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Master's Thesis

Vortex pinning in inhomogeneous type-II superconductors

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Contents

Introduction

Superconductivity is a property achieved by some materials characterized by a complete loss of electrical resistance when they are cooled below a critical temperature. This property was discovered by K. Ohnes in 1911.

A striking feature of superconductors is the so-called *Meissner effect*, which states that superconductors repel applied external magnetic fields and levitate over magnetic sources. However, this property may be lost if the intensity of the external fields is sufficiently large. A particular family of superconductors, called type-II superconductors, are characterized by the existence of a *mixed phase* where superconductivity coexists with magnetic defects given by the partial penetration of the external field. These defects are called *vortices*.

These vortices might move because of their interactions or by external forces. This motion causes energy dissipation due to vortices generating an electric field, which generates an electric resistance and the loss of superconductivity. We can control the motion of vortices by introducing an inhomogeneity like varying the width or temperature across the material. This can create pinning sites for the vortices as it may be less efficient in an energetical sense for vortices to move through inhomogeneous regions.

The Ginzburg–Landau functional in 2D. Critical fields.

The behaviour of superconductors under a pinning effect is modeled by the inhomogeneous *Ginzburg–Landau energy functional*. In the two-dimensional framework, the

Ginzburg-Landau energy functional corresponds to

$$GL_\varepsilon(u,A) := \frac{1}{2}\int_\Omega |\nabla_A u|^2 + \frac{(a_\varepsilon(x)-|u|^2)^2}{2\varepsilon^2} + |h-h_{\mathrm{ex}}|^2. \tag{0.0.1}$$

Here

- $\Omega \subset \mathbb{R}^2$ is a smooth, bounded, and simply connected domain.
- $u : \Omega \to \mathbb{C}$ is called the *order parameter* and its squared modulus (the density of Cooper pairs of superconducting electrons in the Bardeen–Cooper–Schrieffer (BCS) quantum theory [BCS57]) indicates the local state (normal or superconducting) of the material.
- $A : \Omega \to \mathbb{R}^2$ is the *electromagnetic vector potential* of the induced magnetic field $h = \operatorname{curl} A := \partial_{x_1} A_2 - \partial_{x_2} A_1$. Note that in $\mathbb{R}^2$, the curl operator yields a scalar field which agrees with the x_3-coordinate of the usual three-dimensional curl.
- ∇_A denotes the covariant gradient $\nabla - iA$.
- $h_{\mathrm{ex}} > 0$ is a constant that represents the intensity of the *external magnetic field* in the direction perpendicular to Ω.
- $\varepsilon > 0$ is the inverse of the *Ginzburg–Landau parameter* usually denoted κ, a nondimensional parameter depending only on the material. We will be interested in the regime of small ε, corresponding to extreme type-II superconductors.
- a_ε is the *pinning function* that accounts for inhomogeneities in the material. We will assume that $a_\varepsilon \in L^\infty(\Omega)$ and that it takes values in $[b,1]$, where $b \in (0,1)$ is a constant independent of ε. The regions where $a_\varepsilon = 1$ correspond to sites without inhomogeneities (we also say that there is no pinning in these regions).

The configuration $(u_\varepsilon, A_\varepsilon)$ which minimizes (0.0.1) indicates the electromagnetic profile of the superconductor exposed to an external field, while vorticity regions are represented by $u \approx 0$. The vortices themselves are codimension 2 topological defects and they are the limiting set of the vorticity regions as $\varepsilon \to 0$. Vortices also have an associated topological degree (winding number). Heuristically, vortices appear as an intermediate phase in a competition between superconductivity, which is represented by $a_\varepsilon \approx |u|^2$, and the abscense of Meissner effect, which is represented by $h \approx h_{\text{ex}}$. The term $|\nabla_A u|$ seeks to enforce regularity.

Many particular models of a_ε have been studied through the literature. For example a_ε can be a step function taking only two values (say, b and 1) where ε controls the regions where $a_\varepsilon \equiv b$, or a periodic function with oscillations controlled by ε. We refer to [LM99, Kac10, DS21, DSRS23] for results in the aformentioned cases. There are also models where the pinning function can go near or even below zero. We refer to [ABP03, AAB05] for these particular cases. We additionally refer to the works of A. Aftalion, E. Sandier and S. Serfaty [ASS01], which studies the limiting behaviour of a_ε through the lens of homogenization theory.

The homogeneous case $a_\varepsilon \equiv 1$ corresponds to the classical Ginzburg–Landau functional, which was first introduced by V. Ginzburg and L. Landau in [GL50]. Interest from mathematicians started in the 90's after the pioneering works from F. Bethuel, H. Brezis and F. Hélein in [BBH94], where they developed tools for systematic vortex analysis. Research on the homogeneous functional with magnetic field is well documented in the book of S. Serfaty and E. Sandier [SS07].

As the intensity of the external field h_{ex} is increased, the superconducting system undergoes several phase transitions which drastically affects the behaviour of minimizers. There are three main values $H_{c_1}, H_{c_2}, H_{c_3}$ called *critical fields* for which these phase transitions occur.

- When h_{ex} is below H_{c_1}, which is of order $O(|\log \varepsilon|)$, the Meissner effect is present and the material is in the superconducting phase.
- As h_{ex} surpasses H_{c_1}, the external magnetic field starts to penetrate the material. When $H_{c_1} < h_{\text{ex}} < H_{c_2}$, we are in the *mixed phase* and vortex configurations are allowed, where the number of vortices increases with h_{ex}. Vortices

repel each other and, in the homogeneous case, they arrange themselves in a triangular lattice to minimize repulsion. As h_{ex} starts approaching the second critical field H_{c_2}, the vortices start to overlap. This means that superconductivity is lost in the bulk of the material.

- Between H_{c_2} and H_{c_3}, superconductivity remains only near the boundary. After h_{ex} surpasses H_{c_3}, superconductivity is completely destroyed.

It is important to note that these phases present hysteretic behaviour. For example, if we initially observe a superconducting phase and steadily increase h_{ex}, we will not observe vortices even when surpassing the first critical field H_{c_1}. Mathematically, this happens because the local minimality of vortexless configurations remains even when $h_{\text{ex}} > H_{c_1}$. The value H_{sh} that h_{ex} has to surpass to break the stability of vortexless configurations is called *superheating field* and it satisfies $H_{sh} \gg H_{c_1}$. We refer to [Ser99, Rom19a] for stability results of vortexless configurations in the homogeneous problem.

Mathematical tools in the Ginzburg–Landau theory

Before we state the main results of this work, we have to give an overview of the mathematical tools used for this problem. We will fill out the details in the corresponding sections.

Gauge invariance

The Ginzburg–Landau energy admits an invariance under transformations of the form

$$(u, A) \to (ue^{i\phi}, A + \nabla\phi),$$

for a given scalar function ϕ. This is called a *gauge transformation* and the invariance induced is referred to as *gauge invariance*. Many of the physically meaningful quantities in this theory are gauge invariant, including but not limited to the *(weighted)* *free energy* for a weight function η defined in Ω

$$F_{\varepsilon,\eta}(u,A) := \frac{1}{2}\int_\Omega \eta^2|\nabla_A u|^2 + |\operatorname{curl} A|^2 + \eta^4\frac{(1-|u|^2)^2}{2\varepsilon^2},$$

the *superconducting current*

$$j(u,A) := \langle iu\,, \nabla_A u\rangle,$$

the *vorticity measure*

$$\mu(u,A) := \operatorname{curl}(j(u,A)+A) = \operatorname{curl}\langle iu\,, \nabla_A u\rangle + h,$$

where $\langle z\,, w\rangle := \frac{z\overline{w}+\overline{z}w}{2}$ is the scalar product in $\mathbb{C}$ identified with $\mathbb{R}^2$. We also say that two configurations $(u_0, A_0), (u_1, A_1)$ are *gauge equivalent* if (u_1, A_1) can be gauge transformed to (u_0, A_0). Every configuration is gauge equivalent to a special configuration called *Coulomb gauge*, which is characterized by A satisfying

$$\begin{cases} \operatorname{div} A &= 0 \quad \text{in } \Omega \\ A\cdot\nu &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{0.0.2}$$

Energy decoupling

Consider the simpler energy functional without magnetic field in $H^1(\Omega,\mathbb{C})$.

