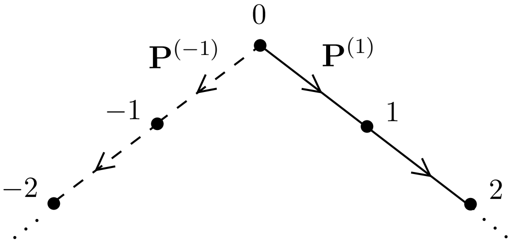


Figure 3.1: The measure ν thought of as a Markov measure on a rooted tree, assigning measure starting from $0 \in \mathbb{Z}$, and applying transition probabilities according to 1 (full lines) and -1 (dashed lines).

Note that if $k \leq 0$ and $\ell \geq 0$

$$\begin{aligned}
\nu([i_k, \ldots, i_\ell\,; k, \ldots, \ell]) &= \mathbf{p}_{i_0} \left(\mathbf{P}^{(-1)}_{i_0,i_{-1}} \ldots \mathbf{P}^{(-1)}_{i_{k+1},i_k}\right) \left(\mathbf{P}^{(1)}_{i_0,i_1} \ldots \mathbf{P}^{(1)}_{i_{\ell-1},i_\ell}\right) \\
&= \mathbf{p}_{i_0} \left(\frac{\mathbf{p}_{i_{-1}}}{\mathbf{p}_{i_0}} \mathbf{P}_{i_{-1},i_0} \ldots \frac{\mathbf{p}_{i_k}}{\mathbf{p}_{i_{k+1}}} \mathbf{P}_{i_k,i_{k+1}}\right) \left(\mathbf{P}_{i_0,i_1} \ldots \mathbf{P}_{i_{\ell-1},i_\ell}\right) \\
&= \mathbf{p}_{i_k} \mathbf{P}_{i_k,i_{k+1}} \ldots \mathbf{P}_{i_{-1},i_0} \ldots \mathbf{P}_{i_{\ell-1},i_\ell} \\
&= \mu([i_k, \ldots, i_\ell\,; k, \ldots, \ell]).
\end{aligned}$$

Thus, μ and ν extend to the same measure on the σ-algebra. In particular, ν is $\mathbb{Z}$-invariant. Also note that the matrix $\mathbf{P}^{(-1)}$ is also stochastic and $\mathbf{p}$ is a left eigenvector. In other words, what we have just shown is that the usual 1-Markov shift on $\mathcal{A}^{\mathbb{Z}}$ can be understood as a tree 1-Markov shift, where the matrix associated to -1 is determined from the matrix associated to 1.

Generalizing this idea, we want to define 1-Markov measures on free groups by changing the way we think of the Cayley graph of $\mathbb{F}_n$. Just like in the case of $\mathbb{F}_n^+$, let $a_1, \ldots, a_n$ be the generators of $\mathbb{F}_n$. We will consider a directed graph $\Gamma = (V, E)$, where $V = \mathbb{F}_n$, and instead of having edges $(g, a_i g)$ for every $g \in \mathbb{F}_n$ and $1 \leq i \leq n$, the directed edges will follow the direction of growth of the distance $d(\varepsilon, g)$ in the word metric associated with $a_1, \ldots, a_n$. Put in a more precise way:

$$(g, a_i g) \in E \iff d(\varepsilon, g) < d(\varepsilon, a_i g),$$

i.e., if, and only if, g does not start with a_i^{-1}. In particular, if $(g, a_i g) \in E$, then necessarily $(a_i^{-1} g, g) \notin E$.

With this definition, it is clear that a finite subtree $T \subseteq \mathbb{F}_n$ has a unique root, namely, the unique element $r(T)$ at minimum distance from ε. All the notation introduced for $\mathbb{F}_n^+$ will be the same.

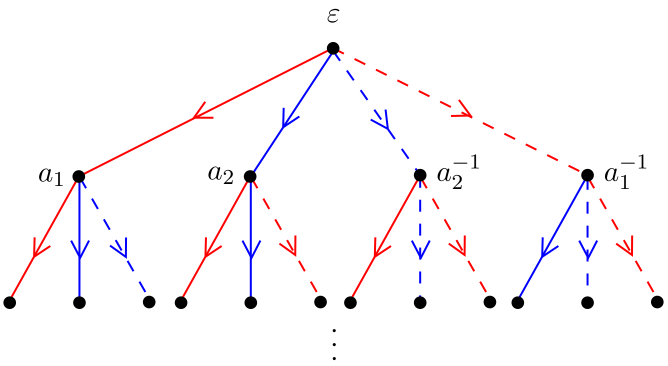


Figure 3.2: How we think the Cayley graph of $\mathbb{F}_2$ for the definition of a 1-Markov measure. Dashed lines represent inverses of the generators of $\mathbb{F}_n$.

Definition 3.2.1. A measure $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{F}_n})$ is 1-**Markov** if there exist a probability vector $\mathbf{p}$ of length $|\mathcal{A}|$ and a family of $|\mathcal{A}| \times |\mathcal{A}|$ stochastic matrices $\{\mathbf{P}^{(i)} : 1 \leq i \leq n\}$ such that

(i) $\mathbf{p}\mathbf{P}^{(i)} = \mathbf{p}$ for every $1 \leq i \leq n$, and

(ii) for every $x \in \mathcal{A}^{\mathbb{F}_n}$ and every finite tree T with root ε we have

$$\mu([x\,;T]) = \mathbf{p}_{x(\varepsilon)} \cdot \prod_{(g,a_i g)\in E(T)} \mathbf{P}^{(i)}_{x(g),x(a_i g)} \cdot \prod_{(g,a_i^{-1}g)\in E(T)} \mathbf{P}^{(-i)}_{x(g),x(a_i^{-1}g)},$$

where $\mathbf{P}^{(-i)}$ is the **reversed transition probability for** $\mathbf{P}^{(i)}$, i.e.,

$$\mathbf{P}^{(-i)}_{k,\ell} = \begin{cases} \dfrac{\mathbf{p}_\ell}{\mathbf{p}_k}\mathbf{P}^{(i)}_{\ell,k} & \text{if } \mathbf{p}_k \neq 0 \\ \boldsymbol{\lambda}_\ell & \text{if } \mathbf{p}_k = 0, \end{cases}$$

for any choice of $\boldsymbol{\lambda} \in [0,1]^{|\mathcal{A}|}$ with $\sum_{\ell\in\mathcal{A}} \boldsymbol{\lambda}_\ell = 1$.

By the exact same argument shown for 1-Markov measures on $\mathbb{F}_n^+$, condition (ii) in the definition suffices to define a measure upon all Borel sets of $\mathcal{A}^{\mathbb{F}_n}$, so a unique measure upon $\mathscr{B}(\mathcal{A}^{\mathbb{F}_n})$ is determined by said condition. It is not obvious, however, whether this definition yields an $\mathbb{F}_n$-invariant measure.

Proposition 3.2.2. *A* 1*-Markov measure* μ *on* $\mathcal{A}^{\mathbb{F}_n}$ *is* $\mathbb{F}_n$*-invariant.*

Proof. We will first check that $\mathbf{P}^{(-i)}$ is a stochastic matrix and that $\mathbf{p}\mathbf{P}^{(-i)} = \mathbf{p}$ for each i. Let, again, J be the set of all $k \in \mathcal{A}$ such that $\mathbf{p}_\ell = 0$. The first property follows directly: since $\mathbf{p}\mathbf{P}^{(i)} = \mathbf{p}$, if $k \in J^{\mathrm{c}}$,

$$\sum_{\ell \in \mathcal{A}} \mathbf{P}^{(-i)}_{k,\ell} = \sum_{\ell \in \mathcal{A}} \frac{\mathbf{p}_\ell}{\mathbf{p}_k} \mathbf{P}^{(i)}_{\ell,k} = \frac{1}{\mathbf{p}_k} \sum_{\ell \in \mathcal{A}} \mathbf{p}_\ell \mathbf{P}^{(i)}_{\ell,k} = \frac{1}{\mathbf{p}_k} \mathbf{p}_k = 1,$$

and if $k \in J$, $\sum_{\ell \in \mathcal{A}} \mathbf{P}^{(-i)}_{k,\ell} = \sum_{\ell \in \mathcal{A}} \boldsymbol{\lambda}_\ell = 1$. For the second property, note that

$$\sum_{k \in \mathcal{A}} \mathbf{p}_k \mathbf{P}^{(-i)}_{k,\ell} = \sum_{k \in J^{\mathrm{c}}} \mathbf{p}_\ell \mathbf{P}^{(i)}_{\ell,k},$$

which is exactly $\mathbf{p}_\ell$ if $\mathbf{p}_\ell = 0$. If $\ell \in J^{\mathrm{c}}$, note that for each $k \in J$ we have

$$0 = \mathbf{p}_k = \sum_{j \in \mathcal{A}} \mathbf{p}_j \mathbf{P}^{(i)}_{j,k} = \sum_{j \in J^{\mathrm{c}}} \mathbf{p}_j \mathbf{P}^{(i)}_{j,k},$$

which implies $\mathbf{P}^{(i)}_{j,k} = 0$ for each $k \in J$ and $j \in J^{\mathrm{c}}$. In particular, $\mathbf{P}^{(i)}_{\ell,k} = 0$ if $k \in J$, so

$$\sum_{k \in \mathcal{A}} \mathbf{p}_k \mathbf{P}^{(-i)}_{k,\ell} = \sum_{k \in J^{\mathrm{c}}} \mathbf{p}_\ell \mathbf{P}^{(i)}_{\ell,k} = \sum_{k \in \mathcal{A}} \mathbf{p}_\ell \mathbf{P}^{(i)}_{\ell,k} = \mathbf{p}_\ell.$$

Therefore, the same argument used in the proof of invariance for 1-Markov measures on $\mathcal{A}^{\mathbb{F}_n}$ shows that for every $x \in \mathcal{A}^{\mathbb{F}_n}$ and every finite tree T with root r (with r not necessarily ε) we have

$$\mu([x\,;T]) = \mathbf{p}_{x(r)} \cdot \prod_{(g,a_i g) \in E(T)} \mathbf{P}^{(i)}_{x(g),x(a_i g)} \cdot \prod_{(g,a_i^{-1} g) \in E(T)} \mathbf{P}^{(-i)}_{x(g),x(a_i^{-1} g)}.$$

We prove now that

$$\mu(g[x\,;T]) = \mu([g \cdot x\,;Tg^{-1}]) = \mu([x\,;T])$$

for every $g \in \mathbb{F}_n$, $x \in \mathcal{A}^{\mathbb{F}_n}$ and finite subtree $T \subseteq \mathbb{F}_n$ by induction on $|T|$. The case $|T| = 1$ is simple: if $T = \{r\}$ then

$$\mu(g[x\,;T]) = \mu([g \cdot x\,;rg^{-1}]) = \mathbf{p}_{(g \cdot x)(rg^{-1})} = \mathbf{p}_{x(r)} = \mu([x\,;T]).$$

Assume now that $|T| > 1$ and $(h', h) \in E(T)$ with $h \in L(T)$. Then $h = a_i^\eta h'$ for some $1 \le i \le n$ and $\eta \in \{1, -1\}$, where h' does not start with $a_i^{-\eta}$. When we deal with the translation by g^{-1} of (h', h), there arise two possible cases depending on g.

Case 1. Assume hg^{-1} does not start with a_i^η. The following several statements hold.

(1) The edge $(hg^{-1}, h'g^{-1})$ belongs to $E(Tg^{-1})$. Indeed, in this case hg^{-1} does not start by a_i^η, and

$$a_i^{-\eta} hg^{-1} = a_i^{-\eta} a_i^\eta h' g^{-1} = h' g^{-1}.$$

(2) The element hg^{-1} is the root of Tg^{-1}: since h is only connected to h' in T, hg^{-1} is only connected to $h'g^{-1}$ in Tg^{-1}, and we already proved that this edge is directed from the first element to the second.

(3) By induction, $\mu(g \cdot x; (T - \{h\})g^{-1}]) = \mu([x\,; T - \{h\}])$, since $T - \{h\}$ is a finite tree with $|T - \{h\}| = |T| - 1$.

(4) We may assume that $\mathbf{p}_{x(t)} > 0$ for every $t \in T$: otherwise $\mathbf{p}_{x(r)} = 0$ or there is a transition from some k with $\mathbf{p}_k > 0$ to some ℓ with $\mathbf{p}_\ell = 0$, which thus has zero probability, implying $\mu([x\,;T]) = 0$ and $\mu([g \cdot x\,; Tg^{-1}]) = 0$.

Putting all together, we obtain

$$\begin{aligned}\mu([g \cdot x\,; Tg^{-1}]) &= \mathbf{p}_{(g\cdot x)(hg^{-1})}\mathbf{P}^{(-\eta i)}_{(g\cdot x)(hg^{-1}),(g\cdot x)(h'g^{-1})} \cdot \left(\frac{\mu([g \cdot x\,; (T - \{h\})g^{-1}])}{\mathbf{p}_{(g\cdot x)(h'g^{-1})}}\right)\\ &= \frac{\mathbf{p}_{x(h)}}{\mathbf{p}_{x(h')}} \cdot \mathbf{P}^{(-\eta i)}_{x(h),x(h')}\mu([x\,; T - \{h\}])\end{aligned}$$

Finally, note that

$$\frac{\mathbf{p}_{x(h)}}{\mathbf{p}_{x(h')}} \cdot \mathbf{P}^{(-\eta i)}_{x(h),x(h')} = \begin{cases} \dfrac{\mathbf{p}_{x(h)}}{\mathbf{p}_{x(h')}} \cdot \left(\dfrac{\mathbf{p}_{x(h')}}{\mathbf{p}_{x(h)}}\mathbf{P}^{(i)}_{x(h'),x(h)}\right) & \text{if } \eta = 1,\\ \mathbf{P}^{(-i)}_{x(h'),x(h)} & \text{if } \eta = -1 \end{cases} = \mathbf{P}^{(\eta i)}_{x(h'),x(h)}.$$

Therefore,

$$\mu([g \cdot x\,; Tg^{-1}]) = \mathbf{P}^{(\eta i)}_{x(h'),x(h)}\mu([x\,; T - \{h\}]) = \mu([x\,; T]).$$

Case 2. Assume hg^{-1} starts with a_i^η. Then

$$a_i^{-\eta}hg^{-1} = a_i^{-\eta}a_i^\eta h'g^{-1} = h'g^{-1},$$

so $hg^{-1} = a_i^\eta h'g^{-1}$ and $h'g^{-1}$ cannot start by $a_i^{-\eta}$, concluding $(h'g^{-1}, hg^{-1}) \in E(Tg^{-1})$. Also, since multiplying by g^{-1} preserves degrees in T, $hg \in L(Tg^{-1})$. This yields

$$\begin{aligned}\mu([g \cdot x\,; Tg^{-1}]) &= \mu([g \cdot x\,; (T - \{h\})g^{-1}])\mathbf{P}^{(\eta i)}_{(g\cdot x)(h'g^{-1}),(g\cdot x)(hg^{-1})}\\ &= \mu([x\,; T - \{h\}])\mathbf{P}^{(\eta i)}_{x(h'),x(h)}\\ &= \mu([x\,; T]).\end{aligned}$$

Cylinder sets upon finite trees generate the Borel σ-algebra, therefore μ is $\mathbb{F}_n$-invariant. □

Remark 3.2.3. A few properties worth keeping in mind from the last proof are listed here. Let $\mu \in \mathcal{M}_{\mathbb{F}_n}(\mathcal{A}^{\mathbb{F}_n})$ be a 1-Markov measure defined by a probability vector $\mathbf{p}$ and transition probabilities $\left\{\mathbf{P}^{(i)} : 1 \leq i \leq n\right\}$.

- Each $\mathbf{P}^{(-i)}$ is a stochastic matrix with $\mathbf{p}\mathbf{P}^{(-i)} = \mathbf{p}$.
- If $\mathbf{p}_k > 0$ and $\mathbf{p}_\ell = 0$, then $\mathbf{P}^{(i)}_{k,\ell} = \mathbf{P}^{(-i)}_{k,\ell} = 0$ for all i.
- If $\mathbf{p}_{x(t)} = 0$ for some $x \in \mathcal{A}^{\mathbb{F}_n}$ and $t \in T$, then $\mu([x\,;T]) = 0$.

As a consequence of our last proposition, we obtain the following corollary.

Corollary 3.2.4. *Let $m \geq 0$. Then, every m-Markov $\mathbb{F}_n^+$-shift is $\mathbb{F}_n$-extensible.*

Proof. We already proved that every m-Markov shift is a factor of a 1-Markov shift, and so by Proposition 2.6.5 it suffices to prove every 1-Markov shift is $\mathbb{F}_n$-extensible. However, given any such measure $\mu \in \mathcal{M}_{\mathbb{F}_n^+}\left(\mathcal{A}^{\mathbb{F}_n^+}\right)$ defined by a probability vector $\mathbf{p}$ and transition probabilities $\left\{\mathbf{P}^{(i)} : 1 \leq i \leq n\right\}$, we have just proven that there is an $\mathbb{F}_n$-invariant measure $\bar{\mu}$ on $\mathcal{A}^{\mathbb{F}_n}$ such that $\bar{\mu}([x\,;F]) = \mu\left(\left[x|_{\mathbb{F}_n^+}\,;F\right]\right)$ for every $x \in \mathcal{A}^{\mathbb{F}_n}$ and finite subtree $T \subseteq \mathbb{F}_n^+$, i.e., $\rho_*\bar{\mu}$ and μ are the same function on the class of cylinders upon finite subtrees, where $\rho\colon \mathcal{A}^{\mathbb{F}_n} \to \mathcal{A}^{\mathbb{F}_n^+}$ is the restriction map $x \mapsto x|_{\mathbb{F}_n^+}$. Therefore, as μ is defined upon the whole σ-algebra, both set functions extend to the same measure. Hence $\rho_*\bar{\mu} = \mu$, as desired. □

3.3 A dense family of $\mathbb{F}_n$-extensible measures

In this section we prove that every measure $\mu \in \mathcal{M}_{\mathbb{F}_n^+}(\mathcal{A}^{\mathbb{F}_n^+})$ is $\mathbb{F}_n$-extensible, by showing that there is a dense collection of $\mathbb{F}_n$-extensible measures. Recall that, if $m \geq 1$ and $\mathcal{B}_m = \mathcal{A}^{B_{m-1}}$, there exist $\mathbb{F}_n^+$-equivariant Borel functions $\varphi_m\colon \mathcal{A}^{\mathbb{F}_n^+} \to (\mathcal{B}_m)^{\mathbb{F}_n^+}$ and $\psi_m\colon (\mathcal{B}_m)^{\mathbb{F}_n^+} \to \mathcal{A}^{\mathbb{F}_n^+}$ with $\psi_m \circ \varphi_m(x) = x$ for all $x \in \mathcal{A}^{\mathbb{F}_n^+}$, and such that if

$$\nu \in \Lambda_m = \left\{\nu' \in \mathcal{M}_{\mathbb{F}_n^+}\left((\mathcal{B}_m)^{\mathbb{F}_n^+}\right) : \operatorname{supp}(\nu') \subseteq \operatorname{im}(\varphi_m)\right\},$$

then $(\psi_m)_*\nu(A) = \nu(\varphi_m(A))$ for all $A \in \mathscr{B}(\mathcal{A}^{\mathbb{F}_n^+})$. Denote by Λ_m^1 the subset of Λ_m of all 1-Markov measures.

Proposition 3.3.1. *The family of measures*

$$\mathcal{D} = \bigcup_{m\geq 1} (\psi_m)_*\Lambda_m^1$$

is dense in $\mathcal{M}_{\mathbb{F}_n^+}\left(\mathcal{A}^{\mathbb{F}_n^+}\right)$.

Proof. Fix an $\mathbb{F}_n^+$-invariant measure μ on $\mathcal{A}^{\mathbb{F}_n^+}$. Let $m \geq 1$ and choose any $\boldsymbol{\lambda} \in [0,1]^{|\mathcal{B}_m|}$ such that $\sum_{\alpha\in\mathcal{B}_m} \boldsymbol{\lambda}_\alpha = 1$. Define, for each $\alpha, \beta \in \mathcal{B}_m$ and $1 \leq i \leq n$,

$$\mathbf{p}_\alpha = \mu([\alpha\,;B_{m-1}]),$$

and

$$\mathbf{P}^{(i)}_{\alpha,\beta} = \begin{cases} \dfrac{\mu\Big([\alpha\,;B_{m-1}] \cap a_i^{-1}[\beta\,;B_{m-1}]\Big)}{\mathbf{p}_\alpha} & \text{if } \mathbf{p}_\alpha > 0, \\ \boldsymbol{\lambda}_\beta & \text{if } \mathbf{p}_\alpha = 0. \end{cases}$$

It is clear that $\mathbf{p}$ is a probability vector, and for each $\alpha \in \mathcal{B}_m$ with $\mathbf{p}_\alpha > 0$,

$$\sum_{\beta\in\mathcal{B}_m} \mathbf{P}^{(i)}_{\alpha,\beta} = \frac{1}{\mathbf{p}_\alpha} \sum_{\beta\in\mathcal{B}_m} \mu\Big([\alpha\,;B_{m-1}] \cap a_i^{-1}[\beta\,;B_{m-1}]\Big) = 1$$

since the sets $\Big\{a_i^{-1}[\beta; B_{m-1}] : \beta \in \mathcal{B}_m\Big\}$ form a partition of $(\mathcal{B}_m)^{\mathbb{F}_n^+}$. Thus, $\mathbf{P}^{(i)}$ is a stochastic matrix for all $1 \leq i \leq n$. Finally, for each $\beta \in \mathcal{B}_m$ we get

$$\sum_{\alpha\in\mathcal{B}_m} \mathbf{p}_\alpha \mathbf{P}^{(i)}_{\alpha,\beta} = \sum_{\alpha\in\mathcal{B}_m} \mu\Big([\alpha; B_{m-1}] \cap a_i^{-1}[\beta; B_{m-1}]\Big) = \mu\Big(a_i^{-1}[\beta\,;B_{m-1}]\Big) = \mathbf{p}_\beta,$$

so $\mathbf{p}\mathbf{P}^{(i)} = \mathbf{p}$ and we conclude $\mathbf{p}$ and $\{\mathbf{P}^{(i)} : 1 \leq i \leq n\}$ define a 1-Markov measure ν_m on $(\mathcal{B}_m)^{\mathbb{F}_n^+}$.