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{(a_\varepsilon - |u|^2)^2}{2\varepsilon^2}.$$

Using direct methods of calculus of variations, we can prove that E_ε admits a minimizer. Moreover, this functional has a *unique positive* minimizer which we will denote by ρ_ε. By using the Euler–Lagrange equation solved by ρ_ε

$$\begin{cases} -\Delta\rho_\varepsilon &= \dfrac{\rho_\varepsilon(a_\varepsilon - \rho_\varepsilon^2)}{\varepsilon^2} \quad \text{in } \Omega \\ \nabla\rho_\varepsilon\cdot\nu &= 0 \qquad\qquad\quad\ \text{on } \partial\Omega, \end{cases} \tag{0.0.3}$$

L. Lassoued and P. Mironescu proved in [LM99] that for *any* $u \in H^1(\Omega, \mathbb{C})$, the following *energy decoupling* holds for any $u \in H^1(\Omega, \mathbb{C})$

$$E_\varepsilon(\rho_\varepsilon u) = E_\varepsilon(\rho_\varepsilon) + \frac{1}{2}\int_\Omega \rho_\varepsilon^2 |\nabla u|^2 + \rho_\varepsilon^4 \frac{(1-|u|^2)^2}{2\varepsilon^2}. \tag{0.0.4}$$

This is a powerful result in the inhomogeneous theory, because this indicates that a inhomogeneous configuration rescaled by ρ_ε behaves like an homogeneous one.

Meissner configuration and Energy splitting

One of the main results of this work is an estimation of the leading order of the first critical field H_{c_1}, which is equivalent to determine for which values of h_{ex} the global minimizer for (0.0.1) corresponds to a vortexless configuration. A good starting point is to look for the configuration which minimizes the energy amongst vortexless configurations. Since, heuristically, vortexless configurations are characterized by $a_\varepsilon \approx |u|^2$, it may be a good idea to find A which minimizes $GL_\varepsilon(\sqrt{a_\varepsilon}, A)$ in a suitable space, but this is not always possible, since a_ε might not be in $H^1(\Omega, \mathbb{C})$. Hence, a good alternative is to minimize $GL(\rho_\varepsilon, A)$, since ρ_ε^2 is essentially a regularization of a_ε.

We identify this minimizer as $h_{\text{ex}} A_\varepsilon^0$, where the h_{ex} rescaling is added for convenience's sake. This special configuration $(\rho_\varepsilon, h_{\text{ex}} A_\varepsilon^0)$ is called the *Meissner configuration*. There are two very important features of the Meissner configuration. First, the energy of any vortexless minimizer is close to the energy of the Meissner configuration. Second, the Ginzburg–Landau energy decomposes as follows

Proposition 0.0.1. *For any configuration* $(\mathbf{u}, \mathbf{A}) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ *and* (u, A) *such that*

$$(\mathbf{u}, \mathbf{A}) = (\rho_\varepsilon u, A + h_{\text{ex}} A_\varepsilon^0),$$

we have

$$GL_\varepsilon(\mathbf{u}, \mathbf{A}) = GL_\varepsilon(\rho_\varepsilon, h_{\text{ex}} A_\varepsilon^0) + F_{\varepsilon,\rho_\varepsilon}(u, A) - \int_\Omega \mu(u, A)\xi_\varepsilon + R_0, \tag{0.0.5}$$

where ξ_ε is the unique solution of the following Dirichlet elliptic problem

$$\begin{cases} -\operatorname{div} \dfrac{\nabla \xi_\varepsilon}{\rho_\varepsilon^2} + \xi_\varepsilon & = & 1 \quad \text{in } \Omega \\ \xi_\varepsilon & = & 0 \quad \text{on } \partial\Omega, \end{cases} \tag{0.0.6}$$

and

$$R_0 := \frac{h_{\mathrm{ex}}^2}{2} \int_\Omega \frac{|\nabla \xi_\varepsilon|^2}{\rho_\varepsilon^2} (|u|^2 - 1). \tag{0.0.7}$$

In the regime of h_{ex} that we are interested in, the term $R_0 = R_0(\varepsilon)$ is negligible. Considering that the energy of the Meissner solution is a good approximation of the energy of vortexless minimizers, this indicates that the presence of vortices strongly depends on the sign of the term $(F_{\varepsilon,\rho_\varepsilon}(u, A) - h_{\mathrm{ex}} \int_\Omega \mu(u, A)\xi_\varepsilon)$.

It is worth mentioning that this splitting was strongly prompted by [BR95, LM99, SS07, Kac10].

Vorticity estimate and Vortex ball construction

The previously introduced *vorticity measure* $\mu(u, A)$ indicates the density of vorticity regions. We have mentioned previously that, as $\varepsilon \to 0$, vorticity regions become point-like. The *vorticity estimate*, popularized by the works on the homogeneous case of R. Jerrard and H. Soner in [JS02] captures this idea. The vorticity estimates states that $\mu(u, A)$ approaches a sum of Dirac masses in a weak sense. These Dirac masses are centered at the vortices and the weight of each vortex is equal to $2\pi d$, where d is the associated degree. This indicates that vortices are *quantized*. We also point out that the presence of inhomogeneities affects only the positions of the vortices and not their degrees.

On the other hand, the *vortex ball construction* gives us a lower bound for $F_{\varepsilon,\rho_\varepsilon}(u, A)$, the *free energy with weight* ρ_ε, in terms of ρ_ε and degrees of the vortices. In the homogeneous case, this construction was introduced independently by E. Sandier [San98] and by R. Jerrard [Jer99]. It can be roughly described as finding a lower bound for $F_{\varepsilon,\rho_\varepsilon}(u, A)$ restricted to an initial collection of balls covering the vortices and letting the balls increase in radii in a way that preserves the lower bound. If during the growth process two balls intersect each other, they are merged into one ball in a way that the bound is preserved. This gives us a precise estimate on the energetic cost of each vortex, which is $\pi\,(\min_B \rho_\varepsilon^2)\,|d||\log\varepsilon| + O(\log|\log\varepsilon|)$ and does not depend on the number of vortices. One of our results will be to indicate the necessary modifications to generalize the construction to the inhomogeneous case.

The Ginzburg–Landau functional in 3D

The Ginzburg–Landau energy functional in three dimensions is defined as

$$GL_\varepsilon^{3\mathrm{D}}(u,A) := \frac{1}{2}\int_\Omega |\nabla_A u|^2 + \frac{(a_\varepsilon - |u|^2)^2}{2\varepsilon^2} + \frac{1}{2}\int_{\mathbb{R}^3} |H - H_{\mathrm{ex}}|^2 \tag{0.0.8}$$

Compared to the 2D functional,

- $\Omega \subseteq \mathbb{R}^3$ is assumed to be smooth, bounded, simply connected domain.
- $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the *electromagnetic vector potential* of the magnetic field $H = \operatorname{curl} A$, which now is a vector field that extends beyond Ω.
- $H_{\mathrm{ex}}\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the *applied external magnetic field.* Its *intensity* h_{ex} is now defined as its norm in an adequate space. Since H_{ex} represents a magnetic field, it is natural to impose that $\operatorname{div} H_{\mathrm{ex}} = 0$. This means that H_{ex} admits a vector potential A_{ex}, which satisfies $\operatorname{curl} A_{\mathrm{ex}} = H_{\mathrm{ex}}$. The intensity h_{ex} now corresponds to the norm of H_{ex} in an appropriate space and we introduce the *normalized external magnetic field* $H_{0,\mathrm{ex}}$ as $\frac{1}{h_{\mathrm{ex}}}H_{\mathrm{ex}}$ and its *normalized potential* $A_{0,\mathrm{ex}}$ as $\frac{1}{h_{\mathrm{ex}}}A_{0,\mathrm{ex}}$. A_{ex} is not unique and it can be chosen such that $\operatorname{div} A_{\mathrm{ex}} = 0$ in Ω and $A_{\mathrm{ex}} \cdot \nu = 0$ on $\partial\Omega$.

Note that the 2D functional is a particular case of (0.0.8), where Ω is a cylinder of infinite length with the 2D domain being its cross-section, while H_{ex} is constant and directed perpendicularly to Ω.

The first noticeable difference between the 2D and 3D functionals is that the magnetic component $|H-H_{\mathrm{ex}}|^2$ is now defined in all of $\mathbb{R}^3$. On the other hand, vortices now have a more geometrically complex structure, as they are 1-dimensional objects now. Some of the tools can be slightly modified to hold in the 3D case, like the energy decoupling or the Meissner configuration. However, some of them, like the vortex ball construction, have to be replaced by more sophisticated tools. We refer to [Riv95, JMS04, BJOS13, RSS23] for further results in the homogeneous problem.

Meissner configuration and Energy splitting in 3D

One of the main difficulties in adapting the energy splitting (0.0.1) is the redefinition of the Meissner configuration. In particular, the functional minimized by the Meissner configuration now includes the energy of the magnetic field $|H-H_{\mathrm{ex}}|^2$ extended beyond Ω. To ensure more regular behavior near the boundary, we minimize in a Coulomb-like gauge $(ue^{i\phi_A}, A) \sim (u, A-\nabla\phi_A)$, where ϕ_A is such that

$$\begin{cases} \operatorname{div} \rho_\varepsilon^2(A-\nabla\phi_A) &= 0 \quad \text{in } \Omega \\ \rho_\varepsilon^2(A-\nabla\phi_A)\cdot\nu &= 0 \quad \text{on } \partial\Omega \end{cases}$$

This is equivalent to writing $A-\nabla\phi_A$ as $\frac{\operatorname{curl} B}{\rho_\varepsilon^2}$, where B satisfies

$$\begin{cases} \operatorname{div} B &= 0 \quad \text{in } \Omega \\ B\times\nu &= 0 \quad \text{on } \partial\Omega \end{cases}$$

Thus, the Meissner configuration in 3D corresponds to $(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi^0_{\varepsilon,\rho_\varepsilon^2}}, h_{\mathrm{ex}}A_\varepsilon^0)$, where $h_{\mathrm{ex}}A_\varepsilon^0$ minimizes $GL_\varepsilon^{3\mathrm{D}}(\rho_\varepsilon e^{i\phi_A}, A)$ and $\phi^0_{\varepsilon,\rho_\varepsilon^2} := \phi_{A_\varepsilon^0,\rho_\varepsilon^2}$. We also denote B_ε^0 the vector field given by the previously stated relation

$$A_\varepsilon^0 - \nabla\phi^0_{\varepsilon,\rho_\varepsilon^2} = \frac{\operatorname{curl} B_\varepsilon^0}{\rho_\varepsilon^2}.$$

The reason we bring attention to B_ε^0 is that this vector field is related to the energy splitting result in $3D$, which is one of our new results stated in a work-in-progress [DVR].

Proposition 0.0.2. *For any configuration* $(\mathbf{u}, \mathbf{A}) \in H^1(\Omega, \mathbb{C}) \times [A_{\mathrm{ex}} + H_{\mathrm{curl}}(\mathbb{R}^3, \mathbb{R}^3)]$ *and* (u, A) *such that*

$$(\mathbf{u}, \mathbf{A}) = (u\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi^0_{\varepsilon,\rho_\varepsilon^2}}, A + h_{\mathrm{ex}}A^0_\varepsilon),$$

we have

$$GL_\varepsilon(\mathbf{u}, \mathbf{A}) = GL_\varepsilon(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi^0_{\varepsilon,\rho_\varepsilon^2}}, h_{\mathrm{ex}}A^0_\varepsilon) + F_{\varepsilon,\rho_\varepsilon}(u, A) - h_{\mathrm{ex}}\int_\Omega \mu(u, A)\cdot B^0_\varepsilon + R_0 \quad (0.0.9)$$

where B^0_ε *is the unique vector field such that*

$$\begin{cases} \operatorname{div} B^0_\varepsilon & = & 0 & \text{in } \Omega \\ \dfrac{\operatorname{curl} B^0_\varepsilon}{\rho_\varepsilon^2} & = & A^0_\varepsilon - \nabla\phi^0_{\varepsilon,\rho_\varepsilon^2} & \text{in } \Omega \\ B^0_\varepsilon \times \nu & = & 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$R_0 := \frac{h_{\mathrm{ex}}^2}{2}\int_\Omega \frac{|\operatorname{curl} B^0_\varepsilon|^2}{\rho_\varepsilon^2}(|u|^2 - 1).$$

Remark 0.0.1. *Note that, in contrast to the 2D case, the vorticity* $\mu(u, A)$ *is now a vector field since the curl of a vector field returns a vector field in 3D.*

Main results

The main results of this work are concerned with *vortexless configurations.* We formally define them as follows

Definition 0.0.1. *We say* $(\mathbf{u}_\varepsilon, \mathbf{A}_\varepsilon) = (\rho_\varepsilon u_\varepsilon, A_\varepsilon + h_{\mathrm{ex}}A^0_\varepsilon)$ *is a weakly vortexless configuration if it satisfies the following asymptotic estimate.*

$$\|\mu(u_\varepsilon, A_\varepsilon)\|_{(C^{0,1}_0(\Omega))^*} = o(1).$$

Additionally, we say that a weakly vortexless configuration $(\mathbf{u}_\varepsilon, \mathbf{A}_\varepsilon)$ *is strongly vor-*

texless if it also satisfies

$$|u_\varepsilon| > c$$

for some $c > 0$ independent of ε.

From here on out, we drop the ε subscript for these configurations, but keep in mind the dependence on ε.

Main results in 2D

The first couple of results are a characterization of the first critical field. These are the main results of our paper [DVR24]. As we have stated before, the presence of vortices is strongly determined by $(F_{\varepsilon,\rho_\varepsilon}(u,A) - h_{\mathrm{ex}} \int_\Omega \mu(u,A)\xi_\varepsilon)$, since the Meissner configuration will no longer be near a minimizer when $(F_{\varepsilon,\rho_\varepsilon}(u,A) - h_{\mathrm{ex}} \int_\Omega \mu(u,A)\xi_\varepsilon) \ll 0$. Using the previously mentioned vortex ball construction and vorticity estimate, by ignoring lower order terms, we deduce that this is possible when

$$\pi \sum_i |d_i| \left((\min_{B_i} \rho_\varepsilon^2) |\log \varepsilon| - 2h_{\mathrm{ex}} \langle \delta_i \,, \xi_\varepsilon \rangle \right) < 0.$$

If ξ_ε is sufficiently regular, which is expected from elliptic regularity on (0.0.6), we can set ξ_ε and ρ_ε^2 in the same point while adding a negligible error to the energy. We conclude that this is equivalent to $h_{\mathrm{ex}} > \frac{|\log \varepsilon|}{2\max_\Omega \psi_\varepsilon}$, where $\psi_\varepsilon = \frac{\xi_\varepsilon}{\rho_\varepsilon^2}$. Our first result formalizes this characterization of the first critical field. We define the *approximated first critical field* $H_{c_1}^\varepsilon$ as

$$H_{c_1}^\varepsilon := \frac{|\log \varepsilon|}{2\max_\Omega \psi_\varepsilon},$$

Theorem 1. *There exist $\varepsilon_0 > 0$ and $K_0 > 0$ such that, for any $\varepsilon < \varepsilon_0$ and any $h_{\mathrm{ex}} \leq H_{c_1}^\varepsilon - K_0 \log|\log \varepsilon|$, the global minimizers $(\mathbf{u}, \mathbf{A})$ of GL_ε in $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ are strongly vortexless configurations and satisfy the following asymptotic estimate*

$$GL_\varepsilon(\mathbf{u}, \mathbf{A}) = GL_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}} A_\varepsilon^0) + o(1).$$

The preceding result characterizes the behavior of global minimizers below $H_{c_1}^{\varepsilon}$. Moreover, an explicit construction also provides a characterization above this value.