We now want to see that $\operatorname{supp}(\nu_m) \subseteq \operatorname{im}(\varphi_m)$. If $y \in (\mathcal{B}_m)^{\mathbb{F}_n^+} - \operatorname{im}(\varphi_m)$ then there exist $s \in \mathbb{F}_n^+$ and $1 \leq i \leq n$ such that $[y(s)\,;B_{m-1}] \cap a_i^{-1}[y(a_i s)\,;B_{m-1}] = \varnothing$. Indeed, if we assume otherwise, for every $s \in \mathbb{F}_n^+$ and $1 \leq i \leq n$ there is an $x \in [y(s); B_{m-1}] \cap a_i^{-1}[y(a_i s); B_{m-1}]$, so for all $\ell < m-1$ and $a \in B_\ell$,

$$y(s)(aa_i) = x(aa_i) = (a_i \cdot x)(a) = y(a_i s)(a).$$

Now define $z \in \mathcal{A}^{\mathbb{F}_n^+}$ by $z(s) = y(s)(\varepsilon)$. Applying iteratively the identity just proven implies that, for $s \in \mathbb{F}_n^+$ and $t \in B_{m-1}$,

$$(\varphi_m(z)(s))(t) = (s \cdot z)(t) = z(ts) = y(ts)(\varepsilon) = y(s)(t),$$

so $\varphi_m(z) = y$, contradicting that $y \notin \operatorname{im}(\varphi_m)$. Hence, the claim that there exist $s \in \mathbb{F}_n^+$ and $1 \leq i \leq n$ such that $[y(s)\,;B_{m-1}] \cap a_i^{-1}[y(a_i s)\,;B_{m-1}] = \varnothing$ was true, which directly implies that $\nu_m([y\,;\{s, a_i s\}]) = \mathbf{p}_{y(s)}\mathbf{P}^{(i)}_{y(s),y(a_i s)} = 0$, so there is an open set containing y with null measure, i.e., $y \notin \operatorname{supp}(\nu_m)$. Thus $\operatorname{supp}(\nu_m) \subseteq \operatorname{im}(\varphi_m)$.

Let $\mu_m = (\psi_m)_*\nu_m$, i.e., $\mu_m(A) = \nu_m(\varphi_m(A))$ for all $A \in \mathscr{B}(\mathcal{A}^{\mathbb{F}_n^+})$. The measure μ_m is $\mathbb{F}_n$-extensible, as it is a push forward of a 1-Markov measure. We just need to prove that $\mu_m \to \mu$ in the weak-* topology. If $F \subseteq \mathbb{F}_n^+$ is finite and $x \in \mathcal{A}^F$, there is some $m \geq 1$ with $F \subseteq B_{m-1}$, so

$$\varphi_m([x\,;F]) = \varphi_m\left(\bigsqcup_{y\in\mathcal{A}^{B_{m-1}-F}} [y \wedge x\,;B_{m-1}]\right) = \operatorname{im}(\varphi_m) \cap \bigsqcup_{y\in\mathcal{A}^{B_{m-1}-F}} [y\wedge x\,;\varepsilon],$$

whence

$$\begin{aligned}\mu_m([x\,;F]) = \nu_m\left(\bigsqcup_{y\in\mathcal{A}^{B_{m-1}-F}} [y \wedge x\,;\varepsilon]\right) &= \sum_{y\in\mathcal{A}^{B_{m-1}-F}} \mathbf{p}_{y\wedge x} \\ &= \sum_{y\in\mathcal{A}^{B_{m-1}-F}} \mu([y\wedge x\,;B_{m-1}]) = \mu([x\,;F]).\end{aligned}$$

As a result, $\mu_m([x;F]) \to \mu([x;F])$ as $m \to \infty$ for every cylinder, implying

$$\lim_{m\to\infty} \int \phi \, d\mu_m = \int \phi \, d\mu$$

for every simple function ϕ. Since simple functions on $\mathcal{A}^{\mathbb{F}_n^+}$ are continuous and dense in the continuous functions, we have $\mu_m \to \mu$ in the weak-* topology.

$\square$

Corollary 3.3.2. *Every measure* $\mu \in \mathcal{M}_{\mathbb{F}_n^+}(\mathcal{A}^{\mathbb{F}_n^+})$ *is* $\mathbb{F}_n$*-extensible.*

Proof. The set $\mathrm{Ext}_{\mathbb{F}_n}(\mathcal{A}^{\mathbb{F}_n^+}, \mathbb{F}_n^+)$ is weak-* closed, and the family

$$\mathcal{D} = \bigcup_{m\geq 1} (\psi_m)_* \Lambda_m^1 \subseteq \mathcal{M}_{\mathbb{F}_n^+}(\mathcal{A}^{\mathbb{F}_n^+})$$

is dense by the last proposition. Since this family consists of factors of 1-Markov measures, which we have already shown to be $\mathbb{F}_n$-extensible, we obtain that $\mathcal{D} \subseteq \mathrm{Ext}_{\mathbb{F}_n}(\mathcal{A}^{\mathbb{F}_n^+}, \mathbb{F}_n^+)$, so

$$\mathcal{M}_{\mathbb{F}_n^+}(\mathcal{A}^{\mathbb{F}_n^+}) = \overline{\mathcal{D}} = \mathrm{Ext}_{\mathbb{F}_n}(\mathcal{A}^{\mathbb{F}_n^+}, \mathbb{F}_n^+).$$

$\square$

3.4 An example of a non-extensible measure

We have already dealt with non-extensibility in the topological case, but we have not provided an example of a non-extensible measure. Of course, one could simply take a topologically non-partially extensible action $S \curvearrowright X$ admitting an invariant measure: this would yield an invariant measure which trivially cannot be extended, as $X_G = \varnothing$. The more interesting question is whether there exist topologically extensible actions admitting invariant non-extensible measures.

Our example will be a Markovian version of the first example of a topologically non-extensible action. Recall that the Baumslag-Solitar group

$$\mathrm{BS}(1,2) = \langle a, b \mid b^{-1}ab = a^{-1}ba \rangle,$$

which will be called G, contains a copy of $\mathbb{F}_2^+$ as the subsemigroup generated by a, b, and is thus an $\mathbb{F}_2^+$-group.

Instead of considering an $\mathbb{F}_2^+$-subshift, i.e., forcing certain configurations with a set of rules, we will work with the full $\mathbb{F}_2^+$-shift and simply make certain configurations have a higher chance of occurring via a 1-Markov measure. Fix $0 < \eta < 1/2$ and let $\mu \in \mathcal{M}_{\mathbb{F}_2^+}\left(3^{\mathbb{F}_2^+}\right)$ be the 1-Markov measure given by the matrices

$$\mathbf{P}^{(a)} = \begin{pmatrix} \eta & 1-2\eta & \eta \\ \eta & \eta & 1-2\eta \\ 1-2\eta & \eta & \eta \end{pmatrix} \quad \text{and} \quad \mathbf{P}^{(b)} = \begin{pmatrix} \eta & \eta & 1-2\eta \\ 1-2\eta & \eta & \eta \\ \eta & 1-2\eta & \eta \end{pmatrix},$$

and the probability vector $\mathbf{p} = (1/3, 1/3, 1/3)$ (which, note, is a left eigenvector for both $\mathbf{P}^{(a)}$ and $\mathbf{P}^{(b)}$). It is not difficult to see that μ is fully supported. As we have seen, 3^G is the topological extension of $3^{\mathbb{F}_2^+}$ via the restriction map $\rho\colon \overline{x} \mapsto \overline{x}|_{\mathbb{F}_2^+}$.

Assume now that there is a measure $\mu_G \in \mathcal{M}_G(3^G, \rho)$. If $\overline{x} \in 3^G$ and $g, g' \in G$ are such that $g' = sg$ with $s \in \{a, b\}$, then

$$\mu_G([\overline{x}\,;\{g, g'\}]) = \mu_G(g^{-1}[g \cdot \overline{x}\,;\{1_G, s\}]) = \mu\left([\rho(g \cdot \overline{x})\,;\{\varepsilon, s\}]\right) = \mathbf{p}_{\overline{x}(g)}\mathbf{P}^{(s)}_{\overline{x}(g),\overline{x}(g')},$$

since $\rho(g \cdot \overline{x})(t) = \overline{x}(tg)$ for all $t \in \mathbb{F}_2^+$. Consider once more the cycle

$$\mathfrak{c} = \{1_G, b, ab, b^{-1}ab = a^{-1}ba, ba, a, 1_G\}$$

in G illustrated in the figure above.

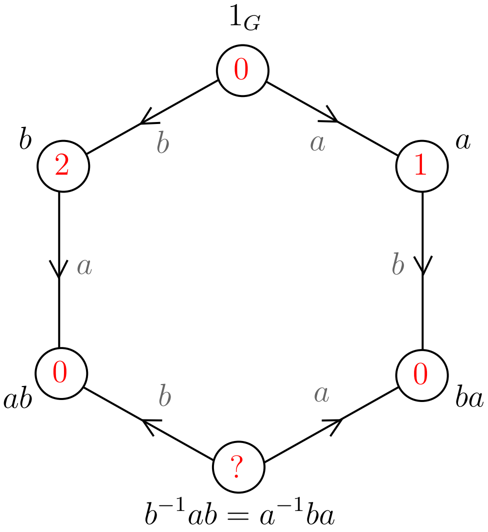


Figure 3.3: The directed structure of the cycle $\mathfrak{c}$ with respect to a and b. In red, it is shown how any colouring of this cycle with the symbols 0, 1 and 2, starting with 0 in 1_G, contains a transition with probability η. The same happens if we start with 1 or 2 at 1_G.

It is easily seen that $\mathfrak{c}$ cannot be coloured with 0, 1 and 2 without containing a transition of probability η, yielding

$$1/3 = \mu([0\,;\varepsilon]) = \mu_G([0\,;1_G]) = \sum_{\tau \in 3^{\mathfrak{c}}:\tau(1_G)=0} \mu_G([\tau;\mathfrak{c}]) \leq \frac{\left|3^{\mathfrak{c}-\{1_G\}}\right|}{3}\eta = 81\eta,$$

which leads to a contradiction if we take $\eta < 1/243$.

Chapter 4

Periodic approximations on groups

We want to discuss an interesting property of semigroups, in which the theory of measure extensions will play a major role, once we have reviewed the group-theoretic version of this property.

First, we need to talk about periodicity and periodic measures for groups. Throughout the chapter, G will denote a countable group.

Definition 4.0.1. Let $G \curvearrowright X$ be a group action. An element $x \in X$ is G-**periodic** if $|Gx| < \infty$, or, equivalently, if $[G : \mathrm{Stab}_G(x)] < \infty$.

The equivalence follows from the orbit-stabilizer theorem, which in a sense tells us that orbits and stabilizers are "two sides of the same coin". Indeed for every $x \in X$, we can define $f_x \colon G \to X$ by $g \mapsto g \cdot x$, so that $\mathrm{Stab}_G(x) = f_x^{-1}(x)$ and $Gx = f_x(G)$. Note that $f_x(g) = f_x(h)$ if and only if $h^{-1}g \in \mathrm{Stab}_G(x)$, so f_x descends to a bijection $G/\mathrm{Stab}_G(x) \to Gx$.

Every G-periodic point x yields a finite-index normal subgroup of G, namely, the stabilizer of x. The relationship between periodicity and finite-index subgroups is, in fact, stronger, as we shall se now.

Definition 4.0.2. Let $H \trianglelefteq G$ be a normal subgroup. A **fundamental domain** of H is a subset $D \subseteq G$ containing exactly one element from each coset of H, so the quotient map $G \to G/H$ restricted to D defines a bijection $D \leftrightarrow G/H$.

Note that if $H \trianglelefteq G$ is a subgroup and D a fundamental domain, then

$$G = \bigsqcup_{d \in D} Hd = \bigsqcup_{d \in D} dH = \bigsqcup_{h \in H} Dh = \bigsqcup_{h \in H} hD.$$

Remark 4.0.3. Let $H \trianglelefteq G$ be a finite index subgroup. A fundamental domain D for H induces G-periodic points in $\mathcal{A}^G$. Take any finite configuration $x \in \mathcal{A}^D$, and define $\overline{x} \in \mathcal{A}^G$ as follows. If $g \in G$, then there are unique elements $d \in D$ and $h \in H$ such that $g = dh$, and we set $\overline{x}(g) = x(d)$. Thus, if g writes as $g = dh$ with $h \in H$ and $d \in D$, for any $h' \in H$, we have

$$(h' \cdot \overline{x})(g) = \overline{x}(dhh') = \overline{x}(d) = \overline{x}(g),$$

yielding $H \subseteq \mathrm{Stab}_G(x)$. Therefore $G\overline{x} = D\overline{x}$ is finite.

We want to speak of periodicity of measures now. Recall that the support $\mathrm{supp}(\mu)$ of a probability measure μ on a Polish space satisfies $\mu(A) = 0$ for every Borel set $A \subseteq X - \mathrm{supp}(\mu)$. Hence, if $\mathrm{supp}(\mu)$ is finite, μ can be written in the form

$$\mu = \sum_{k=1}^{n} \alpha_k \delta_{x_k},$$

with $\sum_{1 \leq k \leq n} \alpha_k = 1$. Indeed, for every $1 \leq k \leq n$ there is an open set $U \subseteq X - \{x_i : i \neq k\}$ containing x_k, so $U - \{x_k\} \subseteq X - \mathrm{supp}(\mu)$ and

$$0 < \mu(U) = \mu(U - \{x_k\}) + \mu(\{x_k\}) = \mu(\{x_k\}),$$

so $\mu(\{x_k\}) > 0$. Take $\alpha_k = \mu(\{x_k\})$. If we request μ to be G-invariant, then $g \cdot x_k \in \{x_1, \ldots, x_n\}$ for every k.

Definition 4.0.4. Let $G \curvearrowright X$ be an action in G-**Top**. A measure $\mu \in \mathcal{M}_G(X)$ is **G-periodic** if $\mathrm{supp}(\mu)$ is finite. In other words, it can be written as

$$\mu = \sum_{k=1}^{n} \alpha_k \frac{1}{|Gx_k|} \sum_{y \in Gx_k} \delta_y,$$

where each x_k is G-periodic and $\sum_{1 \leq k \leq n} \alpha_k = 1$. In this case,

$$\mathrm{supp}(\mu) = \bigcup_{k=1}^{n} Gx_k.$$

For a given action $G \curvearrowright X$, a measure $\mu \in \mathcal{M}_G(X)$ is said to be **ergodic** if every set $A \in \mathscr{B}(X)$ satisfying $\mu(gA \triangle A) = 0$ for all $g \in G$ has measure 0 or 1. We will denote the set of periodic measures of $\mathcal{M}_G(X)$ by $\mathrm{Per}(X, G)$, and the set of periodic ergodic measures by $\mathrm{Perg}(X, G)$.

Definition 4.0.5. A group G is said to have **periodic approximations** (abbreviated PA) if the set $\mathrm{Per}(\mathcal{A}^G, G)$ is weak-* dense in $\mathcal{M}_G(\mathcal{A}^G)$ for every finite alphabet $\mathcal{A}$, and **ergodic periodic approximations** (abbreviated EPA) if the set $\mathrm{Perg}(\mathcal{A}^G, G)$ is weak-* dense in $\mathcal{M}_G(\mathcal{A}^G)$ for every finite alphabet $\mathcal{A}$.

Naturally, the property EPA implies PA; it is still open whether the converse implication holds in the general case. This property has been studied in an equivalent[1]

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[1] The equivalence is stated in [Kec12, p. 467].

formulation by A. Kechris, under the name of "property MD." See [Kec12; BK20] for more details. L. Bowen has contributed to this topic in [Bow03; BTD13]. An important result in [Bow03] (which will be addressed in this thesis) is that $\mathbb{F}_n$ has property PA. We also take care of the fact that every amenable residually finite group has the property PA. Some further details are listed below.

- The group $\mathrm{SL}_n(\mathbb{Z})$ has property PA for $n = 2$, but not for $n \geq 3$ (see [Kec12, p. 466]).
- The group $\mathbb{F}_2 \times \mathbb{F}_2$ does not have property PA, as a consequence of the negative answer to the Connes Embedding Problem (see [BK20; Shr23]).
- Subgroups of groups with property PA have property PA. This is not known for EPA groups, and in fact the statement for EPA implies the equivalence between PA and EPA (see [BK20, p. 27]).
- There is a characterization of this property in terms of free energy densities of measures with respect to sofic approximations. See [Shr23].

4.1 Denseness of periodic measures for $\mathbb{N}$-shifts

We first deal with the classical case, and prove that the set periodic ergodic measures on the full $\mathbb{N}$-shift is dense in the invariant measures. Here, we define

$$\operatorname{Perg}\left(\mathcal{A}^{\mathbb{N}}, \mathbb{N}\right) = \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j x} : x \in X,\ n = \inf\{k > 0 : T^k(x) = x\} \in \mathbb{N} \right\},$$

where $T\colon \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ is the shift map $(Tx)(n) = x(n+1)$, i.e., the associated to $1 \in \mathbb{N}$ in the shift action $\mathbb{N} \curvearrowright \mathcal{A}^{\mathbb{N}}$. Obviously, $\mathbb{N}$ is not a group, but we prefer to give the proof for $\mathbb{N}$ as it is more general, and the same exact argument shows that $\mathbb{Z}$ has property PA. In fact, said argument will be the basis for our proof that a much broader class of groups has the property PA.

Recall that a possible metric compatible with the weak-* topology on the probability measures $\mathcal{M}(X)$ is

$$D(\mu, \nu) = \sum_{k \geq 1} \frac{1}{2^k \|f_k\|_\infty} \left| \int f_k \, d\mu - \int f_k \, d\nu \right|,$$

where $(f_k)_{k \geq 1}$ is a (from now on, fixed) dense sequence of continuous functions. This sequence exists, as the space $(\mathscr{C}(X, \mathbb{R}), \|\cdot\|_\infty)$ is separable for every compact Hausdorff space.

We first prove that $\operatorname{Perg}\left(\mathcal{A}^{\mathbb{N}}, \mathbb{N}\right)$ is dense in the ergodic measures, and then show denseness of ergodic measures on $\mathcal{M}_{\mathbb{N}}\left(\mathcal{A}^{\mathbb{N}}\right)$.

Proposition 4.1.1. *Let $\mathcal{M}_{\mathbb{N}}^{\mathrm{erg}}\Big(\mathcal{A}^{\mathbb{N}}\Big)$ be the set of ergodic invariant probability measures on $\mathcal{A}^{\mathbb{N}}$. Then,*

$$\mathcal{M}_{\mathbb{N}}^{\mathrm{erg}}\Big(\mathcal{A}^{\mathbb{N}}\Big) \subseteq \overline{\mathrm{Perg}(\mathcal{A}^{\mathbb{N}}, \mathbb{N})}.$$

Proof. Let μ be an arbitrary ergodic invariant probability measure, and consider the fixed dense sequence $(f_k)_k$ of continuous functions, which without loss of generality, we may assume does not contain the zero function. By Birkhoff's Theorem, for every $k \geq 1$ we have

$$\frac{1}{n}\sum_{j=0}^{n-1} f_k \circ T^j \to \int f_k \, d\mu$$

μ-a.e. on $\mathcal{A}^{\mathbb{N}}$. Letting C_k be the set where this happens for f_k, we see $\mu\Big(\bigcap_{k\geq 1} C_k\Big) = 1$, and thus there is an $x \in \bigcap_{k\geq 1} C_k$. Now, if $f \in \mathscr{C}(\mathcal{A}^{\mathbb{N}})$ is any continuous function and $\epsilon > 0$, there is a $k \geq 1$ such that $\|f - f_k\|_\infty < \epsilon$. Therefore,

$$\begin{aligned}\left|\frac{1}{n}\sum_{j=0}^{n-1} f \circ T^j(x) - \int f \, d\mu\right| &\leq \left|\frac{1}{n}\sum_{j=0}^{n-1} f \circ T^j(x) - \frac{1}{n}\sum_{j=0}^{n-1} f_k \circ T^j(x)\right| \\ &\quad + \left|\frac{1}{n}\sum_{j=0}^{n-1} f_k \circ T^j(x) - \int f_k \, d\mu\right| + \left|\int f_k \, d\mu - \int f \, d\mu\right| \\ &\leq 2\|f - f_k\| + \epsilon < 3\epsilon\end{aligned}$$

for n large enough. Thus,

$$\frac{1}{n}\sum_{j=0}^{n-1} f \circ T^j(x) \to \int f \, d\mu$$

for every $f \in \mathscr{C}(\mathcal{A}^{\mathbb{N}})$, which implies

$$\nu_n := \frac{1}{n}\sum_{j=0}^{n-1} \delta_{T^j(x)} \xrightarrow{*} \mu \quad \text{as } n \to \infty.$$

The element x need not be $\mathbb{N}$-periodic, and so ν_n might not be $\mathbb{N}$-invariant. We will use this sequence to construct a new sequence $(\mu_n)_n$ in $\mathrm{Perg}(\mathcal{A}^{\mathbb{N}}, \mathbb{N})$ converging to μ. Let $\epsilon > 0$ and define

$$D_m(\eta, \nu) = \sum_{k>m} \frac{1}{2^k \|f_k\|_\infty} \left|\int f_k \, d\eta - \int f_k \, d\nu\right|$$

for every $m \geq 1$. Since $\sup_{\eta,\nu}\{D_m(\eta, \nu)\} \leq 1/2^{m-1}$, we can choose an $M \in \mathbb{N}$ such that $D_M(\eta, \nu) < \epsilon$ for all $\eta, \nu \in \mathcal{M}_{\mathbb{N}}(\mathcal{A}^{\mathbb{N}}, \mathbb{N})$. Now, by uniform continuity of each f_k, we can find an $m \in \mathbb{N}$ such that for all $y, z \in \mathcal{A}^{\mathbb{N}}$ coinciding on the first m coordinates, we have $|f_k(y) - f_k(z)| < \epsilon/2$ for all $1 \leq k \leq M$. Thus, if $N > m$ and y coincides with x on the first N coordinates, we have $T^j(x)$ and $T^j(y)$ coincide on the first m coordinates for every $0 \le j \le N-m$, which implies

$$\begin{aligned}
\left|\frac{1}{N}\sum_{j=0}^{N-1} f_k \circ T^j(x) - \frac{1}{N}\sum_{j=0}^{N-1} f_k \circ T^j(y)\right| &\le \frac{1}{N}\sum_{j=0}^{N-1} |f_k \circ T^j(x) - f_k \circ T^j(y)| \\
&< \frac{\epsilon}{2} + \frac{1}{N}\sum_{j=N-m+1}^{N} |f_k \circ T^j(x) - f_k \circ T^j(y)| \\
&< \frac{\epsilon}{2} + \frac{1}{N}\sum_{j=N-m+1}^{N} 2\|f_k\|_\infty \\
&\le \frac{\epsilon}{2} + \frac{2m}{N}\max\{\|f_k\|_\infty : 1 \le k \le M\}.
\end{aligned}$$

Let $N = N(\epsilon)$ be such that $D(\mu, \nu_N) < \epsilon$ and

$$\left|\frac{1}{N}\sum_{j=0}^{N-1} f_k \circ T^j(x) - \frac{1}{N}\sum_{j=0}^{N-1} f_k \circ T^j(y)\right| < \epsilon \min\{\|f_k\|_\infty : 1 \le k \le M\},$$

whenever x and y coincide on the first N coordinates. Choose any periodic point $\overline{x} \in \mathcal{A}^{\mathbb{N}}$ of least period N which coincides with x on the first N coordinates, and define

$$\mu_N = \frac{1}{N}\sum_{j=0}^{N-1} \delta_{T^j(\overline{x})},$$

which by the above will satisfy

$$\left|\int f_k \, d\nu_N - \int f_k \, d\mu_N\right| < \epsilon \min\{\|f_k\|_\infty : 1 \le k \le M\}.$$

Using this, we get

$$\begin{aligned}
D(\mu, \mu_N) &\le D(\mu, \nu_N) + D(\nu_N, \mu_N) \\
&< \epsilon + \sum_{k \ge 1} \frac{1}{2^k \|f_k\|_\infty}\left|\int f_k \, d\mu_N - \int f_k \, d\nu_N\right| \\
&< 2\epsilon + \sum_{k=1}^{M} \frac{1}{2^k \|f_k\|_\infty}\left|\int f_k \, d\mu_N - \int f_k \, d\nu_N\right| \\
&< 2\epsilon + \epsilon \sum_{k=1}^{M} \frac{1}{2^k} < 3\epsilon.
\end{aligned}$$

Since μ_N is periodic and ergodic, this shows $\mathrm{Perg}(\mathcal{A}^{\mathbb{N}}, \mathbb{N})$ is weak-* dense on $\mathcal{M}_{\mathbb{N}}^{\mathrm{erg}}(\mathcal{A}^{\mathbb{N}})$. □

The next step is to show denseness of ergodic measures in $\mathcal{M}_{\mathbb{N}}(\mathcal{A}^{\mathbb{N}})$, i.e., that the convex set $\mathcal{M}_{\mathbb{N}}(\mathcal{A}^{\mathbb{N}})$ is the **Poulsen simplex**: the unique metrizable convex set with dense extreme points.

Proposition 4.1.2. *We have that* $\overline{\mathcal{M}^{erg}_{\mathbb{N}}(\mathcal{A}^{\mathbb{N}})} = \mathcal{M}_{\mathbb{N}}(\mathcal{A}^{\mathbb{N}})$ *in the weak-* topology.*

Proof. Let $\mu \in \mathcal{M}_{\mathbb{N}}(\mathcal{A}^{\mathbb{N}})$, and define for every $n \geq 1$ the function on the semi-algebra of cylinders of length kn for all $k \geq 1$, by

$$\mu_n\Big([\omega_i, \dots, \omega_{i+kn-1}]\Big) = \prod_{j=0}^{k-1} \mu\Big([\omega_{i+jn}, \dots, \omega_{i+(j+1)n-1}]\Big).$$

Here, $[\omega_i, \dots, \omega_{i+m-1}] = \{x \in \mathcal{A}^{\mathbb{N}} : x_j = \omega_j \text{ for all } i \leq j < i+m\}$. Let $\Omega_n = \mathcal{A}^n$ be the set of words of length n with symbols from $\mathcal{A}$. Clearly $|\Omega_n| = |\mathcal{A}|^n$. Define the bijection $\varphi : \mathcal{A}^{\mathbb{N}} \to \Omega_n^{\mathbb{N}}$ by sending

$$(\omega_0, \omega_1, \omega_3, \dots) \mapsto (\omega_0 \cdots \omega_{n-1}, \omega_n \cdots \omega_{2n-1}, \dots).$$

It is clear that $\varphi \circ T^n = T \circ \varphi$. Let ν be the pushforward $\nu = \varphi_* \mu_n$. We want to see ν measures cylinders as a Bernoulli measure associated to the probability vector $\mathbf{p} \in [0,1]^{\Omega_n}$ which, at a coordinate $\omega = \omega_0 \cdots \omega_{n-1} \in \Omega_n$ is given by $\mathbf{p}_\omega = \mu\Big([\omega_1, \dots, \omega_n]\Big)$. This follows by construction: if $\omega^0, \dots, \omega^{m-1} \in \Omega_n$,

$$\begin{aligned} \nu\Big([\omega^0, \dots, \omega^{m-1}]\Big) &= \mu_n\Big([\omega^0_0, \dots, \omega^0_{n-1}, \omega^1_0, \dots, \omega^1_{n-1}, \dots, \omega^{m-1}_0, \dots, \omega^{m-1}_{n-1}]\Big) \\ &= \prod_{j=0}^{m-1} \mu\Big([\omega^j_0, \dots, \omega^j_{n-1}]\Big) \\ &= \prod_{j=0}^{m-1} \nu\Big([\omega^j]\Big) \end{aligned}$$

Since ν extends to a measure on the whole sigma-algebra generated by the cylinders on $\Omega_n^{\mathbb{N}}$, so does μ_n on $\mathcal{A}^{\mathbb{N}}$. Thus $\Big(\mathcal{A}^{\mathbb{N}}, T^n, \mu_n\Big)$ and $\Big(\Omega_n^{\mathbb{N}}, T, \nu\Big)$ are measure conjugate to each other.

Now for each $n \geq 1$ define the measure

$$\nu_n = \frac{1}{n} \sum_{i=0}^{n-1} T^i_* \mu_n.$$

This measure is T-invariant as a direct consequence of T^n-invariance of μ_n. It is not hard to see that ν_n measures the same as μ_n upon T-invariant subsets, so we can prove that ν_n is ergodic by showing μ_n is mixing upon the generating sets (the cylinders of length kn, $k \in \mathbb{N}$). Let $A = [\omega_0, \dots, \omega_{in-1}]$ and $B = [\tau_0, \dots, \tau_{jn-1}]$ be two sets of positive μ_n-measure. Choose a $k > i$. Let W be the set of words of length $kn - in$. Then,

$$A \cap T^{-kn}(B) = \bigsqcup_{w \in W} [\omega_0, \dots, \omega_{in-1}, w_0, \dots, w_{kn-in-1}, \tau_0, \dots, \tau_{jn-1}],$$

and we have

$$\begin{aligned}\mu_n(A \cap T^{-kn}B) &= \sum_{w \in W} \mu_n\Big([\omega_0, \ldots, \omega_{in-1}]\Big)\mu_n\Big([w_0, \ldots, w_{kn-in-1}]\Big)\mu_n\Big([\tau_0, \ldots, \tau_{jn-1}]\Big) \\ &= \mu_n(A)\mu_n(B)\mu_n\left(\bigsqcup_{w \in W} [w]\right) \\ &= \mu_n(A)\mu_n(B).\end{aligned}$$

Therefore we conclude $\mu_n(A \cap T^{-k}B) \to \mu_n(A)\mu_n(B)$ when $k \to \infty$, so μ_n is mixing.

It just remains to prove the weak-* convergence of ν_n to μ. Since $\nu_n(C) \to \mu(C)$ for every cylinder C and for every $f \in \mathscr{C}(\mathcal{A}^{\mathbb{N}})$ and simple function ϕ we have

$$\left|\int f\, d\nu_n - \int f\, d\mu\right| \leq 2\|f - \phi\|_\infty + \left|\int \phi\, d\nu_n - \int \phi\, d\mu\right|,$$

it suffices to prove that simple functions over cylinders are dense in $\mathscr{C}(\mathcal{A}^{\mathbb{N}})$ with the norm $\|\cdot\|_\infty$. Given any $f \in \mathscr{C}(\mathcal{A}^{\mathbb{N}})$ and $\epsilon > 0$, by uniform continuity we can take an $N \in \mathbb{N}$ such that if $x_i = y_i$ for $0 \leq i \leq N-1$ then $|f(x) - f(y)| < \epsilon$. Letting C_N be the collection of cylinders of length N located at $[0, \ldots, N-1]$ and choosing, for each $C \in C_N$, an element $x_C \in C$, we have

$$\left|f(y) - \sum_{C \in C_N} f(x_C)\mathbf{1}_C(y)\right| \leq \max\{|f(x) - f(x_C)| : x \in C, C \in C_N\} < \epsilon,$$

for all $y \in \mathcal{A}^{\mathbb{N}}$, hence concluding the desired denseness.

□

4.2 A quick review of residually finite groups

The class of residually finite groups will be key for the remaining of this chapter. Here we give a brief review of the concept, based on [CSC10], where the reader can find more details.

Definition 4.2.1. A group G is called **residually finite** if for every element $g \neq 1_G$, there is a finite group F and a morphism $\phi\colon G \to F$ such that $\phi(g) \neq 1_F$.

Remark 4.2.2. The following are examples of residually finite groups.

1. It is clear from the definition that every finite group is residually finite. Just consider the identity morphism to itself.

2. The additive group $\mathbb{Z}$ is residually finite. Indeed, if $n \in \mathbb{Z} - \{0\}$, take any $m > |n|$ and consider the finite group $\mathbb{Z}/m\mathbb{Z}$, with the morphism $\phi : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ given by reduction modulo m. It is now clear that $\phi(n) \neq 0$.

3. The general linear group $\mathrm{GL}_n(\mathbb{Z})$ is residually finite for every $n \geq 1$. To see this, take a matrix $A = (a_{ij}) \in \mathrm{GL}_n(\mathbb{Z})$ with $A \neq I_n$. Choose any $m \in \mathbb{Z}$ with $m > |a_{ij}|$ for all $i, j \in \{1, \ldots, n\}$ and consider the morphism $\phi : \mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z})$ of reduction modulo m on coordinates. Then $\phi(B) = 1_{GL_n(\mathbb{Z}_m)}$ if and only if $B = 1_{GL_n(\mathbb{Z})}$.

Proposition 4.2.3. *The following statements are equivalent:*

(i) *G is residually finite,*

(ii) *for all distinct elements $g, h \in G$ there is a finite group F and a group morphism $\phi\colon G \to F$ such that $\phi(g) \neq \phi(h)$.*

(iii) *for every finite $K \subseteq G$ there is a finite group F and a morphism $\phi\colon G \to F$ such that if $k, k' \in K$, $k \neq k'$, then $\phi(k) \neq \phi(k')$.*

Proof. The equivalence between (i) and (ii) follows from the fact that $g \neq h$ if and only if $gh^{-1} \neq 1_G$, and that (iii) implies (ii) is straightforward.

To see that (ii) implies (iii), take a finite subset $K \subseteq G$. For each pair $k, k' \in K$ of distinct elements, there is a morphism $\phi_{k,k'}\colon G \to F_{k,k'}$ such that $\phi(k) \neq \phi(k')$. Define

$$F = \prod_{k,k' \in K, k \neq k'} F_{k,k'},$$

and the morphism $\phi\colon G \to F$ given by

$$\phi(g) = (\phi_{k,k'}(g))_{k,k' \in K}.$$

It is directly verified now that $\phi(k) \neq \phi(k')$ if $k \neq k'$. □

Definition 4.2.4. A group G is called **divisible** if for each $g \in G$ and integer $n \geq 1$, there exists an $h \in G$ such that $h^n = g$.

For instance, the additive groups $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are divisible. The multiplicative group $\mathbb{C}^\times$ is divisible, whereas the multiplicative group $\mathbb{R}^\times$ is not.

Lemma 4.2.5. *Let G be a divisible group, and $\phi\colon G \to F$ a morphism to a finite group. Then, ϕ is the trivial morphism.*

Proof. Take any $g \in G$. Let $n = |F|$, and $h \in G$ such that $h^n = g$. Then, $\phi(g) = \phi(h)^n = 1_F$. □

Corollary 4.2.6. *If G is a nontrivial divisible group, then G is not residually finite.*

Thus, we obtain our first examples of non-residually finite groups: the underlying additive group of a field of characteristic 0, such as $\mathbb{Q}$ or $\mathbb{C}$. Another (harder to see) examples of non-residually finite groups include the Higman group, which in fact does not have finite quotients, and the Baumslag-Solitar group $\mathrm{BS}(2, 3)$.

Lemma 4.2.7. *Let $H \leq G$ be any subgroup, and define*

$$K = \bigcap_{g \in G} gHg^{-1}.$$

Then, $K \trianglelefteq G$ is a normal subgroup contained in H, and if $[G : H] < \infty$, then $[G : K] < \infty$.

Proof. It is clear that $K \subseteq H$. Now, if $g \in G$, we have

$$gKg^{-1} = \bigcap_{t \in G} gtH(gt)^{-1} = \bigcap_{h \in G} hHh^{-1} = K,$$

and thus K is a normal subgroup.

Consider the action of G on G/H by left multiplication, with its associated morphism $\rho\colon G \to \mathrm{Sym}(G/H)$. Observe that, for all $g \in G$,

$$\mathrm{Stab}(gH) = \{t \in G : tgH = gH\} = \{t \in G : g^{-1}tg \in H\} = gHg^{-1}.$$

Therefore, $\ker(\rho) = \bigcap_{g \in G} \mathrm{Stab}(gH) = K$. Since $G/K \cong \mathrm{im}(\rho) \leq \mathrm{Sym}(G/H)$, the last statement follows. □

Definition 4.2.8. Given a group G, the intersection of all finite index subgroups is called the **residual subgroup** (or **profinite kernel**) of G.

We have the following proposition.

Proposition 4.2.9. *Let G be a group, and N its residual subgroup. Then,*

(i) *N is equal to the intersection of all normal subgroups of finite index of G,*

(ii) *N is a normal subgroup of G.*

Proof. For (i), since $N \subseteq N'$, it suffices to show the inclusion $N' \subseteq N$, with N' the intersection of all normal subgroups of finite index of G. By the previous lemma, however, this inclusion is straightforward.

For (ii), recall that the intersection of a family of normal subgroups is a normal subgroup as well. Indeed, if $n \in \bigcap_{i \in I} N_i$, then, for all $g \in G$, $gng^{-1} \in N_i$ for all $i \in I$. Thus $gng^{-1} \in \bigcap_{i \in I} N_i$. □

Note that, if G is finite, $\{1_G\}$ is a normal subgroup of finite index, and thus $N = \{1_G\}$. In general, we have the following characterization.

Proposition 4.2.10. *A group G is residually finite if and only if its residual subgroup N is trivial.*

Proof. Suppose G is residually finite. Take any $1_G \neq g \in G$ and a morphism $\phi : G \to F$ to a finite group F with $\phi(g) \neq 1_F$. Since the group $G/\ker(\phi) \simeq \operatorname{im}(\phi)$ is finite, $\ker(\phi)$ is of finite index. Now, $g \notin \ker(\phi)$, showing $g \notin N$. We conclude $N = \{1_G\}$.

Conversely, if N is trivial, let $g \in G$ such that $g \neq 1_G$. Since N is the intersection of all normal subgroups of finite index, there is a normal subgroup of finite index $K \trianglelefteq G$ such that $g \notin K$. The canonical morphism $\phi\colon G \to G/K$ satisfies $\phi(g) \neq 1_{G/K}$. $\square$

Observe that, as a consequence of last result, for every finite subset $K \subseteq G$, there is a normal subgroup $H \leq G$ of finite index such that $K \cap H = \varnothing$. We will establish a slightly more powerful result which will be useful later on, and allows us to relate residual finiteness with periodicity.

Proposition 4.2.11. *Let G be a residually finite group. For every finite subset $K \subseteq G$, there is a $H \trianglelefteq G$ of finite index and a fundamental domain D of H such that $K \subseteq D$. In other words, $\pi : G \to G/H$ restricted to K is injective.*

Proof. Let $\phi : G \to F$ be a morphism such that if $k, k' \in K$ are distinct then $\phi(k) \neq \phi(k')$, and let $H = \ker(\phi)$. Clearly $H \trianglelefteq G$ and $[G : H] \leq |F| < \infty$. It is not difficult to see that choosing an arbitrary preimage $d_f \in \phi^{-1}(f)$ for each $f \in \operatorname{im}(\phi) - \phi(K)$ and defining $D' = \{d_f : f \in \operatorname{im}(\phi) - \phi(K)\}$, the set $D = K \sqcup D'$ is a fundamental domain for H containing K. $\square$

Corollary 4.2.12. *If G is residually finite, there exist a sequence of subgroups $(H_n)_n$ and, for each n, a fundamental domain D_n of H_n, such that $D_n \uparrow G$, i.e., $D_n \subseteq D_{n+1}$ for all n, and $\bigcup_n D_n = G$.*

To end this section, we want to show that every group satisfying the property PA must be residually finite. In order to do this, we need to determine the relationship between residual finiteness and periodicity in $G \curvearrowright \mathcal{A}^G$.

Proposition 4.2.13. *A group G is residually finite if, and only if, the set of periodic points of $\mathcal{A}^G$ is dense in $\mathcal{A}^G$ for every finite alphabet $\mathcal{A}$ with $|\mathcal{A}| \geq 2$.*

Proof. Suppose that G is a residually finite group, $x \in \mathcal{A}^S$ is an arbitrary point, and let $K \subseteq G$ be an arbitrary finite set. By residual finiteness, there is a morphism $\theta\colon G \to F$, with F a finite group, such that $\theta(g) \neq \theta(h)$ for all distinct $g, h \in F$. Define a configuration $y\colon F \to \mathcal{A}$ by $y(\theta(g)) = x(g)$ for all $g \in K$, and extend it arbitrarily to the rest of F if needed. As the map $\theta|_K\colon K \to F$ is injective, this is well-defined. We have already mentioned that any configuration $z \in \mathcal{A}^F$ defines a periodic point $\bar{z} \in \mathcal{A}^G$ by $\bar{z}(g) = z(\theta(g))$, and observe that $\overline{y}(g) = y(\theta(g)) = x(g)$ by construction, for all $g \in K$, so $[x\,; K]$ contains a periodic point.

Conversely, if $\mathcal{A}^G$ has dense periodic points and $g, h \in G$ are distinct elements, choose two distinct elements from $\mathcal{A}$, which, for simplicity, will be denoted by 0 and 1. Consider any sequence $x \in \mathcal{A}^G$ such that $x(g) = 0$ and $x(h) = 1$. By our hypothesis of density of periodic points, we may assume x is G-periodic and consider $F = \operatorname{Sym}(Gx)$, which is a finite group. Define a morphism $\theta\colon G \to F$ by sending $g \in G$ to the function $s|_{Gx}\colon Gx \to Gx$ given by $g|_{Gx}(h \cdot x) = gh \cdot x$ for all $h \in G$. By our choice of x, we must have $(g \cdot x)(1_G) = x(g) = 0$ and $(h \cdot x)(1_G) = x(h) = 1$. Hence, $g \cdot x \neq h \cdot x$, and thus $g|_{Sx}$ and $h|_{Sx}$ correspond to different permutations of Gx, that is, $\theta(g) \neq \theta(y)$. ☐

Proposition 4.2.14. *If G has property* PA*, then it is a residually finite group.*

Proof. Assume G is not residually finite, but satisfies property PA. By Proposition 4.2.13, there is a configuration $x \in \mathcal{A}^G$ and a finite subset $F \subseteq G$ such that $[x\,;F]$ does not contain G-periodic elements. Choose a measure $\mu \in \mathcal{M}_G(\mathcal{A}^G)$ such that $\mu([x\,;F]) > 0$ (take, for instance, the Bernoulli measure associated to a positive probability vector), and let $(\mu_n)_{n\geq 1}$ be a sequence converging weakly to μ, such that for every $n \geq 1$,

$$\mu_n = \sum_{i=1}^{n} \frac{\alpha_i}{|Gx_i^n|} \sum_{y \in Gx_i^n} \delta_y,$$

where $\sum_{i=1}^{n} \alpha_i = 1$ and x_i^n is G-periodic for every $1 \leq i \leq n$. By weak convergence, we have that $\mu_n([x\,;F]) \to \mu([x\,;F])$. However, for all $n \geq 1$ and $1 \leq i \leq n$ the set $[x\,;F] \cap Gx_i^n$ is empty, meaning that $\mu_n([x\,;F]) = 0$, a contradiction with our choice of μ. ☐

4.3 Monotilings with good invariance properties

Let $K \subseteq G$ be a finite subset containing 1_G, and $\epsilon > 0$. A finite subset $T \subseteq G$ is called (K, ϵ)-**invariant** if

$$|\{g \in T : Kg \subseteq T\}| \geq \epsilon |T|.$$

The set on the left-hand side will be denoted by $\mathrm{Int}_K(T)$. We also define the set

$$\partial_K T = \{g \in G : Kg \cap T \neq \varnothing \text{ and } Kg \cap (G \setminus T) \neq \varnothing\}.$$

Note that $K^{-1}T = \partial_K T \sqcup \mathrm{Int}_K(T)$ always.

The finite set T is called a **monotile** if there is a subset $C \subseteq G$ such that

$$G = \bigsqcup_{c \in C} Tc.$$

Note that a fundamental domain for a finite index subgroup H of G is a monotile by taking $C = H$. Hence, the existence of large monotiles is related to residual finiteness of G, while the existence of (K, ϵ)-invariant sets for arbitrary K and ϵ is associated to amenability of G, since

$$\lim_{n\to\infty} \frac{|\mathrm{Int}_K(F_n)|}{|F_n|} = 1$$

for a Følner sequence and arbitrary finite K with $1_G \in K$.