Theorem 2. *Assume* $[\rho_\varepsilon]_{C^{0,\alpha}(\Omega)} \leq |\log\varepsilon|^m$ *for some* $m > 0$ *and* $\alpha \in (0,1]$, *where* $[\,\cdot\,]_{C^{0,\alpha}(\Omega)}$ *denotes the Hölder seminorm. Then, there exist* $\varepsilon_0 > 0$ *and* $K^0 > 0$ *such that, for any* $\varepsilon < \varepsilon_0$ *and any* h_{ex} *such that* $H_{c_1}^{\varepsilon} + K^0 \log|\log\varepsilon| \leq h_{\mathrm{ex}} \leq |\log\varepsilon|^N$ *for some* $N \geq 2$, *the global minimizers* $(\mathbf{u},\mathbf{A})$ *of* GL_ε *in* $H^1(\Omega,\mathbb{C}) \times H^1(\Omega,\mathbb{R}^2)$ *are not vortexless configurations. More specifically, there exists a configuration* $(\mathbf{u},\mathbf{A})$ *which is not weakly vortexless such that* $GL_\varepsilon(\mathbf{u},\mathbf{A}) \ll GL_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}}A)$.

Thus, under the assumptions of the previous two theorems, we conclude that

$$H_{c_1} = H_{c_1}^{\varepsilon} + O(\log|\log\varepsilon|).$$

The reason we show that the energy of vortexless minimizers approaches the energy of the Meissner configuration is that vortexless configurations themselves approach the Meissner configuration in an adequate Banach space. This is proved in the following result, which includes values for h_{ex} above the first critical field.

Theorem 3. *Let* $\alpha \in \left(0,\frac{1}{2}\right)$. *There exists* $\varepsilon_0 > 0$ *such that, for any* $\varepsilon < \varepsilon_0$ *and* $h_{\mathrm{ex}} < \varepsilon^{-\alpha}$, *there exists a vortexless local minimizer* $(\mathbf{u},\mathbf{A})$ *for* GL_ε. *Furthermore, if* $(\mathbf{u},\mathbf{A})$ *is in the Coulomb gauge, it approximates the Meissner configuration. More formally,*

(1) $GL_\varepsilon(\mathbf{u},\mathbf{A}) = GL_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}}A_\varepsilon^0) + o(1)$.

(2) *The configuration* (u,A) *such that* $(u,A) = (\rho_\varepsilon^{-1}\mathbf{u}, \mathbf{A} - h_{\mathrm{ex}}A_\varepsilon^0)$ *satisfies*

$$\inf_{\theta\in[0,2\pi]} \left\|u - e^{i\theta}\right\|_{H^1(\Omega,\mathbb{C})} + \|A\|_{H^1(\Omega,\mathbb{R}^2)} = o(1). \tag{0.0.10}$$

(3) *The configuration* $(\mathbf{u},\mathbf{A})$ *satisfies*

$$\left\|\mathbf{A} - h_{\mathrm{ex}}A_\varepsilon^0\right\|_{H^1(\Omega,\mathbb{R}^2)} = o(1). \tag{0.0.11}$$

Moreover, if $\|\nabla\rho_\varepsilon\|_{L^2(\Omega,\mathbb{R}^2)} < \varepsilon^{-\gamma}$ *for some* $\gamma < 1-2\alpha$*, we have for any* $r \in [1,2)$

$$\inf_{\theta\in[0,2\pi]} \left\|\mathbf{u} - \rho_\varepsilon e^{i\theta}\right\|_{W^{1,r}(\Omega,\mathbb{C})} = o(1). \tag{0.0.12}$$

Moreover, we also prove that these locally minimizing vortexless configurations are unique up to gauge equivalence.

Theorem 4. *Assume* $E_\varepsilon(\rho_\varepsilon) \ll \frac{1}{\varepsilon^2}$*. Let* $\alpha \in (0,1)$ *and* $\beta > 0$*. There exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon < \varepsilon_0$*, if* $h_{\text{ex}} \le \varepsilon^{-\alpha}$ *then a pair* $(\mathbf{u},\mathbf{A}) = (\rho_\varepsilon u, A + h_{\text{ex}} A_\varepsilon^0)$ *which locally minimizes* GL_ε *in* $H^1(\Omega,\mathbb{C}) \times H^1(\Omega,\mathbb{R}^2)$ *and satisfies* $F_{\varepsilon,\rho_\varepsilon}(u,A) < \varepsilon^\beta$*, is unique up to a gauge transformation.*

Remark 0.0.2. *The hypotheses of this theorem are verified by the vortexless local minimizer found in Theorem 3. More precisely, given* $\alpha \in \left(0,\frac{1}{2}\right)$*, the vortexless local minimizer* $(\mathbf{u},\mathbf{A})$ *given by Theorem 3 is such that* $F_{\varepsilon,\rho_\varepsilon}(u,A) < \varepsilon^\beta$*, for some constant* $\beta > 0$*.*

As far as we know, analogous results to Theorem 3 and Theorem 4 have only been stated in the homogeneous functional. We refer to [Ser99] and [Rom19a] for these results in 2D and 3D respectively. The preceding theorems also indicate the hysteretic property of the Meissner state, which states that configurations that start vortexless remain vortexless even for some values of $h_{\text{ex}} > H_{c_1}$.

Main result in 3D

Our final result is a lower estimate of the first critical field, which is a result from a work in preparation [DVR]. This is an analogous version of Theorem 1 for the 3D problem. This result is more limited in comparison, which shows the limitations of the 2D tools in the 3D framework. Our lower bound approximant changes in the following way: We define

$$H_{c_1}^{\varepsilon,3\text{D}} := \frac{|\log\varepsilon|}{2\max_{\Gamma\in\mathcal{N}(\Omega)} R_\varepsilon(\Gamma)},$$

where $R_\varepsilon(\Gamma)$ is, roughly speaking, the ratio of the circulation of Γ on the vector field B^0_ε and its length. This is what is recently referred to as an *isoflux problem*. In the inhomogeneous framework, we work on a weighted variant of the isoflux problem studied by C. Román, E. Sandier and S. Serfaty in [RSS23, Section 1.2].