B. Weiss proved in [Wei01] that if G is residually finite and amenable, then for every finite $K \subseteq G$ and $\epsilon > 0$, there is a (K, ϵ)-invariant monotile. Looking closely to the argument, one realizes that the monotile can be chosen to be a fundamental domain for a finite-index subgroup of G. In [CP14], the authors prove that in fact one can construct a Følner sequence of fundamental domains such that every step of the sequence is in turn monotiled by the previous step. One can find more details on this in [CC19] as well.

Here, we give a proof of the result of Weiss using the version of Ornstein-Weiss' Quasitiling Theorem presented in [KL16]. Given $\epsilon > 0$ and $T \subseteq G$, an ϵ-**quasitiling** is a collection $\{T_1, \dots, T_n\}$ of finite subsets of G such that there are subsets $C_1, \dots, C_n$ satisfying the following.

(i) $\bigcup_{k=1}^{n} T_k C_k \subseteq T$, and the collection $\{T_k c : c \in C_k, 1 \leq k \leq n\}$ $(1-\epsilon)$-**covers** T, in the sense that $|\bigcup_{k=1}^{n} T_k C_k| \geq (1-\epsilon)|T|$.

(ii) The collection $\{T_k c : c \in C_k, 1 \leq k \leq n\}$ is ϵ-**disjoint**, meaning that there exist, for all $1 \leq k \leq n$ and $c \in C_k$, a subset $F_{k,c} \subseteq T_k c$ with $|F_{k,c}| \geq (1-\epsilon)|T_k c|$ and the sets $\{F_{k,c} : c \in C_k, 1 \leq k \leq n\}$ is pairwise disjoint.

Theorem 4.3.1 (Ornstein-Weiss' Quasitiling Theorem). *Let $\epsilon \in (0, 1/2)$ and $n \in \mathbb{N}$ be such that $(1 - \epsilon/2)^n < \epsilon$. Let $e_G \in T_1 \subseteq T_2 \subseteq \cdots \subseteq T_n \in \mathcal{F}(G)$ such that $|\partial_{T_{k-1}} T_k| \leq (\epsilon/8)|T_k|$ for all $k = 2, \dots, n$. Then, every $(T_n, \epsilon/4)$-invariant finite subset of G is ϵ-quasitiled by $\{T_1, \dots, T_n\}$.*

The following technical lemma will be useful for us in the proof of Weiss' Theorem 4.3.3.

Lemma 4.3.2. *Let $K \subseteq G$ be a finite subset with $1_G \in K$ and $0 < \epsilon, \delta < 1$. Let $F' \subseteq F$ be finite subsets of G such that $|F'| \geq (1-\delta)|F|$. We have the following.*

(i) *If F is $(K, \epsilon/2)$-invariant and $\delta \leq \epsilon/(2|K|)$, then F' is (K, ϵ)-invariant.*

(ii) *If F' is $(K, \epsilon/2)$-invariant and $\delta \leq \epsilon/(2-\epsilon)$, then F is (K, ϵ)-invariant.*

(iii) *We have*

$$1 - \sqrt{1 - \frac{\epsilon}{2|K|}} \leq \frac{\epsilon}{2|K|} \leq \frac{\epsilon}{2-\epsilon},$$

and if δ is upper bounded by the leftmost term in the above chain of inequalities, then $1 - (1-\delta)^2 \leq \epsilon/(2|K|)$.

Proof. To prove assertion (i), observe that $\mathrm{Int}_K(F') = \mathrm{Int}_K(F) - K^{-1}(F - F')$ and $|F - F'| \leq \delta|F|$, whence we get

$$\begin{aligned} |\mathrm{Int}_K(F')| &\geq |\mathrm{Int}_K(F)| - |K^{-1}(F - F')| \geq (1 - \epsilon/2)|F| - \delta|K||F| \\ &\geq \left(1 - \frac{\epsilon}{2} - \delta|K|\right)|F'| \geq (1-\epsilon)|F'|. \end{aligned}$$

To prove (ii), simply note that

$$|\mathrm{Int}_K F| \geq |\mathrm{Int}_K F'| \geq \left(1-\frac{\epsilon}{2}\right)(1-\delta)|F| \geq \left(1-\frac{\epsilon}{2}\right)\frac{2(1-\epsilon)}{2-\epsilon}|F| = (1-\epsilon)|F|.$$

Finally, to prove (iii), the last statement can be easily verified, the second inequality of the chain is a consequence of $|K| \geq 1$ and $\epsilon > 0$, and the first inequality follows from the fact that the function $x \mapsto 1-\sqrt{1-x}$ is upper bounded by $x \mapsto x$ in $[0,1] \subseteq \mathbb{R}$. ☐

Theorem 4.3.3 (Weiss, 2001). *Let G be a residually finite, amenable group. If $K \subseteq G$ is finite, contains 1_G, and $0 < \epsilon < 1$, there is a finite $T \subseteq G$ such that*

- (i) *T is (K,ϵ)-invariant, and*
- (ii) *T is a fundamental domain of G, i.e., there is a finite group Ω and a morphism $\theta : G \to \Omega$ such that $\theta|_T$ is injective and $\theta(T) = \Omega$.*

In particular, T is a (K,ϵ)-invariant monotile of G.

Proof. Let

$$\delta \leq 1-\sqrt{1-\frac{\epsilon}{4|K|}},$$

so, in particular by Lemma 4.3.2, $\delta \leq \epsilon/(4|K|) \leq \epsilon/(4-\epsilon)$. Let $n \in \mathbb{N}$ be such that $(1-\delta/2)^n < \delta$. By amenability of G, we may recursively choose $(K,\epsilon/4)$-invariant sets $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n$ containing the identity element 1_G, and such that, for each $k < n$,

(1) $|\mathrm{Int}_{F_k} F_{k+1}| \geq (1-\delta/16)|F_{k+1}|$, and

(2) $|F_k^{-1}F_{k+1} - F_{k+1}| \leq (\delta/16)|F_{k+1}|$.

Since G is residually finite, there exist a finite group Ω and a morphism $\theta\colon G \to \Omega$ such that $\theta|_{F_n}$ is injective. We will write $\Theta_k = \theta(F_k)$ for every $k \leq n$.

First of all, note that $\theta(\mathrm{Int}_{F_k} F_{k+1}) \subseteq \mathrm{Int}_{\Theta_k}\Theta_{k+1}$, since if $g \in \mathrm{Int}_{F_k} F_{k+1}$ then $F_k g \subseteq F_{k+1}$, so $\theta(F_k)\theta(g) \subseteq \theta(F_{k+1})$. Hence, by injectivity of θ on F_n,

$$|\mathrm{Int}_{\Theta_k}\Theta_{k+1}| \geq |\theta(\mathrm{Int}_{F_k} F_{k+1})| = |\mathrm{Int}_{F_k} F_{k+1}| \geq \left(1-\frac{\delta}{16}\right)|\Theta_{k+1}|,$$

concluding Θ_{k+1} is $(\Theta_k, \delta/16)$-invariant for all $k < n$.

Next, observe that $\Theta_k^{-1}\theta_{k+1} - \Theta_{k+1} = \theta(F_k^{-1}F_{k+1}) - \theta(F_{k+1}) \subseteq \theta(F_k^{-1}F_{k+1} - F_{k+1})$, so

$$|\Theta_k^{-1}\Theta_{k+1} - \Theta_{k+1}| \leq |\theta(F_k^{-1}F_{k+1} - F_{k+1})| \leq |F_k^{-1}F_{k+1} - F_{k+1}| \leq \frac{\delta}{16}|\Theta_{k+1}|.$$

These two bounds, together with the fact that $\Theta_k^{-1}\Theta_{k+1} = \partial_{\Theta_k}\Theta_{k+1} \sqcup \mathrm{Int}_{\Theta_k}\Theta_{k+1}$ yield, for each $k < n$,

$$\begin{aligned}
|\partial_{\Theta_k}\Theta_{k+1}| &= |\Theta_k^{-1}\Theta_{k+1}| - |\mathrm{Int}_{\Theta_k}\Theta_{k+1}| \\
&= |\Theta_k^{-1}\Theta_{k+1} - \Theta_{k+1}| + |\Theta_{k+1}| - |\mathrm{Int}_{\Theta_k}\Theta_{k+1}| \\
&\leq \left(\frac{\delta}{16} + 1 - \left(1 - \frac{\delta}{16}\right)\right)|\Theta_{k+1}| \\
&= \frac{\delta}{8}|\Theta_{k+1}|.
\end{aligned}$$

Therefore, by Ornstein and Weiss' Quasitiling Theorem ([KL16, Theorem 4.36]), since Ω is $(\Theta_n, \delta/4)$-invariant (as $\mathrm{Int}_K(\Omega) = \Omega$), there exist subsets $C_k = \{c_1^k, \ldots, c_{n_k}^k\} \subseteq \Omega$, $1 \leq k \leq n$, such that

(a) $\left|\bigcup_{k\leq n} \Theta_k C_k\right| \geq (1-\delta)|\Omega|$, and

(b) for each $k \leq n$ and $j \leq n_k$ there is a subset $\Theta_{k,j} \subseteq \Theta_k c_j^k$ such that $|\Theta_{k,j}| \geq (1-\delta)|\Theta_k|$ and the collection $\{\Theta_{k,j} : 1 \leq j \leq n_k, 1 \leq k \leq n\}$ is pairwise disjoint.

Writing $c_j^k = \theta(g_j^k)$ for each j, k, observe that $\theta(F_k g_j^k) = \Theta_k c_j^k$. We define

$$F_{k,j} := F_k g_j^k \cap \theta^{-1}(\Theta_{k,j}) = \left[F_k \cap \theta^{-1}\left(\Theta_{k,j}(c_j^k)^{-1}\right)\right] g_j^k.$$

Fact 1: $\theta(F_{k,j}) = \Theta_{k,j}$. Indeed, it is directly verified that $\theta(F_{k,j}) \subseteq \Theta_{k,j}$, and conversely if ω is an element of $\Theta_{k,j} \subseteq \theta(F_k)c_j^k$, then $\omega = \theta(f)c_j^k = \theta(fg_j^k)$ for some $f \in F_k$. Clearly, $fg_j^k \in F_k g_j^k$, and since $\theta(fg_j^k) = \omega \in \Theta_{k,j}$, we conclude $fg_j^k \in F_k g_j^k \cap \theta^{-1}(\Theta_{k,j}) = F_{k,j}$, obtaining the desired inclusion.

Fact 2: θ is injective when restricted to $\bigsqcup_{k\leq n}\bigsqcup_{j\leq n_k} F_{k,j}$. That it is injective when restricted to each $F_{k,j}$ follows directly by injectivity upon F_n. Now, if $(j,k) \neq (i,m)$ and $s \in F_{k,j}$, $t \in F_{k,j}$, then $\theta(s) \neq \theta(t)$, because $\Theta_{k,j} \cap \Theta_{m,i} = \varnothing$.

As a consequence of these two last facts, we know that the mapping

$$\bigsqcup_{k\leq n}\bigsqcup_{j\leq n_k} F_{k,j} \xrightarrow{\theta} \bigsqcup_{k\leq n}\bigsqcup_{j\leq n_k} \Theta_{k,j}$$

is bijective. Set

$$F' := \bigsqcup_{k\leq n}\bigsqcup_{j\leq n_k} F_{k,j} \quad \text{and} \quad \Theta' := \bigsqcup_{k\leq n}\bigsqcup_{j\leq n_k} \Theta_{k,j}.$$

From (a) and (b) one sees that $|\Theta'| \geq (1-\delta)^2|\Omega|$. We want to see that F' is $(K, \epsilon/2)$-invariant, and then apply Lemma 4.3.2 in several steps to conclude that the set consisting of adding any section of $\Omega - \Theta'$ to F' is (K, ϵ)-invariant. Note that, by *Fact 1* and injectivity of θ on F', we have

$$|F_{k,j}| = |\Theta_{k,j}| \geq (1-\delta)|\Theta_k c_j^k| = (1-\delta)|F_k g_j^k|.$$

Also, since F_k is $(K, \epsilon/4)$-invariant and $\mathrm{Int}_K(F_k g_j^k) = \mathrm{Int}_K(F_k) g_j^k$, the $(K, \epsilon/4)$-invariance of $F_k g_j^k$ follows, which combined with the previous fact yields, by applying part (i) of Lemma 4.3.2,

$$|\mathrm{Int}_K (F_{k,j})| \geq (1 - \epsilon/2) |F_{k,j}| .$$

Therefore,

$$|\mathrm{Int}_K F'| \geq \left| \bigsqcup_{k \leq n} \bigsqcup_{j \leq n_k} \mathrm{Int}_K (F_{k,j}) \right| \geq (1 - \epsilon/2) \sum_{k \leq n} \sum_{j \leq n_k} |F_{k,j}| = (1 - \epsilon/2)|F'|,$$

and we conclude F' is $(K, \epsilon/2)$-invariant. Now choose any set $F_\epsilon \subseteq G$ disjoint from F' such that θ defines a bijection $F_\epsilon \to \Omega - \Theta'$. Define $\eta := 1 - (1 - \delta)^2 > 0$, so by our choice of δ and by Lemma 4.3.2 (part (iii)), we know $\eta \leq \epsilon/(4 - \epsilon) \leq \epsilon/(2 - \epsilon)$. Since $|\Theta'| \geq (1 - \delta)^2 |\Omega|$, we obtain $|F_\epsilon| = |\Omega| - |\Theta'| \leq \eta |\Theta'|$. Also note that

$$(1 - \eta)|F' \sqcup F_\epsilon| \leq |F'| - \eta |F'| + |F_\epsilon| \leq |F'| - \eta |\Theta'| + \eta |\Theta'| = |F'|,$$

so applying part (ii) of Lemma 4.3.2 we obtain the (K, ϵ)-invariance of $F' \sqcup F_\epsilon$. Thus, $T = F' \sqcup F_\epsilon$ is a (K, ϵ) invariant subset, and a fundamental domaing for $\ker(\theta)$. ☐

The following immediate corollary is the reason why we wanted to prove this last result.

Corollary 4.3.4. *Let G be an amenable, residually finite group. Then, there is a Følner sequence of fundamental domains in G.*

4.4 Residually finite amenable groups have ergodic periodic approximations

Let G be a countable, amenable, residually finite group acting on $\mathcal{A}^G$ by the shift action. We want to prove that the G-periodic measures on $\mathcal{A}^G$ are weak-* dense in the G-invariant measures, i.e., that G is a group satisfying property PA.

Fix an increasing sequence $K_n \uparrow G$ of finite subsets of G, and a dense sequence $(f_k)_{k \geq 1}$ of continuous functions on $\mathcal{A}^G$. Recall that the metric

$$D(\nu, \eta) = \sum_{k \geq 1} \frac{1}{2^k} \|f_k\|_\infty \left| \int f_k d\nu - \int f_k d\eta \right|$$

is compatible with the weak-* topology on $\mathcal{M}(\mathcal{A}^G)$, and following the proof of Proposition 4.1.1, we define, for every $m \geq 1$,

$$D_m(\nu, \eta) = \sum_{k > m} \frac{1}{2^k} \|f_k\|_\infty \left| \int f_k d\nu - \int f_k d\eta \right| .$$

Remark 4.4.1. A **tempered Følner sequence** is a Følner sequence $(F_n)_{n\geq 1}$ such that there exists a $b > 0$ with

$$\left|\bigcup_{k=1}^{n-1} F_k^{-1}F_n\right| \leq b|F_n|$$

for every $n > 1$. Every Følner sequence has a tempered subsequence, and for this particular type of Følner sequences, an amenable version of Birkhoff's Ergodic Theorem exists, meaning for an ergodic p.m.p. action $G \curvearrowright (X, \mu)$ and $f \in L^1(X, \mu)$,

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{t\in F_n} f(t \cdot x) = \int_X f \, d\mu.$$

For further details on this topic, a nice exposition of group-theoretical versions of Ergodic Theorems is given in [KL16].

Proposition 4.4.2. *Let $\mathcal{M}_G^{\mathrm{erg}}(\mathcal{A}^G)$ be the set of ergodic G-invariant probability measures on $\mathcal{A}^G$. Then,*

$$\mathcal{M}_G^{\mathrm{erg}}(\mathcal{A}^G) \subseteq \overline{\mathrm{Perg}(\mathcal{A}^G, G)}.$$

Proof. Fix a Følner sequence $(F_n)_n$ of fundamental domains on G, which by the above remark we may assume is tempered. Just as we did in the proof of Proposition 4.1.1, we may choose a generic element $x \in \mathcal{A}^G$ such that Birkhoff's Theorem holds for x, for every continuous function. This yields the weak-* convergence

$$\nu_n := \frac{1}{|F_n|} \sum_{t\in F_n} \delta_{t\cdot x} \xrightarrow{*} \mu.$$

Following the proof of 4.1.1, for any $\epsilon > 0$ we can choose an $M \geq 1$ such that $D_M(\eta, \nu) < \epsilon$ for all $\eta, \nu \in \mathcal{M}_G(\mathcal{A}^G)$, and an $m \geq 1$ such that if y and x coincide on K_m then $|f_k(x) - f_k(y)| < \epsilon/2$ for $1 \leq k \leq M$. If $n > m$ and y coincides with x on F_n, then for every $t \in F_n \cap K_m^{-1}F_n$ we have that $t \cdot x$ and $t \cdot y$ coincide on K_m, so

$$\begin{aligned}
\left|\frac{1}{|F_n|}\sum_{t\in F_n} f_k(t\cdot x) - \frac{1}{|F_n|}\sum_{t\in F_n} f_k(t\cdot y)\right| &\leq \frac{1}{|F_n|}\sum_{t\in F_n} |f_k(t\cdot x) - f_k(t\cdot y)| \\
&< \epsilon/2 + \frac{1}{|F_n|}\sum_{t\in F_n - K_m^{-1}F_n} |f_k(t\cdot x) - f_k(t\cdot y)| \\
&\leq \epsilon/2 + \frac{2}{|F_n|}\sum_{t\in F_n - K_m^{-1}F_n} \|f_k\|_\infty \\
&\leq \epsilon/2 + \frac{2|F_n - K_m^{-1}F_n|}{|F_n|} \max\{\|f_k\|_\infty : 1 \leq k \leq M\}.
\end{aligned}$$

Therefore, since $(F_n)_n$ is Følner, we can choose an $N = N(\epsilon) \in \mathbb{N}$ such that

$$\left|\frac{1}{|F_N|}\sum_{t\in F_N} f_k(t\cdot x) - \frac{1}{|F_N|}\sum_{t\in F_N} f_k(t\cdot y)\right| < \epsilon \cdot \min\{\|f_k\|_\infty : 1 \leq k \leq M\},$$

and such that $D(\mu, \nu_N) < \epsilon$, for every $1 \leq k \leq M$. Now choose a G-periodic element $\bar{x}$ of $\mathcal{A}^G$ which coincides with x on F_N. This can be done by virtue of the fact that F_N is a fundamental domain for a finite-index subgroup of G. Define

$$\mu_N = \frac{1}{|F_N|} \sum_{t \in F_N} \delta_{t \cdot \bar{x}},$$

which by the above will satisfy

$$\left| \int f_k \, d\nu_N - \int f_k \, d\mu_N \right| < \epsilon \cdot \min\{\|f_k\|_\infty : 1 \leq k \leq M\}.$$

Using this, we obtain

$$\begin{aligned} D(\mu, \mu_N) &\leq D(\mu, \nu_N) + D(\nu_N, \mu_N) \\ &< \epsilon + \sum_{k \geq 1} \frac{1}{2^k \|f_k\|_\infty} \left| \int f_k \, d\mu_N - \int f_k \, d\nu_N \right| \\ &< 2\epsilon + \sum_{k=1}^{M} \frac{1}{2^k \|f_k\|_\infty} \left| \int f_k \, d\mu_N - \int f_k \, d\nu_N \right| \\ &< 2\epsilon + \epsilon \sum_{k=1}^{M} \frac{1}{2^k} < 3\epsilon. \end{aligned}$$

□

We now prove denseness of the set of ergodic measures into the set of invariant measures of $\mathcal{A}^G$. This holds for (and, in fact, characterizes) a broader class of groups: those that do not have Kazhdan's property (T), as shown in [GW97].

Proposition 4.4.3. *The set $\mathcal{M}_G^{\mathrm{erg}}(\mathcal{A}^G)$ of ergodic measures is weak-* dense in $\mathcal{M}_G(\mathcal{A}^G)$.*

Proof. Let $\mu \in \mathcal{M}_G(\mathcal{A}^G)$. For a finite-index subgroup $H \leq G$, $[G : H] = n$, set a fundamental domain $D = \{d_1, \ldots, d_n\}$. For each finite $F \subseteq H$, define

$$\mu_D\left([x\,; DF]\right) = \prod_{f \in F} \mu\left([x\,; Df]\right).$$

This function, defined on the semi-algebra of cylinders supported over sets of the form DF with $F \subseteq H$ finite (which are finite disjoint unions of translations of D), extends to a probability measure on the Borel σ-algebra. This can be easily seen through the natural H-equivariant identification between the space $\mathcal{A}^G$ and the space Ω^H, where $\Omega = \mathcal{A}^D$, given by $x \overset{\varphi}{\mapsto} (h \mapsto (h \cdot x)|_D)$. Via this identification, a cylinder supported over DF on $\mathcal{A}^G$ corresponds to a cylinder over F on Ω^H, so $\varphi_* \mu_D$ is a Bernoulli measure on Ω^H.