Now we state our 3D result

Theorem 5. *Assume* $[\rho_\varepsilon]_{C^{0,\alpha}(\Omega)} \leq M|\log\varepsilon|^m$ *for some* $M, m > 0$ *and* $\alpha \in (0,1]$. *Additionally, assume that there exists* $\kappa = \kappa(\varepsilon)$ *such that* ρ_ε *is constant in* $\{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) \leq \kappa\}$. *There exist* $\varepsilon_0, K_0 > 0$ *such that for any* $\varepsilon < \varepsilon_0$ *and* $h_{\rm ex} \leq H^{\varepsilon,3\mathrm{D}}_{c_1} - K_0 \log|\log\varepsilon|$ *such that any global minimizer* $(\mathbf{u}, \mathbf{A}) = (\rho_\varepsilon u e^{ih_{\rm ex}\phi^0_{\varepsilon,\rho_\varepsilon^2}}, A + h_{\rm ex}A^0_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times [A_{\rm ex} + H_{\rm curl}(\mathbb{R}^3, \mathbb{R}^3)]$ *for* GL_ε *is weakly vortexless and satisfies the following asymptotic estimate*

$$GL^{3\mathrm{D}}_\varepsilon(\mathbf{u}, \mathbf{A}) = GL^{3\mathrm{D}}_\varepsilon(\rho_\varepsilon e^{ih_{\rm ex}\phi^0_{\varepsilon,\rho_\varepsilon^2}}, h_{\rm ex}A^0_\varepsilon) + o(1)$$

An important remark is that in the homogeneous case, strong vortexlessness holds (see [Rom19a, Theorem 1.1]). Therefore, we can add that Theorem 1 directly generalizes to the 3D problem in the homogeneous case.

This document is organized as follows: In Chapter 1, we present the mathematical tools for the 2D functional GL_ε, including notation and some preliminary results on topics like function spaces, norms and differential forms. Chapter 2 presents the tools for vortex analysis like the vortex ball construction and the energy splitting. Chapter 3 contains the proofs of the estimations for the first critical field, Theorem 1 and Theorem 2. In Chapter 4 we prove results on the stability of the Meissner configurations, Theorem 3 and Theorem 4. Finally, Chapter 5 contains an overview of the 3D problem and a proof of Theorem 5 that characterizes the first critical field.

Chapter 1

Mathematical theory for the 2D inhomogeneous Ginzburg–Landau functional

Our main goal in this chapter is to characterize the minimizers of GL_ε and develop on relevant estimates for minimizing configurations which will be used throughout the following chapters.

1.1 Notation and Preliminaries

1.1.1 Vector Algebra and Calculus

We denote the dot product in $\mathbb{R}^n$ by $x \cdot y$ and the usual Euclidean norm by $|x|$. We can also identify the inner product of $\mathbb{C}$

$$\langle z , w \rangle := \frac{z\overline{w} + \overline{z}w}{2}$$

in $\mathbb{R}^2$. The interaction between $\mathbb{C}$ and $\mathbb{R}^2$ appears in terms like the expansion of $|\nabla_A u|^2$.

The divergence operator in $\mathbb{R}^n$ is defined in Cartesian coordinates by

$$\operatorname{div} A = \sum_{i=1}^{n} \partial_{x_i} A_i.$$

In $\mathbb{R}^3$, the curl (rotational) operator in Cartesian coordinates corresponds

$$\operatorname{curl} A = (\partial_{x_2} A_3 - \partial_{x_3} A_2, \partial_{x_3} A_1 - \partial_{x_1} A_3, \partial_{x_1} A_2 - \partial_{x_2} A_1).$$

In $\mathbb{R}^2$, the curl operator is the x_3 coordinate of the 3D curl

$$\operatorname{curl} A = \partial_{x_1} A_2 - \partial_{x_2} A_1.$$

curl can be understood as the dual of the operator $\nabla^{\perp} := (-\partial_{x_2}, \partial_{x_1})$, since it satisfies the following identity for any A and compactly supported ϕ

$$\int_{\Omega} A \cdot \nabla^{\perp} \phi = \int_{\Omega} \operatorname{curl} A \phi.$$

The preceding equation can be used as the definition of the curl operator in the sense of distributions. Moreover, we have the following operator identity for any scalar function ϕ

$$\operatorname{curl}(\phi \nabla^{\perp}) = \operatorname{div}(\phi \nabla) \tag{1.1.1}$$

We will use ∇ as the gradient operator for scalar-valued functions and D as the total derivative (Jacobian) operator for vector-valued functions.

1.1.2 Function spaces and Norms

In this section, Ω is an open subset of $\mathbb{R}^n$ with smooth boundary and X represents a Banach algebra (like $\mathbb{R}^n$ or $\mathbb{C}^n$). We will describe the most common spaces in this theory, which are L^p spaces, Hölder spaces and Sobolev spaces.

Given a Banach space X, we denote its dual space, the space of bounded linear functionals in X, by X^* and we use $\langle \cdot\,,\cdot\rangle$ to denote the duality product. The norm of the dual space is defined by

$$\|f\|_{X*} := \sup_{x\in X} \frac{\langle f\,,x\rangle}{\|x\|_X} = \sup_{x\in X, \|x\|_X=1} \langle f\,,x\rangle.$$

For $p \in [1,\infty)$, we define the space $L^p(\Omega, X)$ as

$$L^p(\Omega, X) := \left\{ f\colon \Omega \to X\colon \int_\Omega \|f\|_X^p < \infty \right\}.$$

For $p=\infty$, $L^\infty(\Omega, X)$ is the space of functions $f\colon \Omega \to X$ which are bounded when excluding measure zero sets. In the particular case $X = \mathbb{R}$, we use the simpler notation $L^p(\Omega)$. $L^p(\Omega, X)$ is a Banach space with the norm

$$\|f\|_{L^p(\Omega,X)} := \left(\int_\Omega \|f\|_X^p \right)^{\frac{1}{p}}$$

for $p \in [1,\infty)$ and

$$\|f\|_{L^\infty(\Omega,X)} := \operatorname*{ess\,sup}_\Omega f,$$

where $\operatorname{ess\,sup}_\Omega$ is the supremum in Ω excluding measure zero sets. For any $p \in [1,\infty]$, $L^p(\Omega, X)$ is a Banach space.

If $p \in [1,\infty)$, the dual space of $L^p(\Omega, X)$ is isometrically isomorphic to $L^q(\Omega, X)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Thus, if $p \in (1,\infty)$, $L^p(\Omega, X)$ is reflexive. Moreover, $L^2(\Omega, X)$ is a Hilbert space with inner product

$$\langle f\,,g\rangle_{L^2(\Omega,X)} := \int_\Omega f\overline{g}.$$

For $\alpha \in (0,1]$, we define the space of α-Hölder continuous functions $C^{0,\alpha}(\Omega, X)$ as

$$C^{0,\alpha}(\Omega, X) := \left\{ f\colon \Omega \to X\colon \sup_{x\neq y\in\Omega} \frac{\|f(x)-f(y)\|_X}{|x-y|^\alpha} < \infty \right\}.$$

This is a nonreflexive Banach space with the norm

$$\|f\|_{C^{0,\alpha}(\Omega,X)} := \|f\|_{L^\infty(\Omega,X)} + \sup_{x\neq y\in\Omega} \frac{\|f(x)-f(y)\|_X}{|x-y|^\alpha}.$$

We also denote the α-Hölder seminorm as

$$[f]_{C^{0,\alpha}(\Omega,X)} := \sup_{x\neq y\in\Omega} \frac{\|f(x)-f(y)\|_X}{|x-y|^\alpha}.$$

For $k \in \mathbb{N}, p \in [1,\infty]$, we define the Sobolev space $W^{k,p}(\Omega,X)$ as

$$W^{k,p}(\Omega,X) := \{f\colon \Omega \to X\colon D^\alpha f \in L^p(\Omega,X) \text{ for all } |\alpha| \leq k\},$$

where the derivatives are taken in the sense of distributions. This is a Banach space with the norm

$$\|f\|_{W^{k,p}(\Omega,X)} := \sum_{|\alpha|\leq k} \|D^\alpha f\|_{L^p(\Omega,X)}.$$