Note that μ_D is H-invariant: if $h \in H$, $x \in \mathcal{A}^G$ and $F \subseteq H$ is finite, by G-invariance of μ we get

$$\begin{aligned}\mu_D(h[x\,;DF]) = \mu_D([h\cdot x\,;DFh^{-1}]) &= \prod_{f\in F}\mu([h\cdot x;Dfh^{-1}])\\ &= \prod_{f\in F}\mu(h[x\,;Df]) = \mu_D([x\,;DF]).\end{aligned}$$

Define the measure

$$\nu_D = \frac{1}{|D|}\sum_{d\in D} d_*\mu_D.$$

To verify the G-invariance of ν_D, let $g \in G$ be a fixed element. Since the collection $\{Dh : h \in H\}$ is a partition of G, for any $d \in D$, there exist two unique elements $\alpha(d) \in D$ and $\beta(d) \in H$ such that $gd = \alpha(d)\beta(d)$, thus making sense to define corresponding functions $\beta\colon D \to H$ and $\alpha\colon D \to D$ such that $gd = \alpha(d)\beta(d)$ for all $d \in D$. Note that α must be injective, since if $\alpha(d) = \alpha(d')$ then

$$dH = g^{-1}\alpha(d)\Big(\beta(d)H\Big) = g^{-1}\alpha(d')\Big(\beta(d')H\Big) = d'H,$$

so $d = d'$. Thus, α is bijective and we have

$$g_*\nu_D = \frac{1}{|D|}\sum_{d\in D}(gd)_*\mu_D = \frac{1}{|D|}\sum_{d\in D}\alpha(d)_*(\beta(d)_*\mu_D) = \frac{1}{|D|}\sum_{d\in D}\alpha(d)_*\mu_D = \nu_D$$

as a consequence of the H-invariance of μ_D.

We want to see ν_D is ergodic. Since ν_D and μ_D measure the same at G-invariant sets, we can check ergodicity of μ_D to conclude for ν_D. In fact, we will prove that μ_D is mixing (see Appendix C). For any two finite subsets $F, K \subseteq H$, if $h \in H - F^{-1}K$, then $DF \cap DKh^{-1} = \varnothing$. Therefore, if $\mu_D([x\,;DF])\mu_D([y\,;DK]) > 0$, by the definition and H-invariance of μ_D,

$$\mu_D([x\,;DF]\cap h[y\,;DK]) = \mu_D([x\,;DF]\cap[h\cdot y\,;DKh^{-1}]) = \mu_D([x\,;DF])\mu_D([y\,;DK]),$$

showing the mapping $h' \mapsto \Big(\mu_D([x\,;DF]\cap h'[y\,;DK]) - \mu_D([x\,;DF])\mu_D([y\,;DK])\Big)$ vanishes at infinity. Since mixing can be checked upon generators of the σ-algebra, we conclude $H \curvearrowright (\mathcal{A}^G, \mu_D)$ is mixing, in particular ergodic. This implies $G \curvearrowright (\mathcal{A}^G, \mu_D)$ is ergodic, as desired.

It just remains to approximate μ by measures of the type ν_D. Since G is residually finite and amenable, by Theorem 4.3.3 we can find a Følner sequence of fundamental domains $D_n \uparrow G$. Let us see that $\nu_{D_n} \xrightarrow{*} \mu$ by checking setwise convergence for cylinders and then arguing by approximation by simple functions. If $C \subseteq G$ is finite, then $d \in D_n$ satisfies $Cd \cap (G - D_n) \neq \varnothing$ if, and only if, $d \in D_n \cap C^{-1}(G - D_n) = D_n - C^{-1}D_n$, and thus we can decompose D_n as

$$D_n = [D_n \cap \mathrm{Int}_C(D_n)] \sqcup [D_n - C^{-1}D_n],$$

where $\mathrm{Int}_C(D_n) = \{g \in G : Cg \subseteq D_n\}$. The sequence $(D_n)_{n\geq 1}$ is Følner and C^{-1} is finite, so

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{|D_n|} \sum_{d \in D_n - C^{-1}D_n} \mu_{D_n}(d^{-1}[x\,;C]) &= \lim_{n\to\infty} \frac{1}{|D_n|} \sum_{d \in D_n - C^{-1}D_n} \mu_{D_n}([d^{-1}\cdot x; Cd]) \\
&\leq \lim_{n\to\infty} \frac{|D_n - C^{-1}D_n|}{|D_n|} \\
&= 0.
\end{aligned}$$

On the other hand, if $d \in \mathrm{Int}_C(D_n)$ then $Cd \subseteq D_n$, so $\mu_{D_n}([d\cdot x; Cd]) = \mu([dx; Cd])$. Again, as $(D_n)_{n\geq 1}$ is Følner, we have $|D_n|^{-1}|\mathrm{Int}_C(D_n)| \to 1$ as $n \to \infty$. Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \nu_{D_n}([x\,;C]) &= \lim_{n\to\infty} \frac{1}{|D_n|} \sum_{d\in D_n} d_*\mu_{D_n}([x\,;C]) \\
&= \lim_{n\to\infty} \frac{|\mathrm{Int}_C(D_n)|}{|D_n|}\mu([x\,;C]) \\
&= \mu([x\,;C]),
\end{aligned}$$

and we conclude $\nu_{D_n}(C) \to \mu(C)$ for every cylinder C, which implies convergence in expectation for every simple function. We already proved that simple functions are dense in $(\mathscr{C}(\mathcal{A}^{\mathbb{N}}), \|\cdot\|_\infty)$. The same argument shows that simple functions are dense in $(\mathscr{C}(\mathcal{A}^G), \|\cdot\|_\infty)$, implying in consequence the weak-* convergence of ν_{D_n} to μ. □

4.5 Free groups of finite rank have periodic approximations

In this section, we follow L. Bowen's proof in [Bow03] of the denseness of periodic measures for the full $\mathbb{F}_n$-shift, presenting a re-written and fully detailed version of the proof. We also show denseness of periodic measures for any $\mathbb{F}_n$-subshift of finite type. Here, ε will denote the identity of $\mathbb{F}_n$ (not to be confused with ϵ, which will denote a positive real number).

Definition 4.5.1. Let $\mu, \nu \in \mathcal{M}_G(\mathcal{A}^G)$, $r \in \mathbb{N}$ and $\epsilon > 0$. We say ν is a (r,ϵ)-**approximation** of μ if for all $x \in \mathcal{A}^G$,

$$|\nu([x\,;B(\varepsilon,r)]) - \mu([x\,;B(\varepsilon,r)])| < \epsilon.$$

Note that if ν_j is a $(r_j, 0)$-approximation of μ and $r_j \to \infty$ as $j \to \infty$, then $\nu_j \xrightarrow{*} \mu$. Also, if ν_j is a (r, ϵ_j)-approximation of μ with $\epsilon_j \to 0$ as $j \to \infty$, any weak-* limit point of $(\nu_j)_j$ is a $(r, 0)$-approximation of μ. Indeed, if ν is such a measure, since the function $\mathbf{1}_{B(\varepsilon,r)}$ is continuous we have

$$\nu(B(\varepsilon, r)) = \lim_{j\to\infty} \mu_j(B(\varepsilon, r)) = \mu(B(\varepsilon, r)).$$

Therefore, in order to prove that a family $\mathcal{F} \subseteq \mathcal{M}_G(\mathcal{A}^G)$ is weak-* dense, it suffices to show for every $\mu \in \mathcal{M}_G(\mathcal{A}^G)$, $r \in \mathbb{N}$ and $\epsilon > 0$, there is a (r, ϵ)-approximation of μ from $\mathcal{F}$.

We will need the following previous results in order to prove the main theorem of this section.

Lemma 4.5.2. *Let* $\mathbf{A} \in \mathrm{Mat}_{n\times k}(\mathbb{Q})$*,* $\mathbf{B} \in \mathrm{Mat}_{m\times k}(\mathbb{R})$*,* $\mathbf{b} \in \mathbb{Q}^n$ *and* $\mathbf{x} \in \mathbb{R}^k$ *be such that* $\mathbf{A}\mathbf{x} = \mathbf{b}$ *and* $\mathbf{x}_i \geq 0$ *for each* $1 \leq i \leq k$*. Then, for every* $\epsilon > 0$ *there is a* $\mathbf{x}' \in \mathbb{Q}^k$ *such that*

(i) $\mathbf{A}\mathbf{x}' = \mathbf{b}$*,*

(ii) $|(\mathbf{B}\mathbf{x}')_i - (\mathbf{B}\mathbf{x})_i| < \epsilon$ *for all* $1 \leq i \leq m$*, and*

(iii) ${\mathbf{x}'}_i = 0$ *whenever* $\mathbf{x}_i = 0$*.*

Proof. Let $\epsilon > 0$. We can construct $\mathbf{x}'$ in the context of the following optimization problem:

$$\begin{cases} \text{minimize:} & \|\mathbf{B}\mathbf{y} - \mathbf{B}\mathbf{x}\| \\ \text{subject to:} & \mathbf{A}'\mathbf{y} \leq \mathbf{b}', \end{cases}$$

where

$$\mathbf{A}' = \begin{pmatrix} \mathbf{A} \\ -\mathbf{A} \\ -\mathbf{1}_{\mathbf{k}\times\mathbf{k}} \end{pmatrix} \quad \text{and} \quad \mathbf{b}' = \begin{pmatrix} \mathbf{b} \\ -\mathbf{b} \\ \mathbf{0} \end{pmatrix}.$$

The region $P \subseteq \mathbb{R}^k$ defined by the constraint $\mathbf{A}'\mathbf{y} \leq \mathbf{b}'$ is a (non-empty) polyhedron. Thus, by the Main Theorem for Polyhedra (see, for instance, [Zie12, chap. 1]), there exist finite sets $U, V \subseteq \mathbb{R}^k$ such that

$$P = \mathrm{conv}_{\mathbb{R}}(U) + \mathrm{cone}_{\mathbb{R}}(V) = \left\{ \sum_{\mathbf{u}\in U} \alpha_{\mathbf{u}}\mathbf{u} + \sum_{\mathbf{v}\in V} \beta_{\mathbf{v}}\mathbf{v} \;\middle|\; \alpha_{\mathbf{u}}, \beta_{\mathbf{v}} \geq 0, \sum_{\mathbf{u}\in U} \alpha_{\mathbf{u}} = 1 \right\}.$$

Moreover, since $\mathbf{A}'$ and $\mathbf{b}'$ are rational, we have that $U, V \subseteq \mathbb{Q}^k$. We already know that $\|\mathbf{B}\mathbf{y} - \mathbf{B}\mathbf{x}\|$ reaches its minimum at $\mathbf{y} = \mathbf{x} \in P$. Write

$$\mathbf{x} = \sum_{\mathbf{u}\in U} \alpha_{\mathbf{u}}\mathbf{u} + \sum_{\mathbf{v}\in V} \beta_{\mathbf{v}}\mathbf{v}$$

with non-zero coefficients, and note that, since $\mathbf{u}, \mathbf{v} \geq 0$, we have $\mathbf{u}_i = \mathbf{v}_i = 0$ for all $\mathbf{u} \in U$ and $\mathbf{v} \in V$ whenever $\mathbf{x}_i = 0$. Now, by continuity of $\mathbf{B}$, there is a $\delta > 0$ such that if $\|\mathbf{y} - \mathbf{x}\| < \delta$, then $|(\mathbf{B}\mathbf{y})_i - (\mathbf{B}\mathbf{x})_i| < \epsilon$ for all $1 \leq i \leq m$. Choose, for each $u \in U$ and $v \in V$, rational numbers p_u and q_v close enough to α_u and β_v, respectively, so that

$$\mathbf{x}' := \sum_{\mathbf{u}\in U} p_{\mathbf{u}}\mathbf{u} + \sum_{\mathbf{v}\in V} q_{\mathbf{v}}\mathbf{v},$$

satisfies $\|\mathbf{x}'-\mathbf{x}\| < \delta$. Observe that $\mathbf{x}' \in \mathbb{Q}^k$, $\mathbf{x}'_i = 0$ whenever $\mathbf{x} = 0$, and the remaining conditions are satisfied by construction. □

Lemma 4.5.3. *Let Q be a finite directed graph such that every vertex has an equal amount of vertices going out of and into it. Then Q is weakly connected (i.e., in an undirected sense) if and only if it is strongly connected (i.e., in a directed sense).*

Proof. Let $\text{in}(A) = \{(v,w) \in E : w \in A, v \notin A\}$ and $\text{out}(A) = \{(v,w) \in E : v \in A, w \notin A\}$ for every subset $A \subseteq V$. First, we want to prove that for all $A \subseteq V$ we have $\text{in}(A) = \text{out}(A)$ by induction on the cardinality of A. If $|A| = 1$, we are done by the hypothesis. Now assume the claim holds for a subset A of cardinality n. Choose any element $v \in A$, and let k, j be the number of edges going from $A - \{v\}$ to v, and from v to $A - \{v\}$, respectively. By the inductive step, $\text{in}(A - \{v\}) = \text{out}(A - \{v\})$, so

$$\begin{aligned}\text{in}(A) &= (\text{in}(A - \{v\}) - j) + (\text{in}(v) - k)\\ &= (\text{out}(A - \{v\}) - j) + (\text{out}(v) - k)\\ &= \text{out}(A).\end{aligned}$$

Now, assume Q is weakly connected, and let $v \in V$ be any vertex. Define $A_v \subseteq V$ to be the set of all vertices w such that there is a directed path from v to w. If $A_v = V$ we are done. If not, by connectedness there is an edge joining A_v with $V - A_v$ in either direction, and since $\text{in}(A_v) = \text{out}(A_v)$, we conclude there is an edge going out of A_v, contradicting the definition of A_v. Therefore $A_v = V$, and since v is arbitrary, V is strongly connected. □

Now, we proceed and prove the main result.

Theorem 4.5.4 ([Bow03]). *The free group of rank n, $\mathbb{F}_n$, has property* PA, *i.e.,* $\text{Per}(\mathcal{A}^{\mathbb{F}_n}, \mathbb{F}_n)$ *is weak-* dense in* $\mathcal{M}_{\mathbb{F}_n}\left(\mathcal{A}^{\mathbb{F}_n}\right)$.

Proof. Let $r \in \mathbb{N}$ and $\epsilon > 0$. To economize notation, we will denote the set $[x\,; B(\varepsilon, r)]$ simply as $[x]$, for every $x \in \mathcal{A}^{\mathbb{F}_n}$. Write $\mathbb{F}_n = \langle g_1, \ldots, g_n \rangle$. First, we define a directed, labeled finite graph $\Gamma = (V, E)$, where

$$V = \{[x] : x \in \mathcal{A}^{\mathbb{F}_n}\},$$

$$E = \{e(x,i) : 1 \leq i \leq n\},$$

where $e(x,i) = ([x], [g_i \cdot x])$ will be labeled by i. Note that $[g_i \cdot x]$ and $g_i[x]$ are not necessarily equal.

Next, we define a weight function w on Γ by setting

$$w([x]) = \mu([x]),$$

$$w(e(x,i)) = \mu([x\,; B(\varepsilon, r) \cup B(g_i, r)]).$$

The function w satisfies two essential properties.

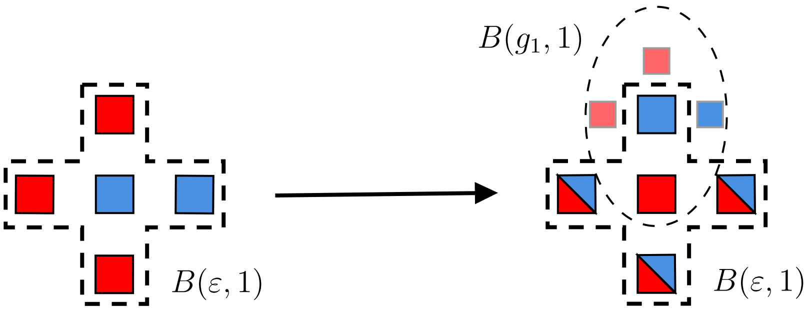

Figure 4.1: There is an i-labeled edge going out of $[x]$ for every $\mathcal{A}$-coloring of $B(\varepsilon, r) - B(g_i, r)$. Here an illustration for $n = k = 2$ and $r = i = 1$.

1. We have that $\sum_{v \in V} w(v) = 1$. This is straightforward from the fact that the cylinders of length r form a partition of $\mathcal{A}^{\mathbb{F}_n}$.

2. For every $v \in V$, if we define the sets $E_i^+(v) = \{e(x,i) \in E : v = [x]\}$ and $E_i^-(v) = \{e(x,i) \in E : v = [g_i \cdot x]\}$ of i-labeled edges going out of and into v, respectively, then
$$\sum_{e \in E_i^+(v)} w(e) = w(v) = \sum_{f \in E_i^-(v)} w(f).$$
Indeed, there is one i-labeled edge going out of $v = [x]$ for each $\mathcal{A}$-coloring of $B(\varepsilon, r) - B(g_i, r)$ (check Figure 4.1), so the first sum can be re-written as
$$\sum_{\omega \in \mathcal{A}^{B(g_i,r)-B(\varepsilon,r)}} \mu([x\,; B(g_i, r)] \cap [\bar{\omega}; B(\varepsilon, r) - B(g_i, r)]) = \mu([x\,; B(g_i, r)]) = w(v),$$
where $\bar{\omega}$ is any extension of ω to $\mathcal{A}^{\mathbb{F}_n}$, and we use the fact that μ is $\mathbb{F}_n$-invariant. Similarly, there is one i-labeled edge going into $v = [x]$ for each $\mathcal{A}$-coloring of $B(\varepsilon, r) - B(g_i^{-1}, r)$, so we get the second equality by the same argument.

If μ is ergodic and periodic, then it consists of a single finite orbit, which means every set has measure j/m, with m the length of the orbit and $j \leq m$ the number of elements of the orbit belonging to the set. Therefore, in this context, the weights of all vertices and edges will be rational numbers. The strategy of this proof goes as follows. We want to find a function w_p on Γ satisfying properties 1 and 2, which takes values in the non-negative rational numbers, such that $w_p(e) = 0$ for every $e \in E$ with $w(e) = 0$, and satisfying $|w_p(v) - w(v)| < \epsilon$ for all $v \in V$. Then, we want to use this function w_p to construct a periodic (r, ϵ)-approximation of μ.

For the first part, note that properties 1 and 2 can be summarized as

$$\begin{pmatrix} \mathbf{s} \\ \mathbf{B'} \\ \mathbf{B} \end{pmatrix} \mathbf{w^E} = \begin{pmatrix} 1 \\ \mathbf{0} \\ \mathbf{w^V} \end{pmatrix}.$$

Here, $\mathbf{w}^{\mathbf{V}}$ and $\mathbf{w}^{\mathbf{E}}$ are the column vectors of weights of vertices and edges, respectively, and $\mathbf{s}$ is the $1 \times |E|$ row vector defined by

$$\mathbf{s}_e = \mathbf{1}_{E_1^+}(e) := \begin{cases} 1 & \text{if } e \in E_1^+ \\ 0 & \text{otherwise,} \end{cases}$$

with $E_1^+ := \bigcup_{v \in V} E_1^+(v)$. The $|V| \times |E|$ matrix $\mathbf{B}$ consists of the row $\mathbf{1}_{E_1^+(v)}$ for each position $v \in V$, and the $2n|V| \times |E|$ matrix $\mathbf{B}'$ consists of the row $\mathbf{1}_{E_1^+(v)} - \mathbf{1}_{E_i^*(v)}$ at position $(v, i, *)$, with $1 \leq i \leq n$, $v \in V$ and $* \in \{+, -\}$.

By Lemma 4.5.2, there is a $\mathbf{x}' \in \mathbb{Q}^{|E|}$ such that w_p defined by $\mathbf{w}_{\mathbf{p}}^{\mathbf{E}} = \mathbf{x}'$ and $\mathbf{w}_{\mathbf{p}}^{\mathbf{V}} = \mathbf{B}\mathbf{x}'$ satisfies the desired conditions. We use now w_p to construct the desired (r, ϵ)-approximation of μ. First, choose a natural number N such that Nw_p is integer-valued. Let Q_0 be an edge-less graph of N vertices. Since $\sum_{v \in V} Nw_p(v) = N$, we can divide the vertices into $|V|$ groups, where the group labeled v contains exactly $Nw_p(v)$ vertices. Now, for a given vertex $v \in V$, since

$$\sum_{e \in E_i^+(v)} Nw_p(e) = Nw_p(v),$$

we can choose, for each i and $x \in \mathcal{A}^{\mathbb{F}_n}$ such that $v = [x]$, a set $b_1(x, i)$ of $Nw_p(e(x, i))$ vertices labeled v. Moreover, we choose these sets so that, for fixed v and i, they partition the set of v-labeled vertices, i.e., if $[g_i \cdot x] \neq [g_i \cdot y]$, then $b_1(x, i) \cap b_1(y, i) = \varnothing$. Next, since

$$\sum_{e \in E_i^-(v)} Nw_p(e) = Nw_p(v),$$

we can choose, for each i and $x \in \mathcal{A}^{\mathbb{F}_n}$ such that $v = [g_i \cdot x]$, a set $b_2(x, i)$ of $Nw_p(e(x, i))$ vertices labeled v, satisfying $b_2(x, i) \cap b_2(y, i) = \varnothing$ whenever $[x] \neq [y]$. Now, for each $x \in \mathcal{A}^{\mathbb{F}_n}$ and $1 \leq i \leq n$, let $B_{x,i} : b_1(x, i) \to b_2(x, i)$ be a bijection, and define the graph Q whose vertex set is Q_0 and has an edge labeled $e(x, i)$ going from v to $B_{x,i}(v)$ for every $x \in \mathcal{A}^{\mathbb{F}_n}$, $1 \leq i \leq n$ and $v \in Q_0$.

Let $Q_1, \ldots, Q_m$ the connected components of Q, which by Lemma 4.5.3 are also strongly connected. Let $S = \{g_1, \ldots, g_n\}$. For each $1 \leq j \leq m$ and $v \in V(Q_j)$, let $C_j^v \colon \mathrm{Cay}(\mathbb{F}_n; S) \to Q_j$ be a label-preserving surjective graph homomorphism determined by $C_j^v(\varepsilon) = v$. The maps C_j can be constructed by looking at non-backtracking paths in Q_j, and we may assume C_j respects the labels, i.e., the edge $(g, g_j g)$ maps to an edge labeled j in Q_j, since the latter satisfies $|\mathrm{in}(v)| = |\mathrm{out}(v)| = n$ for every $v \in V(Q_j)$.