If $p \in (1,\infty)$, $W^{k,p}(\Omega,X)$ is reflexive. Moreover, $W^{k,2}(\Omega,X)$ is a Hilbert space, denoted by $H^k(\Omega,X)$, with inner product

$$\langle f\,,g\rangle_{H^k(\Omega,X)} := \int_\Omega \sum_{|\alpha|\leq k} D^\alpha f\overline{D^\alpha g}.$$

The following spaces are present in the 3D problem. The space $H_{\mathrm{curl}}(\Omega,\mathbb{R}^3)$ is the space of L^2 vector fields with L^2 curl. This is a Hilbert space with associated inner product

$$\langle A\,,V\rangle_{H_{\mathrm{curl}}(\Omega,\mathbb{R}^3)} := \int_\Omega \operatorname{curl} A\cdot\operatorname{curl} V + A\cdot V.$$

We denote the space of compactly supported infinitely differentiable functions by $C_0^\infty(\Omega,X)$. This is a dense space in $L^p(\Omega,X)$. The closure of $C_0^\infty(\Omega,X)$ in the $W^{k,p}(\Omega,X)$ norm is denoted by $W_0^{k,p}(\Omega,X)$, which essentially corresponds to the space of $W^{k,p}$ functions which vanish in $\partial\Omega$. More specifically, these functions do not leave a boundary term when they are integrated by parts.

We use the suffix σ to indicate that the elements of a vector field have zero divergence. We dedicate a special mention to the space $C^\infty_{0,\sigma}(\Omega, \mathbb{R}^n)$, the space of compactly supported vector fields with zero divergence, which we explore in the following subsection.

1.1.3 Boundary conditions

One of the most useful tools in this work is integration by parts, which comes in multiple forms depending on the differential operator and the boundary condition. The notion of boundary conditions in the framework of Sobolev spaces is given by trace operators. In this work, we will mainly use the following forms of integration by parts

$$\int_\Omega A \cdot \nabla f = -\int_\Omega (\operatorname{div} A) f + \int_{\partial\Omega} f A \cdot \nu.$$

$$\int_\Omega B \cdot \operatorname{curl} V = \int_\Omega \operatorname{curl} B \cdot V - \int_{\partial\Omega} (B \times V) \cdot \nu,$$

where ν is the unit normal vector on $\partial\Omega$ pointing outwards.

This motivates the following characterization of boundary conditions.

- If a vector field satisfies $A \cdot \nu = 0$ on $\partial\Omega$, then for any function f

$$\int_\Omega A \cdot \nabla f = -\int_\Omega (\operatorname{div} A) f.$$

- If a vector field satisfies $B \times \nu = 0$ on $\partial\Omega$, then for any vector field V

$$\int_\Omega B \cdot \operatorname{curl} V = \int_\Omega \operatorname{curl} B \cdot V.$$

We will use the suffix N on a space to indicate that their elements satisfy $A \cdot \nu = 0$ on $\partial\Omega$, while we will use the suffix T if they satisfy $A \times \nu = 0$. For example, $W^{1,p}_N(\Omega, \mathbb{R}^3)$ denotes the space of vector fields in $W^{1,p}(\Omega, \mathbb{R}^3)$ such that $A \cdot \nu = 0$ on $\partial\Omega$. A particularly important subspace is $L^p_{\sigma,N}(\Omega,\mathbb{R}^3)$, the space of zero divergence vector fields with tangential boundary conditions, since it is used for decomposition purposes in Section 5. We take this opportunity to state a couple of classical characterizations of this space which are satisfied under certain topological conditions on Ω.

Lemma 1.1.1. *Let $A \in L^p_{\sigma,N}(\Omega,\mathbb{R}^3)$.*

1. [AS13, Theorem 4.3] *If Ω is a simply connected bounded domain, there exists a unique $B_A \in W^{1,p}_{\sigma,T}(\Omega,\mathbb{R}^3)$ such that $A = \operatorname{curl} B_A$.*
2. [Gal94, Lemma III.2.2] *The space $C^\infty_{0,\sigma}(\Omega,\mathbb{R}^3)$ is L^p-dense in $L^p_{\sigma,N}(\Omega,\mathbb{R}^3)$.*

1.1.4 Important inequalities

- *Hölder inequality:* For $\frac{1}{p}+\frac{1}{q}=1$, $f \in L^p(\Omega,X)$, $g \in L^q(\Omega,X)$:

$$\|fg\|_{L^1(\Omega,X)} \leq \|f\|_{L^p(\Omega,X)} \|g\|_{L^q(\Omega,X)}\,.$$

- *Poincaré inequality:* For $p \in [1,\infty)$, there exists $C = C(\Omega,n,p) > 0$ such that for any $f \in W^{1,p}_0(\Omega)$

$$\|f\|_{L^p(\Omega)} \leq C\,\|\nabla f\|_{L^p(\Omega,\mathbb{R}^n)}\,.$$

- *Poincaré–Wirtinger inequality:* For $p \in [1,\infty)$, there exists $C = C(\Omega,n,p) > 0$ such that for any $f \in W^{1,p}(\Omega)$

$$\left\| f - \left(\frac{1}{\Omega}\int_\Omega f\right)\right\|_{L^p(\Omega)} \leq C\,\|\nabla f\|_{L^p(\Omega,\mathbb{R}^n)}\,.$$

- *Sobolev embedding, $p < n$:* For $p \in [1,n)$, p^* such that $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$, there exists $C = C(\Omega,n,p)$ such that

$$\|f\|_{L^{p^*}(\Omega)} \leq C\,\|f\|_{W^{1,p}(\Omega)}\,.$$

  Moreover, for any $q < p^*$, the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact, that is, any bounded sequence in $W^{1,p}(\Omega)$ has a convergent subsequence in $L^q(\Omega)$.

- *Sobolev embedding, $p = n$:* For $p = n$ and any $q \in (1, \infty)$, there exists $C = C(\Omega, p, q)$ such that
$$\|f\|_{L^q(\Omega)} \leq C \|f\|_{W^{1,p}(\Omega)} .$$
Moreover, the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact, that is, any bounded sequence in $W^{1,p}(\Omega)$ has a convergent subsequence in $L^q(\Omega)$.
- *Sobolev embedding, $p > n$ (also known as Morrey inequality):* For $p \in (n, \infty)$, $p_* = 1 - \frac{n}{p}$ there exists $C = C(\Omega, n, p)$ such that
$$\|f\|_{C^{0,p_*}(\Omega)} \leq C \|f\|_{W^{1,p}(\Omega)}$$

1.1.5 Differential forms and currents

An *n-current* Γ is a bounded linear functional defined on the space of smooth n-forms with the L^∞ norm. As before, we denote the duality product between Γ and an n-form ω by $\langle \Gamma , \omega \rangle$. We will mostly work with 1-forms and 2-forms in $\mathbb{R}^3$, which means that we can extrapolate the L^p theory for vector spaces to these forms, since 1-forms and 2-forms can be identified as vector fields in $\mathbb{R}^3$.

The *mass of* Γ is defined by

$$|\Gamma| := \sup_{\omega \in \mathcal{D}^n(\Omega, \mathbb{R}^3)} \frac{\langle \Gamma , \omega \rangle}{\|\omega\|_{L^\infty(\Omega, \mathbb{R}^3)}},$$

where $\mathcal{D}^n(\Omega, \mathbb{R}^3)$ is the space of compactly supported n-forms.

n-currents are a generalization of n-dimensional manifolds that act on n-forms through integration. We say Γ is an integer multiplicity rectifiable n-current if there exists an n-dimensional manifold M, an orientation n-form ξ and an integer-valued function θ such that
$$\langle M , \omega \rangle := \int_\Gamma \theta(x) \langle \omega(x) , \xi(x) \rangle d\mathcal{H}^n(x),$$

where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure. The pair (M, θ) is associated to a geometrical object called *varifold* and is denoted by $v(M, \theta)$.