Defining, for each $1 \leq j \leq m$, the map $\phi_j \colon V(Q_j) \to \mathcal{A}$ given by $\phi_j([x]) = x(\varepsilon)$, we have that the $\mathcal{A}$-coloring ϕ_j of Q_j lifts to a $\mathcal{A}$-coloring $\hat{\phi}_j$ of $\mathbb{F}_n$ by $\hat{\phi}_j(g) = \phi_j(C_j(g))$. Observe that for each j there is a finite subset $F_j \subseteq \mathbb{F}_n$ such that $C_j^v \colon F_j \to V(Q_j)$ is

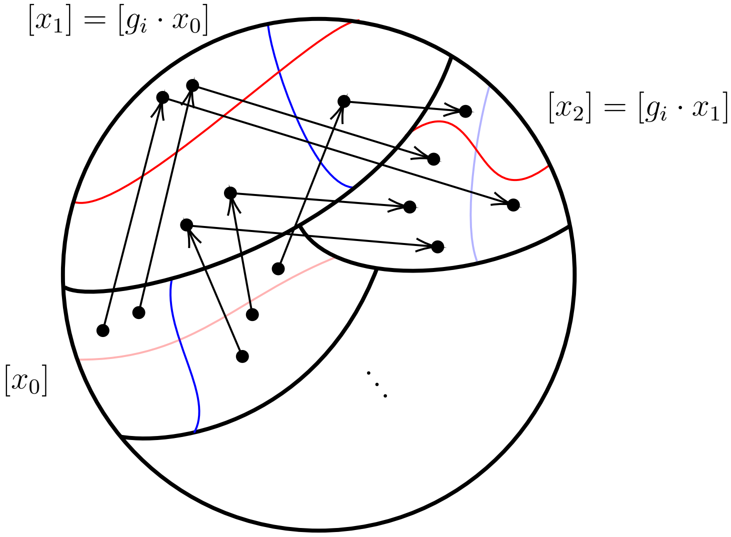

Figure 4.2: A representation of the graph Q. In blue, the partition corresponding to b_1, and in red the one corresponding to b_2. Every vertex is labeled with $[x]$ for some $x \in \mathcal{A}^{\mathbb{F}_n}$, and there is exactly one i-labeled edge going out and one i-labeled edge going into it.

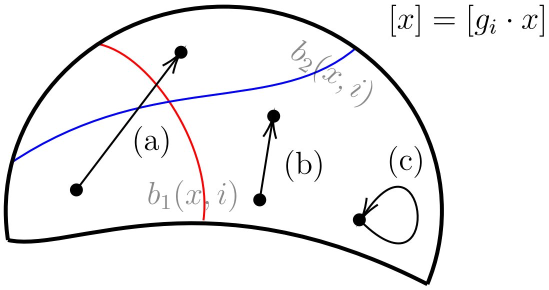

Figure 4.3: All of the following situations may occur when $[x] = [g_i \cdot x]$. In this case, the edge $e(x, i)$ is a self-loop of $[x]$. Depending on the choice of the partition sets $b_1(x, i)$ and $b_2(x, i)$, and the choice of the bijection $B_{x,i}$, one might observe an edge going from a vertex of $b_1(x, i)$ into (a) a vertex out of $b_1(x, i)$, (b) a different vertex of $b_1(x, i)$, or (c) itself.

bijective, so for a given $g \in \mathbb{F}_n$, there is a $\hat{g} \in F_j$ such that $C_j^v(g) = C_j^v(\hat{g})$, yielding

$$\begin{aligned}
(g \cdot \hat{\phi}_j)(h) &= \phi_j(C_j^v(hg)) = \phi_j\left(C_j^{C_j^v(g)}(h) C_j^v(g)\right) \\
&= \phi_j\left(C_j^{C_j^v(g)}(h) C_j^v(\hat{g})\right) = \phi_j\Big(C_j^v\Big(h\hat{g}\Big)\Big) \\
&= \Big(\hat{g} \cdot \hat{\phi}_j\Big)(h).
\end{aligned}$$

Therefore, $\mathbb{F}_n \hat{\phi}_j = F_j \hat{\phi}_j$, so $\hat{\phi}_j$ has finite $\mathbb{F}_n$-orbit. Let λ_j be the $\mathbb{F}_n$-invariant measure supported over the orbit of $\hat{\phi}_j$, and set

$$\lambda := \frac{1}{N} \sum_{j=1}^{m} |V(Q_j)| \lambda_j.$$

It is clear from the definition that λ is a periodic measure. We shall see now that λ is a (r, ϵ)-approximation. In order to do this, we need to understand how the measures λ_j work. Observe that, for any given $v \in V(Q_j)$ labeled $[x]$, when we look the $\mathcal{A}$-coloring ϕ_j at the ball of radius r and center v in the graph Q_j, it "looks like $[x]$". More precisely, for the lifting $\hat{\phi}_j$ of ϕ_j, there is a $g \in \mathbb{F}_n$ such that $(g \cdot \hat{\phi}_j)|_{B(\varepsilon,r)} = x|_{B(\varepsilon,r)}$. Therefore, $\lambda_j([x])$ consists exactly of the (normalized) amount of translates of $\hat{\phi}_j$ which coincide with x at $B(\varepsilon, r)$, i.e., the amount of vertices of Q_j labeled $[x]$, which will be denoted by $N_j([x])$, divided by $|V(Q_j)|$. Using this, we see that

$$\begin{aligned}
\lambda([x\,; B(\varepsilon, r)]) &= \frac{1}{N} \sum_{j=1}^{m} |V(Q_j)| \lambda_j([x\,; B(\varepsilon, r)]) \\
&= \frac{1}{N} \sum_{j=1}^{m} N_j([x]) \\
&= \frac{1}{N} N w_p([x]) \\
&= w_p([x]),
\end{aligned}$$

so λ is a (r, ϵ)-approximation of μ.

□

Remark 4.5.5. Note that the function $C_j^v \colon \mathrm{Cay}(\mathbb{F}_n, S) \to Q_j$ induces a group morphism $\sigma_j \colon \mathbb{F}_n \to \mathrm{Sym}(V(Q_j))$ where the image σ^g of an element $g \in \mathbb{F}_n$ sends $u \in V(Q_j)$ to the vertex $\sigma^g(u)$ which results from following the path dictated by g starting at u. Since $|V(Q_j)| \to \infty$ as $r \to \infty$, we can see the residual finiteness of $\mathbb{F}_n$ playing a role in the proof of the theorem.

Corollary 4.5.6. *If $X \subseteq \mathcal{A}^{\mathbb{F}_n}$ is a subshift of finite type, then $\mathrm{Per}(X, \mathbb{F}_n)$ is weak-* dense in $\mathcal{M}_{\mathbb{F}_n}(X)$.*

Proof. Let $E \subseteq \mathbb{F}_n$ be a finite subset, and $x_1, \dots, x_m \in \mathcal{A}^{\mathbb{F}_n}$ be such that $x \in X$ if and only if for every $g \in \mathbb{F}_n$ there is an $1 \leq i \leq m$ with $(g \cdot x)|_E = (x_i)|_E$. Let $Y = \{x \in \mathcal{A}^{\mathbb{F}_n} : \exists 1 \leq i \leq m \text{ with } x|_E = (x_i)|_E\} \supseteq X$. Note that X is the biggest subshift contained in Y.

We can think of a measure $\mu \in \mathcal{M}_{\mathbb{F}_n}(X)$ as a measure in $\mathcal{M}_{\mathbb{F}_n}(\mathcal{A}^{\mathbb{F}_n})$ whose support is contained in Y, so we may choose an $\epsilon > 0$, an $r \in \mathbb{N}$ and a periodic measure $\lambda \in \mathcal{M}_{\mathbb{F}_n}(\mathcal{A}^{\mathbb{F}_n})$ such that $E \subseteq B(\varepsilon, r)$ and λ is a (r, ϵ)-approximation of μ. Let us show that $\operatorname{supp}(\lambda) \subseteq Y$. If $x|_E \neq (x_i)i|_E$ for all $1 \leq i \leq m$, then $[x\,; B(\varepsilon, r)] \cap Y = \varnothing$, implying $w([x]) = 0$. This, by construction of the graph Γ and of the function w_p, implies that $w_p([x]) = 0$, so

$$\lambda_j([x\,; B(\varepsilon, r)]) = \frac{N w_p([x])}{|V(Q_j)|} = 0.$$

Therefore, $\lambda([x\,; B(\varepsilon, r)]) = 0$, and we conclude $x \notin \operatorname{supp}(\lambda)$. Now, if $x \in \operatorname{supp}(\lambda)$, $C \subseteq \mathbb{F}_n$ and $g \in \mathbb{F}_n$,

$$\lambda([g \cdot x\,; C]) = \lambda(g[x\,; Cg]) = \lambda([x\,; Cg]) > 0,$$

so $g \cdot x \in \operatorname{supp}(\lambda)$ and $\operatorname{supp}(\lambda)$ is $\mathbb{F}_n$-invariant in consequence. This shows $\operatorname{supp}(\lambda) \subseteq X$, since X is the biggest invariant subset of Y. $\square$

Remark 4.5.7 ([Coh20]). It is known there exist aperiodic SFT's on the free group of rank 2, $\mathbb{F}_2$. We want to show an example now. A **Ponzi flow** on a graph $\Gamma = (V, E)$ is a function $\Phi : V \times V \to \mathbb{Z}$ such that

(i) if $(v, w) \notin E$, then $\Phi(v, w) = 0$,

(ii) for all $v, w \in V$, $\Phi(v, w) = -\Phi(w, v)$,

(iii) Φ is bounded,

(iv) for every $w \in V$, $\sum_{v:(v,w)\in E} \Phi(v, w) > 0$.

Further information about Ponzi flows and their relation to amenability can be found in [BW92; Sta09]. We will call a Ponzi flow Φ **normalized** if $\operatorname{im}(\Phi) \subseteq \{-1, 0, 1\}$ and $\Phi(E) \subseteq \{-1, 1\}$. Let a, b be the generators for $\mathbb{F}_2$, and define $\mathcal{A}$ as the set of functions $\{a, a^{-1}, b, b^{-1}\} \to \{-1, 1\}$. For a given normalized Ponzi flow Φ, we define an element $x_\Phi \in \mathcal{A}^{\mathbb{F}_2}$ such that for every $t \in \mathbb{F}_2$ and $s \in \{a, a^{-1}, b, b^{-1}\}$, $x_\Phi(t)(s) = \Phi(ts, t)$. Interpreting $\Phi(ts, t) = 1$ as an arrow going from the vertex ts to the vertex t, such a configuration can be interpreted as if the symbol on each vertex had three or four arrows pointing inwards, as the example shown by Figure 4.4.

Let $X = \{x_\Phi : \Phi \text{ is a normalized Ponzi flow on } \mathbb{F}_2\} \subseteq \mathcal{A}^{\mathbb{F}_2}$. Note that X is an aperiodic SFT, since a Ponzi configuration x_Φ with a finite-index stabilizer H would yield a Ponzi flow on the quotient $\mathbb{F}_2/H$, namely, the flow given by $\bar{\Phi}(Ht, Hg) = \Phi(t, g)$

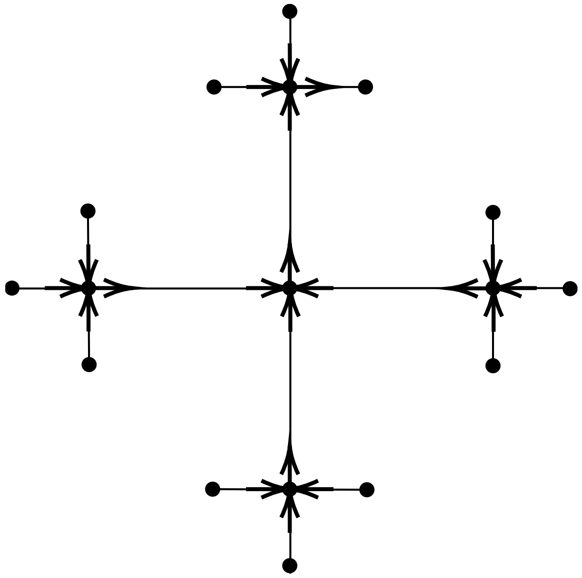

Figure 4.4: A representation of a configuration x_Φ, with Φ a normalized Ponzi flow.

(which is well-defined as a consequence of the H-invariance of x_Φ). However, finite groups (and more generally amenable groups) do not admit Ponzi flows.

The fact that X is an aperiodic SFT does not contradict Corollary 4.5.6, as there are no invariant probability measures supported on X. To see this, note that we may assume X consists only of configurations with exactly three arrows pointing to the vertex (since a configuration admits at most one symbol containing four inward arrows and the group is countable, the measure assigned to this configuration would be zero). Define, for $s \in \{a, a^{-1}, b, b^{-1}\}$, $X(s) := \{x \in X : x(1_{\mathbb{F}_2})(s) = -1\}$, i.e., the set of configurations that have an arrow going out of the identity in the direction of s. Then,

$$s^{-1}X(s) = \bigsqcup_{t \neq s^{-1}} X(t),$$

so $X = X(a^{-1}) \sqcup a^{-1}X(a) = X(b^{-1}) \sqcup b^{-1}X(b)$ is a paradoxical decomposition for X, and there cannot be any invariant measure in consequence.

Chapter 5

Periodic approximations on semigroups

The aim of this chapter is to define and study an analogous property to that of PA, but for semigroups. We start by introducing an adequate notion of periodicity for semigroup actions, as well as establishing the relation between periodic and extensible measures. Given the major role played by residual finiteness in the group-theoretical case, we introduce this notion for semigroups, to then pass to study periodic approximations for residually finite amenable semigroups.

5.1 Periodicity and extensibility

The definition of a periodic configuration in the context of a semigroup action $S \curvearrowright X$ is not straightforward. Emulating the definition for groups, we could say that $x \in X$ is periodic if $|Sx| < \infty$. However, our main purpose is to support invariant measures upon periodic orbits, and we have a first obstacle in the following simple observation.

Lemma 5.1.1. *Let $S \curvearrowright X$ be an action in S-**Top**, and $x \in X$ be such that $|Sx| < \infty$. Then, there is a measure $\mu \in \mathcal{M}_S(X)$ with $\operatorname{supp}(\mu) = Sx$ if, and only if, $tSx = Sx$ for every $t \in S$, i.e., every $t\colon Sx \to Sx$ is a bijection.*

Proof. If $t\colon Sx \to Sx$ is not bijective, there is a $y \in Sx$ such that $t^{-1}(y) \cap (Sx) = \varnothing$. Take now any invariant measure $\mu \in \mathcal{M}_S(X)$ with $\operatorname{supp}(\mu) \subseteq Sx$. Then $\mu(X - Sx) = 0$, so

$$\mu(\{y\}) = \mu(t^{-1}(y)) = \mu(t^{-1}(y) \cap Sx) = 0.$$

This implies $\operatorname{supp}(\mu) \subsetneq Sx$, whence no invariant measure can be supported upon the whole Sx.

Conversely, assume each t acts as a bijection of Sx and define, for every $A \in \mathscr{B}(X)$,

$$\mu(A) = \frac{|A \cap Sx|}{|Sx|}.$$

Define $\varphi_t \colon t^{-1}A \cap Sx \to A \cap Sx$ by $y \mapsto t \cdot y$. It is injective as a consequence of t being injective upon Sx. Now choose any $y \in A \cap Sx$. Since $tSx = Sx$, there is an $y' \in Sx$ with $t \cdot y' = y \in A$, so $y' \in t^{-1}A$ as well, showing surjectivity of φ_t. Thus, μ satisfies

$$\mu(t^{-1}A) = \frac{|t^{-1}A \cap Sx|}{|Sx|} = \frac{|A \cap Sx|}{|Sx|} = \mu(A),$$

as we wanted. $\square$

Now, a natural way of keep going on is to wonder whether there is an S-invariant subset $O \subseteq Sx$ such that for all $s \in S$, $s|_O \colon O \to O$ is bijective. However, this is not necessarily the case. Consider $S = \mathbb{F}_2^+ = \langle a, b\rangle_+$, acting on $2^{\mathbb{F}_2^+}$ by $(s \cdot x)(t) = x(ts)$. Let $x \in 2^{\mathbb{F}_2^+}$ be the configuration defined by $x(as) = 1$ and $x(bs) = 0$ for all $s \in \mathbb{F}_2^+$. The orbit of this configuration does not satisfy the desired condition, and thus there is no subset of this orbit which can support an invariant measure. So, although these kind of orbits exhibit a valid form of periodicity, they will not be taken in consideration here.

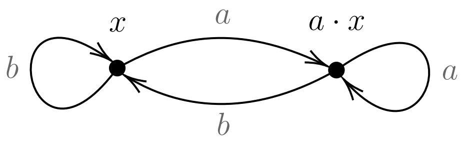

Figure 5.1: A diagram of the $\mathbb{F}_2^+$-orbit of x.

Definition 5.1.2. Let $S \curvearrowright X$ be a semigroup action. An element $x \in X$ will be called **pre-periodic** if $|Sx| < \infty$, and **periodic** if $|Sx| < \infty$ and for all $t \in S$, $tSx = Sx$.

Remark 5.1.3. The last example given shows as well that this notion of periodicity is not equivalent to (it is strictly stronger than) the condition that for all $s, t \in S$, $s \in Stx$ (i.e., every pair on elements of the orbit being in the orbit of each other), since this last condition is satisfied by the configuration whose orbit is the one shown in Figure 5.1.

Also, the condition of $tSx = Sx$ does not necessarily imply $|Sx| < \infty$ (although in the $S = \mathbb{N}$ case it does). Just take, for instance, the element $x \in 2^{\mathbb{N}^2}$ given by $x(n, m) = 1$ if and only if $n = m$. This configuration has infinite translates, and every element of $\mathbb{N}^2$ acts bijectively upon this orbit.

Remark 5.1.4. In the context of group actions, we saw that every periodic element induces a finite-index subgroup (namely, $\mathrm{Stab}(x)$), and a tiling of the group via translates of any fundamental domain.

For semigroups, if $x \in X$ has finite orbit, it induces a left-compatible equivalence relation on S by

$$s \sim t \iff s \cdot x = t \cdot x,$$

and a congruence on S by

$$s \sim t \iff sr \cdot x = tr \cdot x \text{ for all } r \in S.$$

Moreover, both relations have **finite index**, i.e., finitely many equivalence classes, so in order to admit periodic points, a semigroup must contain finite-index congruences. Note that, in the group case, the analogous to the left-compatible relation is the one associated with the stabilizer (i.e., $g \sim h$ if, and only if, $gh^{-1} \in \operatorname{Stab}(x)$). If moreover the stabilizer is a normal subgroup, both the left-compatible relation and the congruence induced by a periodic element x coincide with the relation induced by $\operatorname{Stab}(x)$.

The following property of periodic orbits will be relevant for us. While the proof for groups is completely trivial, here it requires a simple but slightly different argument, thus making it worth to mention.

Proposition 5.1.5. *If $x \in X$ is S-periodic, then the action $S \curvearrowright Sx$ is transitive, i.e., for every $s_1, s_2 \in S$ there is a $t \in S$ with $ts_1 \cdot x = s_2 \cdot x$.*

Proof. Assume there exists an $t \in S$ such that $x \notin S(t \cdot x)$. Since Sx is finite, there exist $1 \leq k < n$ in $\mathbb{N}$ such that $t^n \cdot x = t^k \cdot x$. Let

$$k = \min\{i \geq 1 : \text{ there is a } j > i \text{ with } t^j \cdot x = t^i \cdot x\}.$$

By definition, there is an $n > k$ with $t^n \cdot x = t^k \cdot x$, and we must have $t^{n-1} \cdot x \neq t^{k-1} \cdot x$. Now, note that

$$t(t^{n-1} \cdot x) = t^n \cdot x = t^k \cdot x = t(t^{k-1} \cdot x),$$

contradicting the injectivity of t upon Sx.

Take now two arbitrary elements $s_1, s_2 \in S$. We just proved there is a $t \in S$ with $ts_1 \cdot x = x$, so $(s_2 t)s_1 \cdot x = s_2 \cdot x$, as desired. ☐

It is important to know the behaviour of images of G-orbits via S-equivariant maps.

Proposition 5.1.6. *Let G be a group containing S as a subsemigroup, and let $G \curvearrowright Y$ and $S \curvearrowright X$ be two actions. If $\pi : Y \to X$ is an S-equivariant map and $y \in Y$ is such that $|Gy| < \infty$, then $\pi(y)$ is S-periodic.*

Proof. First, clearly $S\pi(y) = \pi(Sy) \subseteq \pi(Gy)$, so $|S\pi(y)| \leq |Gy| < \infty$ and $\pi(y)$ is pre-periodic. Now, if $s, t \in S$, then the set $\{s^n ty : n \geq 1\}$ is finite, which means there are $k > j$ such that $s^k ty = s^j ty$. Since the action on Y is given by a group, we have that $ty = s^{k-j} ty$ and $s^{k-j} \in S$. Therefore,

$$t\pi(y) = s(s^{k-j-1} ty) \in s(S\pi(y)),$$

so $s : S\pi(y) \to S\pi(y)$ is surjective, and hence bijective by finiteness of $S\pi(y)$. ☐

Definition 5.1.7. Let $S \curvearrowright X$ be an action in S-**Top**. A measure $\mu \in \mathcal{M}_S(X)$ is **ergodic** if every set $A \in \mathscr{B}(X)$ such that $\mu(s^{-1}A\triangle A) = 0$ for all $s \in S$ satisfies $\mu(A) \in \{0, 1\}$.

Just as in the group-theoretical case, a **periodic measure** in $\mathcal{M}_S(X)$ will be a measure $\mu \in \mathcal{M}_S(X)$ such that $\operatorname{supp}(\mu)$ is finite. The set of periodic measures will be denoted by $\operatorname{Per}(X, S)$, and the set of periodic ergodic measures in $\mathcal{M}_S(X)$ will be denoted as $\operatorname{Perg}(X, S)$. We have the following characterizations of periodic measures.

Proposition 5.1.8. *Let $\mu \in \mathcal{M}_S(X)$. Then,*

(i) $\mu \in \operatorname{Per}(X, S)$ *if, and only if,* $\operatorname{supp}(\mu)$ *is a finite disjoint union of periodic S-orbits,*

(ii) $\mu \in \operatorname{Perg}(X, S)$ *if, and only if,* $\operatorname{supp}(\mu)$ *corresponds to a single periodic S-orbit.*

Proof. Note that two distinct periodic orbits must have empty intersection. Indeed, if $y \in Sx_1 \cap Sx_2$, there are elements $s_1, s_2 \in S$ such that $y = s_1 \cdot x_1 = s_2 \cdot x_2$. Since the action $S \curvearrowright Sx$ is transitive, we may choose an element $t \in S$ with $ts_1 \cdot x_1 = x_1$, so $ts_2 \cdot x_2 = x_1$, which implies $Sx_1 \subseteq Sx_2$. By symmetry, we get $Sx_2 \subseteq Sx_1$, and so both orbits coincide.

To prove (i), simply note that if $x \in \operatorname{supp}(\mu)$ then

$$\mu(\{s \cdot x\}) = \mu(s^{-1}(s \cdot x)) \geq \mu(\{x\}) > 0,$$

so $\operatorname{supp}(\mu)$ is S-invariant. This means $Sx \subseteq \operatorname{supp}(\mu)$ for all $x \in \operatorname{supp}(\mu)$, which implies the statement.

Finally, we prove (ii). Assume $\mu \in \operatorname{Perg}(X, S)$. By the definition of periodic measure, we just need to show there cannot be two distinct S-orbits contained in the support of μ. Assume $x_1, x_2 \in X$ are S-periodic elements such that $Sx_1 \sqcup Sx_2 \subseteq \operatorname{supp}(\mu)$. Then, there exist $0 < \alpha_1, \alpha_2 < 1$ such that $\mu(Sx_i) = \alpha_i$ for $i \in \{1, 2\}$. Let $t \in S$. Since $Sx_1 \cap Sx_2 = \varnothing$, we have $t^{-1}(Sx_1) \cap t^{-1}(Sx_2) = \varnothing$, and moreover $\operatorname{supp}(\mu) - t^{-1}(Sx_1) = \varnothing$. Also note that $Sx_1 \subseteq t^{-1}(Sx_1)$, as x_1 is S-periodic. Therefore,

$$\mu(t^{-1}(Sx_1)\triangle Sx_1) = \mu(t^{-1}(Sx_1) - Sx_1) + \mu(Sx_1 - t^{-1}Sx_1) = 0,$$

but $\mu(Sx_1) = \alpha_1 \in (0, 1)$, a contradiction. Hence $\operatorname{supp}(\mu)$ consists of a single orbit.

Conversely, if $\operatorname{supp}(\mu) = Sx$ with x S-periodic and $\mu(s^{-1}A\triangle A) = 0$ for all $s \in S$, then $Sx \cap (A - s^{-1}A) = \varnothing$ for all $s \in S$, and we have two cases. First, if $Sx \cap A = \varnothing$ then $\mu(A) = 0$. Second, if $t \cdot x \in Sx \cap A$, necessarily we must have $t \cdot x \in s^{-1}A$ as well, obtaining $st \cdot x \in A$ for every $s \in S$. Since the action of S upon Sx is transitive, this implies $Sx \subseteq A$, and so $\mu(A) = 1$. Thus, μ is ergodic. $\square$

As a consequence of last proposition, a periodic measure can always be written in the form

$$\mu = \sum_{k=1}^{n} \frac{\alpha_k}{|Sx_k|} \sum_{y \in Sx_k} \delta_y,$$

where $\sum_{k\leq n}\alpha_k = 1$, with $n = 1$ if the measure is moreover ergodic.

A first key observation is that periodic measures are extensible to the free S-group.

Proposition 5.1.9. *Assume that S is embedded into the free S-group (G,γ), and let $S \curvearrowright X$ be an element in S-**Top**. Then, $\mathrm{Per}(X,S) \subseteq \mathrm{Ext}_G(X,S)$.*

Proof. Note that $\mathrm{Ext}_G(X,S)$ is a convex subset of $\mathcal{M}_S(X)$, so it suffices to show that the measures supported upon a single orbit are extensible. Let $\mu \in \mathrm{Per}(X,S)$ be given by

$$\mu = \frac{1}{|Sx|}\sum_{y\in Sx}\delta_y,$$

where $x \in X$ is S-periodic. For every $s \in S$, define $s^{-1}y$ formally as the only element of $s^{-1}(\{y\}) \cap Sx$. Let $F^+ = F(S\cup S^{-1})^+ = (S\cup S^{-1})^*$ be the free semigroup on the alphabet $S \cup S^{-1}$ (note that we are not using the relators of S in the definition of F^+); thus, we may define a semigroup action $F^+ \curvearrowright X$ by extending the definition of $s\cdot y$ on $S\cup S^{-1}$ to all F^+ by concatenation. We want to see this action descends to an action of the free S-group G, for which it suffices to show that two words in F^+ that define the same element in G, act as the same permutation of Sx. Since for all $s \in S$ and $y, z \in Sx$ we have $s \cdot y = z$ if and only if $y = s^{-1}\cdot z$, two words $w, w' \in F^+$ satisfy $w \cdot y = w' \cdot y$ for every $y \in Sx$ if and only if $w^{-1}w' \cdot y = y$ for all such y, where $w^{-1} = s_n^{-\epsilon_n}\cdots s_1^{-\epsilon_1}$ if $w = s_1^{\epsilon_1}\cdots s_n^{\epsilon_n}$. Therefore, we need to check that every word $w \in F^+$ which is the identity in G fixes the set Sx.

Define the relation $\sim$ on F^+ as follows:

$$w_1 \sim w_2 \iff \forall y \in Sx, w_1\cdot y = w_2 \cdot y.$$

Suppose that $w_1 \sim w_2$. We have that, for any $s \in S$,

$$w_1 s\cdot y = w_1\cdot(s\cdot y) = w_2\cdot(s\cdot y) = (w_2 s)\cdot y,$$

and a similar equality holds if we replace s by s^{-1}. Analogously,

$$sw_1\cdot y = s\cdot(w_1\cdot y) = s\cdot(w_2\cdot y) = sw_2\cdot y,$$

and again the same holds for s^{-1}. Thus, if $w_1 \sim w_2$ then $w_1 s^{\pm1} \sim w_2 s^{\pm 1}$ and $s^{\pm1}w_1 \sim s^{\pm1}w_2$. Naturally, by induction, this extends to any element of F^+, thus making $\sim$ a congruence on F^+.

Furthermore, since $ss^{-1}\cdot y = y = s^{-1}s\cdot y$ for all $y \in Sx$ and $s \in S$, we have $w^{-1}w\cdot y = y = ww^{-1}\cdot y$ for all $y \in Sx$. This means $[w]_\sim[w^{-1}]_\sim = [ww^{-1}]_\sim = [1_S]_\sim$ and similarly $[w^{-1}]_\sim[w]_\sim = [1_S]_\sim$. Hence, the quotient semigroup $Q := F^+/\sim$ is actually a group, where the inverse of $[w]_\sim$ is just $[w^{-1}]_\sim$.

Clearly the inclusion $\iota\colon S \to F^+$ is a semigroup morphism, so the composition $\pi_\sim\circ\iota\colon S \to Q$ is a semigroup morphism sending $s \mapsto [s]_\sim$. As every element of Q may be written as a product of elements $[s^{\pm1}]_\sim = [s]_\sim^{\pm1}$, for $s \in S$, we see that $(Q, \pi_\sim\circ\iota)$ is an

S-group (note that, in general, the morphism $\pi_\sim \circ \iota$ is not injective). Letting (G, γ) be the free S-group, by the universal property we get a group morphism $\theta\colon G \to Q$. Any element $w \in F^+$ which represents the identity on G must be mapped to the identity $[1_S]_\sim$ of Q, which is the same as saying it fixes every element of Sx.

We have a well defined action $G \curvearrowright Sx$. For a given $y \in Sx$, let $\overline{y} \in X_G$ be given by $\overline{y}(g) = g \cdot y$. Note that if $t, g \in G$ then

$$(t \cdot \overline{x})(g) = \overline{x}(gt) = gt \cdot x = \overline{t \cdot x}(g),$$

so $t \cdot \overline{x} = \overline{t \cdot x}$. This shows $|G\,\overline{x}| = |Sx| < \infty$, so $\overline{x}$ is G-periodic and the measure

$$\overline{\mu} = \frac{1}{|Sx|} \sum_{y \in G\,\overline{x}} \delta_y$$

is periodic and G-invariant. Finally, note that

$$\overline{\mu}(\pi^{-1}A) = \frac{1}{|Sx|} \sum_{y \in G\,\overline{x}} \delta_y(\pi^{-1}A) = \frac{1}{|Sx|} \sum_{y' \in Sx} \delta_{y'}(A) = \mu(A).$$

□

Example 5.1.10. Note that if we remove the assumption that G is the free S-group then last proposition does not necessarily hold. An example of this has already been given in Example 2.5.1, which exhibits a system consisting of an $\mathbb{F}_2^+$-periodic orbit with empty topological $\mathrm{BS}(1,2)$-extension. Of course, this means that the corresponding $\mathbb{F}_2^+$-periodic measure cannot be extended.

Lemma 5.1.11. *If $\overline{\mu} \in \mathcal{M}(X_G, G)$ has compact support, then supp$(\pi_*\overline{\mu}) = \pi($supp$(\overline{\mu}))$. In particular, we have the following:*

(i) $\pi_*(\mathrm{Per}(X_G, G)) \subseteq \mathrm{Per}(X, S)$, *and*

(ii) *if* $\overline{\mathrm{Per}(X_G, G)} = \mathcal{M}(X_G, G)$, *then* $\mathrm{Ext}_G(X, S) \subseteq \overline{\mathrm{Per}(X, S)}$.

Proof. We prove the main statement first. Let $x \in \mathrm{supp}(\pi_*\bar{\mu})$ and fix a countable sequence $(U_n)_{n\geq 1}$ of open neighborhoods of x with $\bigcap_n U_n = \{x\}$. For every n, $\pi_*\bar{\mu}(U_n) > 0$, or, equivalently, there is a $\bar{x}_n \in \mathrm{supp}(\bar{\mu})$ with $\pi(\bar{x}_n) \in U_n$. By compactness, we can take a convergent subsequence $\bar{x}_{n_k} \to \bar{x} \in \mathrm{supp}(\bar{\mu})$ as $k \to \infty$. By continuity of π we get $\pi(\bar{x}_{n_k}) \to \pi(\bar{x})$, but since $\pi(\bar{x}_{n_k}) \in U_{n_k}$, we conclude $\pi(\bar{x}) = x$, so $x \in \pi(\mathrm{supp}(\bar{\mu}))$.

For the opposite inclusion, let $\bar{x} \in \mathrm{supp}(\bar{\mu})$. If U is an open subset of X containing $\pi(\bar{x})$, then $\bar{x} \in \pi^{-1}U$ and

$$\pi_*\bar{\mu}(U) = \bar{\mu}(\pi^{-1}U) > 0,$$

so $\pi(x) \in \mathrm{supp}(\pi_*\bar{\mu})$, following the desired inclusion.

Now we prove (i). If $\bar{\mu} \in \mathrm{Per}(X_G, G)$ then

$$\mathrm{supp}(\bar{\mu}) = \bigcup_{i=1}^{n} Gx_i$$

so by the equality proven above we have

$$\operatorname{supp}(\pi_*\bar{\mu}) = \bigcup_{i=1}^{n} \pi(Gx_i),$$

which is a S-periodic orbit by Proposition 5.1.6. Finally, by (i) and weak-* continuity of π_* we have

$$\operatorname{Ext}_G(X,S) = \operatorname{im}(\pi_*) = \pi_*\left(\overline{\operatorname{Per}(X_G,G)}\right) \subseteq \overline{\operatorname{Per}(X,S)}.$$

□

Proposition 5.1.12. *Let $S \overset{\alpha}{\curvearrowright} (X,\mu)$ a p.m.p. action and $\mu_G \in \mathcal{M}_G(X_G)$ with $\pi_*\mu_G = \mu$. Then $\operatorname{supp}(\mu_G) \subseteq (\operatorname{supp}(\mu))_G$. In particular, if the support of a measure does not have a topological G-extension, then this measure cannot be G-extensible.*

Proof. By Lemma 5.1.11 we know that

$$\pi|_{\operatorname{supp}(\mu_G)} \colon \operatorname{supp}(\mu_G) \to \operatorname{supp}(\mu)$$

is surjective, so we just need to prove that $\operatorname{supp}(\mu_G) = (\operatorname{supp}(\mu))_G$ if we consider the latter as a subset of X_G. Note that if $\bar{x} = (x_g)_{g\in G} \in \operatorname{supp}(\mu_G) \subseteq X_G$ and $U \subseteq X$ is an open set containing x_t for $t \in G$, then $\pi_t^{-1}U$ is an open set containing $\bar{x}$, which means

$$\mu(U) = \mu_G(\pi_t^{-1}U) > 0,$$

so $x_t \in \operatorname{supp}(\mu)$ for all $t \in G$. Hence $\operatorname{supp}(\mu_G) \subseteq (\operatorname{supp}(\mu))_G$. □

5.2 Residual finiteness in semigroups

Definition 5.2.1. A semigroup S is **residually a finite group** (resp. **residually a finite semigroup**) if for every pair of distinct elements $s, t \in S$ there is a finite group (resp. semigroup) F and a semigroup morphism $\theta\colon S \to F$ such that $\theta(s) \neq \theta(t)$.

Remark 5.2.2. It is clear that being residually a finite group always implies being residually a finite semigroup. Furthermore, we have the converse in the case where S is a group, since the image of a group via a semigroup morphism is always a group. In this situation, both notions coincide with the classic notion of residual finiteness for groups. However, for general semigroups these notions differ: every finite non-bicancellative semigroup is residually a finite semigroup but not residually a finite group. Indeed, if S is such a semigroup, $\theta\colon S \to F$ is a morphism to a group F, and $a, b, c \in S$ are such that $ab = ac$ and $b \neq c$, then $\theta(a)\theta(b) = \theta(a)\theta(c)$ so $\theta(b) = \theta(c)$.

Most of the usual characterisations of residual finiteness in groups translate to the semigroup scenario. In particular:

Proposition 5.2.3. *A semigroup S is residually a finite group (resp. semigroup) if, and only if, it can be embedded in a Cartesian product of finite groups (resp. semigroup).*

Proof. Suppose S is residually a finite semigroup. For every $s, t \in S$ with $s \neq t$, there exists a finite semigroup $F_{s,t}$ and a semigroup morphism $\theta_{s,t} \colon S \to F_{s,t}$ such that $\theta_{s,t}(s) \neq \theta_{s,t}(t)$. Hence, if we define the semigroup:

$$T = \prod_{\substack{s,t \in S \\ s \neq t}} F_{s,t},$$

and the semigroup morphism $\iota \colon S \to T$ defined coordinate-wise by $\iota(u) = (\theta_{s,t}(u))_{s \neq t \in S}$, it is easy to see that, for any $s, t \in S$, $\iota(s)_{s,t} \neq \iota(t)_{s,t}$, so ι embeds S into a semigroup with the desired property.

Conversely, any subsemigroup S of a product $\prod_{i \in I} F_i$ of finite semigroups is residually a finite semigroup; indeed, if $s \neq t \in S$, then for some index $j \in I$ we must have $s_j \neq t_j$, so the projection map $\pi_j \colon \prod_{i \in I} F_i \to F_j$ is a semigroup morphism to a finite semigroup which satisfies $\pi_j(s) \neq \pi_j(t)$.

The corresponding equivalence for the case where S is residually a finite group follows immediately from the fact that we can choose each of the finite semigroups $F_{s,t}$ (or F_i) as a finite group. $\square$

Since any subgroup of a product of finite groups is a residually finite group, we immediately obtain the following:

Corollary 5.2.4. *A semigroup is residually a finite group if and only if it can be embedded in a residually finite group. In particular, such a semigroup is necessarily bicancellative.*

Remark 5.2.5. We already know that if S is a semigroup which is residually a finite group, then it can be embedded in a residually finite group. However, this does not imply residual finiteness of the free S-group. An example of this would be the Baumslag-Solitar semigroup

$$\mathrm{BS}(2,3)^+ = \langle a, b \mid ab^2 = b^3 a \rangle,$$

which is residually a finite group (as proven in [Jac02, Theorem 4.5]), while the corresponding Baumslag-Solitar group $\mathrm{BS}(2,3)$ is known to be non-Hopfian (in particular, non-residually finite). See [BS62].

Remark 5.2.6 (Higman semigroup). An example of a semigroup which is embeddable in a group but not residually a finite group is the subsemigroup H^+ of the Higman group,

$$H = \langle a, b, c, d \mid ba = ab^2, cb = bc^2, dc = cd^2, ad = da^2 \rangle,$$

generated by a, b, c, d. The original proof (see [Hig51]) of the fact that the Higman group does not have finite quotients (and thus cannot be residually finite) relies on equalities of the form $yx^n = x^n y^{2^n}$ where x and y are two generators chosen consecutively; naturally, these equalities also hold in H^+. If there were a non-trivial semigroup epimorphism $\varphi\colon H^+ \to F$, where F is a finite group, then these equalities would imply that, if $n_a, \ldots, n_d$ are the respective orders of the four generators (which cannot be all equal to 1 if F is non-trivial), then $n_b \mid 2^{n_a} - 1, n_c \mid 2^{n_b} - 1$ and so on. Then an elemental number-theoretical argument leads to a contradiction with the hypothesis that $n_a, \ldots, n_d$ are not all 1, and thus F is forced to be a trivial group.

A semigroup S such that for every pair of distinct elements $x, y \in S$, there is some $a \in S$ with $ax \neq ay$, is called a **left reductive** semigroup. Equivalently, S is left reductive if, and only if, the action $S \curvearrowright \mathcal{A}^S$ is **faithful** in the sense that whenever $s \neq t$ there exists a configuration $x \in \mathcal{A}^S$ such that $s \cdot x \neq t \cdot x$. In particular, every monoid and every left cancellative semigroup is left reductive. This condition becomes relevant if we want to state the following result in maximum generality.

Proposition 5.2.7. *Let S be a left reductive semigroup. Then:*

- (i) *S is residually a finite semigroup if, and only if, the set of pre-periodic points of $\mathcal{A}^S$ is dense in $\mathcal{A}^S$ for every finite alphabet $\mathcal{A}$.*
- (ii) *S is residually a finite group if, and only if, the set of periodic points of $\mathcal{A}^G$ is dense in $\mathcal{A}^G$ for every finite alphabet $\mathcal{A}$.*

Proof. We shall only prove (ii), as the proof of (i) is essentially the same, except by replacing any relevant group by an appropriate semigroup (e.g. replacing the symmetric group by the corresponding full transformation monoid, and so on).

Suppose that S is residually a finite group, $x \in \mathcal{A}^S$ is an arbitrary point, and let $U \subset S$ be an arbitrary finite set. As S is residually a finite group, for every pair of distinct elements $s, t \in S$ there exists a semigroup morphism $\theta_{s,t}\colon S \to F_{s,t}$ for which $\theta_{s,t}(s) \neq \theta_{s,t}(t)$; thus, we may define a semigroup morphism $\theta\colon S \to \prod_{s \neq t \in U} F_{s,t}$ for which $\theta(s) \neq \theta(t)$ for any $s, t \in U$, $s \neq t$, by sending any $u \in S$ to the tuple $(\theta_{s,t}(u))_{s \neq t \in U}$. As there are finitely many pairs (s, t) with $s \neq t$ and $s, t \in U$, the group $F = \prod_{s \neq t \in U} F_{s,t}$ is necessarily finite.

Define a configuration $\hat{x}\colon F \to \mathcal{A}$ by $\hat{x}(\theta(s)) = x(s)$ for all $s \in U$, extending it arbitrarily to the rest of F if needed. As the map $\theta|_U\colon U \to F$ is injective, this is well-defined. Note, now, that every $y \in \mathcal{A}^F$ defines a point $\bar{y} \in \mathcal{A}^S$ by $\bar{y}(s) = y(\theta(s))$ for all $s \in S$. Such a point satisfies the equality $s \cdot \bar{y} = \overline{\theta(s) \cdot y}$ for any $s \in S$, and is thus periodic, because F is a group and $\mathcal{A}^F$ is finite. In particular, if $y = \hat{x}$, the corresponding point in $\mathcal{A}^S$ satisfies the equality $\bar{y}|_U = x|_U$. This shows that, for any $x \in \mathcal{A}^S$ and any finite $U \subset S$, the cylinder $[x|_U]$ must contain at least one periodic point, and thus the set of periodic points is dense.

Conversely, if $\mathcal{A}^S$ has dense periodic points and $x, y \in S$ are distinct elements, we proceed as follows. By left reductiveness, there exists $a \in S$ such that $ax \neq ay$. Choose two distinct elements from $\mathcal{A}$, which, for simplicity, will be denoted 0 and 1, and consider any sequence $\omega \in \mathcal{A}^S$ such that $\omega(ax) = 0$ and $\omega(ay) = 1$. By our hypothesis of density of periodic points, we may assume ω is S-periodic and consider $F = \mathrm{Sym}(S\omega)$, which is a finite group. The definition of S-periodicity yields a morphism $\theta\colon S \to F$ by sending $s \in S$ to the function $s|_{S\omega}\colon S\omega \to S\omega$ given by $s|_{S\omega}(t \cdot \omega) = st \cdot \omega$ for all $t \in S$.

By our choice of ω and a, we must have $(x \cdot \omega)(a) = \omega(ax) = 0$ and $(y \cdot \omega)(a) = \omega(ay) = 1$. Hence, $x \cdot \omega \neq y \cdot \omega$, and thus $x|_{S\omega}$ and $y|_{S\omega}$ correspond to different permutations of Sx, that is, $\theta(x) \neq \theta(y)$. ☐

Since a left cancellative semigroup is automatically left reductive, any semigroup that embeds into a group is necessarily left reductive. In particular, as this holds up for any semigroup that is residually a finite group, such a semigroup is forced to be left reductive. In the non-left reductive case, however, the associated full shift may have dense periodic points, but this does not (and cannot) imply that the semigroup is residually a finite group.