The *boundary of an n-current* Γ is denoted by $\partial\Gamma$. This is an $(n-1)$-current and its definition is motivated by Stokes' theorem

$$\langle \partial\Gamma, \omega \rangle := \langle \Gamma, d\omega \rangle.$$

1.2 Ginzburg–Landau equations

A natural starting point to study the minimization of a functional is to deduce its Euler–Lagrange equations, which are solved by minimizing configurations. The Euler–Lagrange equations associated to GL_ε are

$$\begin{cases} -(\nabla_A)^2 u &= \dfrac{u(a_\varepsilon - |u|^2)}{\varepsilon^2} & \text{in } \Omega \\ -\nabla^\perp h &= \langle iu, \nabla_A u \rangle & \text{in } \Omega \\ h &= h_{\text{ex}} & \text{on } \partial\Omega \\ \nabla_A u \cdot \nu &= 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.1}$$

where $(\nabla_A)^2 = (\operatorname{div} - iA)(\nabla_A) = (\Delta + |A|^2 - i(\operatorname{div} A + 2A \cdot \nabla))$ and ν is the unit normal vector pointing outward from Ω. We will refer to (1.2.1) as the *Ginzburg–Landau equations*. Just like GL_ε, these equations are also gauge invariant. This means that any solution of (1.2.1) can be gauge-transformed into a solution (u, A) on the Coulomb gauge. One of the advantages of this particular choice of gauge lies in some elliptic regularity estimates, as we shall see later on. We can also immediately deduce from the second boundary condition that in the Coulomb gauge

$$\nabla u \cdot \nu = 0 \text{ on } \partial\Omega \tag{1.2.2}$$

The elliptic structure of (1.2.1) also gives us a useful estimate on $|u|$.

Proposition 1.2.1. *Let* (u, A) *be a solution of* (1.2.1)*. Then we have in* $\overline{\Omega}$

$$\max_{\Omega} |u|^2 \le \|a_\varepsilon\|_{L^\infty(\Omega)} \le 1. \tag{1.2.3}$$

Proof. The proof is an adaptation of the proof for [SS07, Proposition 3.9]. By gauge invariance, we may assume that A satisfies (0.0.2). By expanding $(\nabla_A)^2 u$ in the first equation and rearranging, we obtain

$$-\Delta u = \frac{u(a_\varepsilon - |u|^2)}{\varepsilon^2} - 2i(A \cdot \nabla)u - |A|^2 u. \tag{1.2.4}$$

If we take inner product with u on both sides, we get

$$-\langle \Delta u \,, u\rangle = |u|^2 \frac{(a_\varepsilon - |u|^2)}{\varepsilon^2} - 2\langle i(A \cdot \nabla)u \,, u\rangle - |A|^2|u|^2.$$

Now, consider the identity

$$\frac{1}{2}\Delta |u|^2 = \langle \Delta u \,, u\rangle + |\nabla u|^2.$$

By inserting (1.2.4) and considering that

$$|\nabla_A u|^2 = |\nabla u|^2 + 2\langle i(A \cdot \nabla)u \,, u\rangle + |A|^2|u|^2,$$

we deduce

$$-\frac{1}{2}\Delta |u|^2 = \frac{|u|^2(a_\varepsilon - |u|^2)}{\varepsilon^2} - |\nabla_A u|^2 \le \frac{|u|^2(a_\varepsilon - |u|^2)}{\varepsilon^2} \le \frac{|u|^2(\|a_\varepsilon\|_{L^\infty(\Omega)} - |u|^2)}{\varepsilon^2}.$$

If $x_0 \in \Omega$ is a maximal point of $|u|^2$, then $\nabla |u|(x_0) = 0$ and $-\Delta |u|^2(x_0) \ge 0$.

On the other hand, if x_0 is a maximal point of $|u|^2$ in $\partial\Omega$, then $\nabla |u| \cdot \tau = 0$, where τ is the unit tangent vector of $\partial\Omega$. On the other hand, (1.2.2) implies that $\nabla u \cdot \nu = 0$. In particular, $\nabla |u| \cdot \nu = 0$. Therefore,

$$\nabla |u| = (\nabla |u| \cdot \tau)\tau + (\nabla |u| \cdot \nu)\nu = 0.$$

We may argue similarly to deduce that $-\Delta|u|(x_0) \geq 0$. Thus, we conclude similarly that

$$|u|^2 \leq \|a_\varepsilon\|_{L^\infty(\Omega)} .$$

□

1.3 Lassoued–Mironescu energy decoupling

As we have mentioned in the overview of the tools, there is a particular minimizer in $H^1(\Omega, \mathbb{C})$ that plays a role in an energy decoupling result. Recall the energy functional without magnetic components.

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{(a_\varepsilon - |u|^2)^2}{2\varepsilon^2}. \tag{1.3.1}$$

Proposition 1.3.1. *There exists a unique non-negative minimizer ρ_ε for* (1.3.1)*. This minimizer is a solution of* (0.0.3) *and satisfies the following decoupling property for any $u \in H^1(\Omega, \mathbb{C})$*

$$E_\varepsilon(\rho_\varepsilon u) = E_\varepsilon(\rho_\varepsilon) + \frac{1}{2}\int_\Omega \rho_\varepsilon^2|\nabla u|^2 + \rho_\varepsilon^4\frac{(1 - |u|^2)^2}{2\varepsilon^2}. \tag{1.3.2}$$

Proof. To prove that a minimizer exists, we aim to use the direct methods of the calculus of variations. The energy E_ε is clearly coercive in $H^1(\Omega, \mathbb{C})$. Moreover, the term $\int_\Omega |\nabla u|^2$ is H^1-continuous and convex, which implies that is H^1-weakly lower semicontinuous. On the other hand, the term $\int_\Omega \frac{(a_\varepsilon - |u|^2)^2}{2\varepsilon^2}$ is L^4-continuous, which implies that it is H^1-weakly continuous by the Rellich-Kondrachov theorem. Therefore, E_ε is weakly lower semicontinuous and coercive in $H^1(\Omega, \mathbb{C})$ and we conclude that it must admit a minimizer in this space. Moreover, if u_ε minimizes E_ε, we have $E_\varepsilon(|u_\varepsilon|) \leq E_\varepsilon(u_\varepsilon)$. Therefore, E_ε admits a non-negative minimizer.

Now we prove the energy decoupling. Let ρ_ε be a non-negative minimizer for E_ε.