Example 5.2.8 (Non-reductive case). Consider the two-element **left zero** semigroup $(L, *)$, where $L = \{a, b\}$ and $s * t = s$ for all $s, t \in L$, and define $S = \mathbb{Z} \times L$. Observe that, for every $n \in \mathbb{Z}$ and $s \in L$, we have

$$(n, s) \cdot (0, a) = (n, s) = (n, s) \cdot (0, b),$$

hence S is not a left reductive semigroup. However, $\mathcal{A}^S$ has dense periodic points. To see this, take any configuration $x \in \mathcal{A}^S$, and consider two sequences $(x_n^a)_{n\geq 1}, (x_n^b)_{n\geq 1} \subseteq \mathrm{Per}(\mathcal{A}^{\mathbb{Z}})$ such that $x_n^a \to x|_{\mathbb{Z}\times\{a\}}$ and $x_n^b \to x|_{\mathbb{Z}\times\{b\}}$ as $n \to \infty$. Define now the sequence $(x_n)_{n\geq 1} \subseteq \mathcal{A}^S$, where $x_n|_{\mathbb{Z}\times\{a\}} = x_n^a$ and $x_n|_{\mathbb{Z}\times\{b\}} = x_n^b$ for each $n \geq 1$. It is straightforward that $x_n \to x$ as $n \to \infty$. Let us see that each x_n is S-periodic.

Indeed, if p_n is the least common multiple of the least periods of x_n^a and x_n^b, we have that

$$\begin{aligned}
((p_n, s) \cdot x_n)(m, t) &= x_n(m + p_n, t * s) \\
&= x_n(m + p_n, t) \\
&= x_n^t(m + p_n) \\
&= x_n^t(m) \\
&= x_n(m, t),
\end{aligned}$$

and from here it easily follows that $Sx_n = \{(m, t) \cdot x_n : 0 \leq m < p_n, t \in L\}$, which is finite set, and each map $y \mapsto (m, t) \cdot y$ is a bijection from Sx_n to itself with inverse $y \mapsto (p_n - m, t) \cdot y$, so x_n is a periodic point.

It is already clear that, in general, if S is residually a finite group, the free S-group need not be residually finite, as we pointed out earlier. However, things turn out to be simpler in the reversible case.

Proposition 5.2.9. *Let S be a left reversible semigroup, and G be the group of right fractions of S. Then, S is residually a finite group if, and only if, G is residually finite.*

Proof. That G being residually finite implies S being residually a finite group is straightforward. Now assume S is residually a finite group. We know that this is equivalent to the existence of an embedding $\iota\colon S \to H$, where H is a residually finite group. This implies that G is a subgroup of H, and thus it is residually finite. ☐

5.3 Semigroups with periodic approximations

We end this chapter by introducing the definition of periodic approximations on semigroups, and generalizing the results proven in the case of groups in Chapter 4.

Definition 5.3.1. A semigroup S is said to have **periodic approximations** (abbreviated PA) if the set $\operatorname{Per}(\mathcal{A}^S, S)$ is weak-* dense in $\mathcal{M}_S(\mathcal{A}^S)$ for every finite alphabet $\mathcal{A}$, and **ergodic periodic approximations** (abbreviated EPA) if the set $\operatorname{Perg}(\mathcal{A}^S, S)$ is weak-* dense in $\mathcal{M}_S(\mathcal{A}^S)$ for every finite alphabet $\mathcal{A}$

Residual finiteness turns out to be also a necessary condition for a semigroup to have the property PA. The proof is the same as the one for the group-theoretical case, but we include it for completeness.

Proposition 5.3.2. *Let S be a left reductive semigroup. If S is* PA*, then it is residually a finite group.*

Proof. Assume S is not residually a finite group. By Proposition 5.2.7, there is a configuration $x \in \mathcal{A}^S$ and a finite subset $F \subseteq S$ such that there is no S-periodic element in $[x\,;F]$. Choose a measure $\mu \in \mathcal{M}_S(\mathcal{A}^S)$ such that $\mu([x\,;F]) > 0$ (take, for instance, the Bernoulli measure associated to a positive probability vector), and let $(\mu_n)_{n\geq 1}$ be a sequence converging weakly to μ, such that for every $n \geq 1$,

$$\mu_n = \sum_{i=1}^{n} \frac{\alpha_i}{|Sx_i^n|} \sum_{y \in Sx_i^n} \delta_y,$$

where $\sum_{i=1}^n \alpha_i = 1$ and x_i^n is S-periodic for every $1 \leq i \leq n$. By weak convergence, we have that $\mu_n([x\,;F]) \to \mu([x\,;F])$. However, for all $n \geq 1$ and $1 \leq i \leq n$ the set $[x\,;F] \cap Sx_i^n$ is empty, meaning that $\mu_n([x\,;F]) = 0$, a contradiction with our choice of μ. ☐

When the free S-group is PA, we reach some interesting conclusions as a consequence of Lemma 5.1.11 and Proposition 5.1.9. Combining both results together yields the following key result.

Corollary 5.3.3. *Let S be an embeddable semigroup such that the free S-group G is* PA*. Then,*

$$\operatorname{Ext}_G(\mathcal{A}^S, S) = \overline{\operatorname{Per}(\mathcal{A}^S, S)}.$$

In particular, when we know that all S-invariant measures on $\mathcal{A}^S$ are G-extensible, we can conclude that the semigroup S has property PA.

Proposition 5.3.4. *The following statements hold.*

(i) *If S is a left reversible semigroup such that the group G of right fractions has property* PA *(resp.* EPA*), then S has property* PA *(resp.* EPA*).*

(ii) *The semigroup $\mathbb{F}_n^+$ has property* PA.

Proof. We prove first assertion (i). We already know that if S is reversible, then $\mathrm{Ext}_G(\mathcal{A}^S, S) = \mathcal{M}_S(\mathcal{A}^S)$, whence it follows immediately that S is PA if G is PA. Now, if G is EPA and $\mu \in \mathcal{M}_S(\mathcal{A}^S)$, there is a sequence $(\bar{\mu}_n)_{n\geq 1}$ in $\mathcal{M}_G(\mathcal{A}^G)$ such that each $\bar{\mu}_n$ is G-periodic and ergodic, and $\pi_*\bar{\mu}_n \xrightarrow{*} \mu$, so it suffices to show that each $\pi_*\bar{\mu}_n$ is S-periodic and ergodic. We have already proven that

(i) for any $\bar{\nu}$ in $\mathcal{M}_G(\mathcal{A}^G)$ with compact support, $\mathrm{supp}(\pi_*\bar{\nu}) = \pi(\mathrm{supp}(\bar{\nu}))$, and

(ii) if $\bar{x} \in \mathcal{A}^G$ is G-periodic, then $\pi(\bar{x})$ is S-periodic.

Therefore, since $\mathrm{supp}(\bar{\mu}_n) = G\bar{x}_n$ for some G-periodic element $\bar{x}_n$ from $\mathcal{A}^G$, we get that $\mathrm{supp}(\pi_*\bar{\mu}_n) = \pi(G\bar{x}_n)$ is S-periodic, from where it follows that $\pi_*\bar{\mu}_n$ is S-periodic and ergodic.

The proof of (ii) follows directly from Corollary 5.3.3, since we have already proven in Chapter 3 that every measure in $\mathcal{M}_{\mathbb{F}_n^+}(\mathcal{A}^{\mathbb{F}_n^+})$ is $\mathbb{F}_n$-extensible, $\mathbb{F}_n$ is the free $\mathbb{F}_n^+$-group and it is PA. □

Recall that every amenable, residually finite group is an EPA group (§4.4). We want to establish an analogous result for bicancellative semigroups. The proof gathers several facts from this and previous sections, and makes use of the fact that amenability is inherited from semigroups to their S-groups, as proven in Proposition 1.4.10.

Theorem 5.3.5. *Let S be a left amenable semigroup which is residually a finite group. Then, S is a* EPA *semigroup.*

Proof. By our hypothesis, S is bicancellative and left amenable, hence left reversible and so it embeds into its group G of right fractions. Since S is left amenable and residually a finite group, by Lemma 1.4.10 and proposition 5.2.9 we conclude that G is amenable and residually finite. Now, every such group is a PA group, so by Corollary 5.3.3 S is a PA semigroup. More precisely:

(1) we have that $\mathrm{Ext}_G(\mathcal{A}^S, S) = \overline{\mathrm{Per}(\mathcal{A}^S, S)}$ as a consequence of G having the property PA, and

(2) the equality $\mathrm{Ext}_G(\mathcal{A}^S, S) = \mathcal{M}_S(\mathcal{A}^S)$ holds as a consequence of G being the group of right fractions.

□

Appendices

A Cayley graphs

Cayley graphs play a crucial role in certain branches of Group Theory, such as Geometric Group Theory, as they are the main way of viewing groups as geometric objects.

Definition A.1. Let G be a group and $S \subseteq G$ a finite generating set for G. The **left** and **right Cayley graphs** of G with respect to S are defined, respectively, as

$$\mathrm{Cay}_{\mathrm{L}}(G;S) = \Big\{\{g, sg\} : g \in G, s \in (S \cup S^{-1}) - \{1_G\}\Big\}, \text{ and}$$

$$\mathrm{Cay}_{\mathrm{R}}(G;S) = \Big\{\{g, gs\} : g \in G, s \in (S \cup S^{-1}) - \{1_G\}\Big\}.$$

Note that for a given $g \in G$ and $s \in (S \cup S^{-1}) - \{1_G\}$, $s^{-1}(sg) = g$, thus making sense to define $\mathrm{Cay}_{\mathrm{L}}(G;S)$ as an undirected graph. This is not the case when we pass to semigroups, where Cayley graphs will be of intrinsic directed nature.

Definition A.2. Let S be a semigroup generated by a finite set $B \subseteq S$. Adjoin an identity element $1_S \in S$ if necessary. The **left** and **right Cayley graphs** of S with respect to B are defined, respectively, as

$$\mathrm{Cay}_{\mathrm{L}}(S;B) = \Big\{(s, ts) : s \in S, t \in B - \{1_S\}\Big\}, \text{ and}$$

$$\mathrm{Cay}_{\mathrm{L}}(S;B) = \Big\{(s, st) : s \in S, t \in B - \{1_S\}\Big\}.$$

It is common to see right Cayley graphs more often in the literature, as in this case a word $\omega \in S$ can be located in the corresponding Cayley graph by reading from left to right, just as in english (and all left-to-right script languages). Nevertheless, we will mostly work with left Cayley graphs, as these are compatible with the (left) shift action $S \curvearrowright \mathcal{A}^S$ given by $(s \cdot x)(t) = x(ts)$, where $\mathcal{A}$ is any set.

If G is a group generated by S, we give G a metric structure as follows. For two different elements $g, h \in G$, define

$$d_S(g,h) = \min\Big\{n \geq 1 : gh^{-1} = s_1 \cdots s_n, s_i \in (S \cup S^{-1})\Big\},$$

i.e., d_S is the distance associated to the usual graph metric of $\mathrm{Cay}_L(G;S)$. In the case of semigroups, the same metric is considered, forgetting the directed structure of the graph.

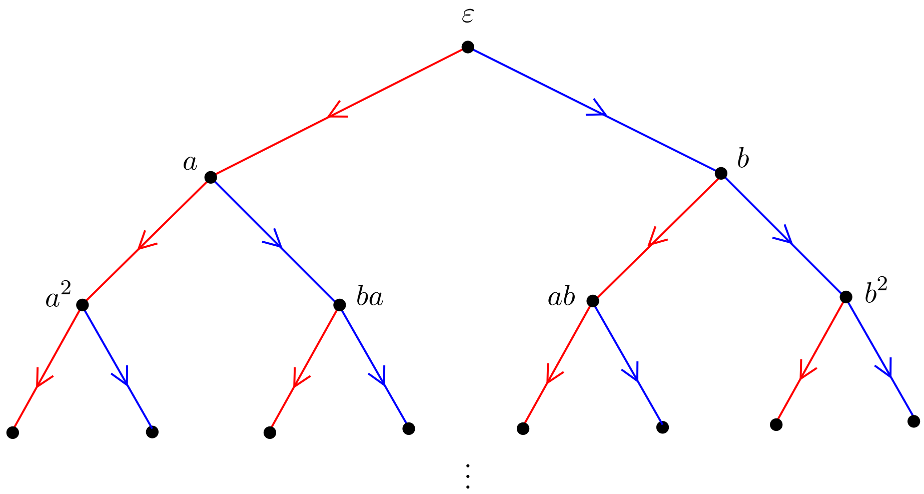

Figure 2: The right Cayley graph of the free semigroup on two generators, $\mathbb{F}_2^+$. The distance between the elements ba and ab is 4, even though there is no directed path from one to the other in the Cayley graph.

B Measure theory

The goal of this appendix is to give a brief review of extension of measures from semi-algebras to σ-algebras, and construction of measures on product spaces.

B.1 Extending measures on (semi-)algebras

Throughout this section, X is a set, $\mathscr{C}$ is a semi-algebra of subsets of X, and $\mathscr{A}(\mathscr{C})$ denotes the algebra generated by $\mathscr{C}$, which consists of all finite disjoint unions of elements in $\mathscr{C}$. The following results can be found in [BR07].

Proposition B.1. *Any finitely additive measure μ on $\mathscr{C}$ extends to a unique finitely additive measure on $\mathscr{A}(\mathscr{C})$. This extension is countably additive on $\mathscr{A}(\mathscr{C})$ exactly when μ is countably additive on $\mathscr{C}$.*

Definition B.2. A finitely additive probability measure μ defined over an algebra $\mathscr{A}$ is said to be **continuous** at $A \in \mathscr{A}$ if for every $A_1 \supseteq A_2 \supseteq \cdots$ in $\mathscr{A}$ with $\bigcap_{n\geq 1} A_n = A$ (this will be denoted as $A_n \downarrow A$),

$$\lim_{n\to\infty} \mu(A_n) = \mu(A).$$

Proposition B.3. *Let $\mathscr{A}$ be an algebra of sets in X, and $\mu : \mathscr{A} \longrightarrow [0,1]$ be a finitely additive probability measure continuous at $\varnothing$. Then μ is countably additive on $\mathscr{A}$.*

Definition B.4. Let $\mathscr{A}$ be an algebra of subsets of X and $\mu : \mathscr{A} \longrightarrow [0,1]$ a finitely additive probability measure. Define, for each $A \subseteq X$, the **exterior measure**

$$\mu^*(A) = \inf \left\{ \sum_{i\geq 1} \mu(A_i) : A \subseteq \bigcup_{i\geq 1} A_i \text{ and } A_i \in \mathscr{A} \right\}.$$

Given any $E \subseteq X$, we say E is **μ^*-measurable** if for every $A \subseteq X$ we have

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^{\mathrm{c}})$$

and denote by $\mathcal{M}^*$ the collection of such sets.

Theorem B.5 (Carathéodory's Extension Theorem). *Let $\mathscr{A}$ be an algebra of subsets of X and μ a countably additive measure on $\mathscr{A}$. Then, $\sigma(\mathscr{A}) \subseteq \mathcal{M}^*$, there is a unique extension of μ to a probability measure $\bar{\mu}$ on $\sigma(\mathscr{A})$, and $\bar{\mu}|_{\sigma(\mathscr{A})} = \mu^*|_{\sigma(\mathscr{A})}$*

Thus, we obtain two useful tools which will help us extend a measure μ defined on a semi-algebra $\mathscr{C}$: to prove σ-additivity of μ upon $\mathscr{C}$, or to prove continuity at $\varnothing$ of the extension of μ to $\mathscr{A}(\mathscr{C})$.

B.2 Product σ-algebras and Kolmogorov's Extension Theorem

For a more detailed and general version of the results shown here, the reader may check [Tao11, §2.4].

Definition B.6. Let $\{(X_i, \mathscr{B}_i)\}_{i \in I}$ be a collection of measurable spaces indexed by an arbitrary set I. The **product σ-algebra** in $\prod_{i \in I} X_i$, denoted by $\prod_{i \in I} \mathscr{B}_i$, is the σ-algebra generated by the sets of the form $\prod_{i \in I} A_i$, with $A_i \in \mathscr{B}_i$ for every $i \in I$ and $A_i \neq X_i$ only for i in a finite subset of I.

Remark B.7. For a finite subset $J \subseteq I$, let $X^J = \prod_{j \in J} X_j$ and $\pi_J : \prod_{i \in I} X_i \longrightarrow X^J$ be the natural projection. Then the above definition of the product σ-algebra can be reformulated as the one generated by the sets of the type $\pi_J^{-1}(B)$ with $B \in \prod_{j \in J} \mathscr{B}_j$ and $J \subseteq I$ finite.

Definition B.8. For I a set and $(X, \mathscr{B})$ a measurable space, suppose we have a collection $\{\mu_J : J \subseteq I \text{ finite}\}$ of probability measures, each μ_J defined on X^J. The family is called **consistent** if

$$\mu_J(\pi_J(\pi_K^{-1}(A))) = \mu_K(A)$$

for every $A \in \prod_{k \in K} \mathscr{B}$, whenever $K \subseteq J$.

From now on, consider X to be a Polish space, and $\mathscr{B}(X)$ the Borel σ-algebra.

Lemma B.9. *Let I be any set and consider the space $\Omega = X^I$ with the product topology. If I is countable, then*

$$\mathscr{B}(\Omega) = \prod_{i \in I} \mathscr{B}_i.$$

If I is uncountable, then $\prod_{i \in I} \mathscr{B}_i \subsetneq \mathscr{B}(\Omega)$.

Theorem B.10 (Kolmogorov's Extension Theorem). *Let I be an arbitrary set, X a Polish space, and consider $\Omega = X^I$ with the product σ-algebra. Given a consistent family of probability measures $\{\mu_J\}$, there is a unique probability measure μ on Ω such that, for every finite $J \subseteq I$ and $A \in \mathscr{B}(X^J)$, we have*

$$\mu(\pi_J^{-1}(A)) = \mu_J(A).$$

C Ergodicity and mixing for group actions

Recall that a p.m.p. $\mathbb{N}$-action $\mathbb{N} \curvearrowright (X, \mu)$ is said to be

(i) **ergodic** if every set $A \in \mathscr{B}(X)$ satisfying $\mu(A \triangle T^{-1}A) = 0$, we have $\mu(A) \in \{0, 1\}$, and

(ii) **mixing** if for every pair of sets $A, B \in \mathscr{B}(X)$ with $\mu(A)\mu(B) > 0$, we have

$$\lim_{n\to\infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B),$$

where $T\colon X \to X$ is the map associated to $1 \in \mathbb{N}$. A great place to get started in these topics is [Wal81].

Let now G be a countable group. If X is a Polish space and μ a probability measure on the Borel sets of X, a **probability measure preserving** (p.m.p.) action $G \curvearrowright (X, \mu)$ is an action $\alpha\colon G \times X \to X$ such that each map $\alpha(g, \cdot)$ is Borel measurable and $\mu(gA) = \mu(A)$ for all $A \in \mathscr{B}(X)$. A subset $A \in \mathscr{B}(X)$ is called G-invariant if $gA = A$ for all $g \in G$. A reference for studying p.m.p. group actions is [KL16]. Here, we introduce a couple of definitions, and some results that will be used in Chapter 4.

Definition C.1. A p.m.p. action $G \curvearrowright (X, \mu)$ is called **ergodic** if every set $A \in \mathscr{B}(X)$ satisfying $\mu(gA \triangle A) = 0$ for all $g \in G$ has measure 0 or 1.

A common (and useful) characterization of ergodicity is the following.

Proposition C.2. *A p.m.p. action $G \curvearrowright (X, \mu)$ is ergodic if, and only if, for every $A, B \in \mathscr{B}(X)$ with $\mu(A)\mu(B) > 0$, there is a $g \in G$ such that $\mu(A \cap gB) > 0$.*

Definition C.3. A function $f\colon G \to \mathbb{C}$ is said to **vanish at infinity** if for every $\epsilon > 0$ there is a finite subset $F \subseteq G$ such that $|f(g)| < \epsilon$ whenever $g \in G - F$.

Definition C.4. A p.m.p. action $G \curvearrowright (X, \mu)$ is called **mixing** if for all $A, B \in \mathscr{B}(X)$ with $\mu(A)\mu(B) > 0$, the function $g \mapsto (\mu(A \cap gB) - \mu(A)\mu(B))$ vanishes at infinity.

It is well known that every mixing p.m.p. action is ergodic. This is specially useful, as the mixing property can be checked upon generators. The following proposition is the group-theoretical analogous of [Wal81, Theorem 1.17, part (iii)].

Proposition C.5. *Let $\mathscr{C}$ be a semi-algebra generating $\mathscr{B}(X)$. A p.m.p action $G \curvearrowright (X, \mu)$ is mixing if, and only if, for all $A, B \in \mathscr{C}$ with $\mu(A)\mu(B) > 0$, the function $g \mapsto (\mu(A \cap gB) - \mu(A)\mu(B))$ vanishes at infinity.*

Proof. Since the defining property of mixing holds for elements of $\mathscr{C}$, and the algebra $\mathscr{A}$ generated by $\mathscr{C}$ consists of disjoint unions of elements in $\mathscr{C}$, it follows that $g \mapsto (\mu(A \cap gB) - \mu(A)\mu(B))$ vanishes at infinity for every pair of elements of $\mathscr{A}$ with positive measure.

The collection of sets $B \in \mathscr{B}(X)$ such that for every $\epsilon > 0$ there is a $B \in \mathscr{A}$ with $\mu(A \triangle B) < \epsilon$ is a σ-algebra containing $\mathcal{A}$, thus it is the whole of $\mathscr{B}(X)$. Let $\epsilon > 0$, $A, B \in \mathscr{B}(X)$, and choose $A', B' \in \mathscr{A}$ with $\mu(A \triangle A') < \epsilon$ and $\mu(B \triangle B') < \epsilon$. A straightforward calculation shows that

$$(A \cap gB) \triangle (A' \cap gB') \subseteq (A \triangle A') \cup (gB \triangle gB'),$$

so $\mu((A \cap gB) \triangle (A' \cap gB')) < 2\epsilon$. Therefore,

$$\begin{aligned} |\mu(A \cap gB) - \mu(A)\mu(B)| &\leq |\mu(A \cap gB) - \mu(A' \cap gB')| + |\mu(A' \cap gB') - \mu(A')\mu(B')| \\ &\quad + |\mu(A')\mu(B') - \mu(A)\mu(B')| + |\mu(A)\mu(B') - \mu(A)\mu(B)| \\ &\leq 4\epsilon + |\mu(A' \cap gB') - \mu(A')\mu(B')|, \end{aligned}$$

yielding the desired conclusion that $g \mapsto (\mu(A \cap gB) - \mu(A)\mu(B))$ vanishes at infinity. $\square$

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