We start by determining $E_\varepsilon(\rho_\varepsilon u)$

$$\begin{aligned} E_\varepsilon(\rho_\varepsilon u) &= \frac{1}{2}\int_\Omega |\nabla(\rho_\varepsilon u)|^2 + \frac{(a_\varepsilon - |\rho_\varepsilon u|^2)^2}{2\varepsilon^2} \\ &= \frac{1}{2}\int_\Omega (|u|^2|\nabla\rho_\varepsilon|^2 + \rho_\varepsilon^2|\nabla u|^2 + 2u\rho_\varepsilon\nabla\rho_\varepsilon\cdot\nabla u) + \frac{(a_\varepsilon - |\rho_\varepsilon|^2 + \rho_\varepsilon^2(1-|u|^2))^2}{2\varepsilon^2}. \end{aligned}$$

After expanding the rightmost square terma and rearranging, we obtain that

$$\begin{aligned} E_\varepsilon(\rho_\varepsilon u) &= E_\varepsilon(\rho_\varepsilon) + \frac{1}{2}\int_\Omega \rho_\varepsilon^2|\nabla u|^2 + \rho_\varepsilon^4\frac{(1-|u|^2)}{2\varepsilon^2} \\ &\quad + \frac{1}{2}\int_\Omega (|u|^2-1)|\nabla\rho_\varepsilon|^2 + 2u\rho_\varepsilon\nabla\rho_\varepsilon\cdot\nabla u + \rho_\varepsilon^2\frac{(a-|\rho_\varepsilon|^2)(1-|u|^2)^2}{\varepsilon^2}. \end{aligned} \tag{1.3.3}$$

While on the other hand, by using $\rho_\varepsilon(1-|u|^2)$ as a test function in (0.0.3), we have the equality

$$\int_\Omega \nabla\rho_\varepsilon\cdot(\nabla\rho_\varepsilon(1-|u|^2) - 2\rho_\varepsilon u\nabla u) = \int_\Omega \rho_\varepsilon^2\frac{(a_\varepsilon-\rho_\varepsilon^2)(1-|u|^2)^2}{\varepsilon^2}$$

By inserting this term into (1.3.3), we see that the second integral vanishes, which yields our decoupling result.

$$E_\varepsilon(\rho_\varepsilon u) = E_\varepsilon(\rho_\varepsilon) + \frac{1}{2}\int_\Omega \rho_\varepsilon^2|\nabla u|^2 + \rho_\varepsilon^4\frac{(1-|u|^2)}{2\varepsilon^2}.$$

Finally, we prove that there exists a unique non-negative minimizer. Suppose p_1, p_2 are non-negative minimizers for E_ε. Using the decoupling result (1.3.2)

$$\begin{aligned} 0 = E_\varepsilon(p_1) - E_\varepsilon(p_2) &= E_\varepsilon\left(p_2\frac{p_1}{p_2}\right) - E_\varepsilon(p_2) \\ &\overset{(1.3.2)}{=} \frac{1}{2}\int_\Omega p_2^2\left|\nabla\frac{p_1}{p_2}\right|^2 + p_2^4\frac{\left(1-\left|\frac{p_1}{p_2}\right|^2\right)^2}{2\varepsilon^2}. \end{aligned}$$

Thus, we have $\left|\frac{p_1}{p_2}\right| = 1$. Since p_1, p_2 are non-negative, we deduce that $p_1 = p_2$.

□

This decoupling result holds even if ∇u is replaced by the covariant gradient $\nabla_A u$ instead, that is, if we consider the following energy functional

$$E_\varepsilon(u, A) := \frac{1}{2}\int_\Omega |\nabla_A u|^2 + \frac{(a_\varepsilon - |u|^2)^2}{2\varepsilon^2},$$

we have the following result.

Lemma 1.3.1. *For any* $(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$, *we have*

$$E_\varepsilon(\rho_\varepsilon u, A) = E_\varepsilon(\rho_\varepsilon) + \frac{1}{2}\int_\Omega \rho_\varepsilon^2 |\nabla_A u|^2 + \rho_\varepsilon^4 \frac{(1-|u|^2)^2}{2\varepsilon^2}. \tag{1.3.4}$$

Proof. Expanding the square on $|\nabla_A(\rho_\varepsilon u)|^2$ we have

$$|\nabla_A(\rho_\varepsilon u)|^2 = |\nabla(\rho_\varepsilon u)|^2 + \rho_\varepsilon^2 |A|^2 |u|^2 - 2\rho_\varepsilon \langle \nabla(\rho_\varepsilon u)\,, iAu\rangle.$$

Combining with (1.3.2), we find

$$\begin{aligned} E_\varepsilon(\rho_\varepsilon u, A) = E_\varepsilon(\rho_\varepsilon) + \frac{1}{2}\int_\Omega \rho_\varepsilon^2 |\nabla u|^2 + \rho_\varepsilon^4 \frac{(1-|u|^2)^2}{2\varepsilon^2} \\ + \rho_\varepsilon^2 |A|^2 |u|^2 - 2\rho_\varepsilon(\langle \rho_\varepsilon \nabla u\,, iAu\rangle + \langle u \nabla \rho_\varepsilon\,, iAu\rangle). \end{aligned}$$

Since $\langle u\,, iu\rangle = 0$ and $\nabla \rho_\varepsilon$ and A are real-valued vector fields, $\langle u\nabla\rho_\varepsilon\,, iAu\rangle = 0$. Thus, the RHS is equal to

$$\begin{aligned} E_\varepsilon(\rho_\varepsilon) + \frac{1}{2}\int_\Omega \rho_\varepsilon^2(|\nabla u|^2 + |A|^2|u|^2 - 2\langle \nabla u\,, iAu\rangle) + \rho_\varepsilon^4 \frac{(1-|u|^2)^2}{2\varepsilon^2} \\ = E_\varepsilon(\rho_\varepsilon) + \frac{1}{2}\int_\Omega \rho_\varepsilon^2 |\nabla_A u|^2 + \rho_\varepsilon^4 \frac{(1-|u|^2)^2}{2\varepsilon^2}. \end{aligned}$$

□

1.3.1 Regularity properties of ρ_ε

The following are a couple of useful estimates for ρ_ε.

Proposition 1.3.2. *For any $x \in \Omega$, we have*

$$\sqrt{b} \leq \rho_\varepsilon(x) \leq 1 \tag{1.3.5}$$

Furthermore, we have

$$\|\nabla \rho_\varepsilon\|_{L^\infty(\Omega,\mathbb{R}^2)} \leq \frac{C}{\varepsilon}$$

for some $C > 0$ which does not depend on ε.

Proof. Observe that $\max\{\rho_\varepsilon, 1\} \in H^1(\Omega, \mathbb{C})$. We are thus allowed to use it as a test function in (0.0.3), which yields

$$0 \leq \int_{\{\rho_\varepsilon > 1\}} |\nabla \rho_\varepsilon|^2 = \int_{\{\rho_\varepsilon > 1\}} \rho_\varepsilon^2 \frac{(a_\varepsilon - \rho_\varepsilon^2)}{\varepsilon^2} + \int_{\{\rho_\varepsilon < 1\}} \rho_\varepsilon \frac{(a_\varepsilon - \rho_\varepsilon^2)}{\varepsilon^2}. \tag{1.3.6}$$

Since $(a_\varepsilon - \rho_\varepsilon^2) < 0$ when $\rho_\varepsilon > 1$, we deduce that

$$\int_{\{\rho_\varepsilon > 1\}} \rho_\varepsilon^2 \frac{(a_\varepsilon - \rho_\varepsilon^2)}{\varepsilon^2} \leq \int_{\{\rho_\varepsilon > 1\}} \rho_\varepsilon \frac{(a_\varepsilon - \rho_\varepsilon^2)}{\varepsilon^2}.$$

Inserting this into (1.3.6) and using the equation solved by ρ_ε yields

$$\begin{aligned}
\int_{\{\rho_\varepsilon > 1\}} |\nabla \rho_\varepsilon|^2 &\leq \int_\Omega \rho_\varepsilon \frac{(a_\varepsilon - \rho_\varepsilon^2)}{\varepsilon^2} \\
&\overset{(0.0.3)}{=} \int_\Omega -\Delta \rho_\varepsilon \\
&= -\int_{\partial\Omega} \nabla \rho_\varepsilon \cdot \nu \\
&\overset{(0.0.3)}{=} 0.
\end{aligned}$$

Therefore we have

$$\int_{\{\rho_\varepsilon > 1\}} |\nabla \rho_\varepsilon|^2 = 0,$$

which implies $|\{\rho_\varepsilon > 1\}| = 0$ and thus, $\rho_\varepsilon \leq 1$. The other inequality follows analogously by using $\min\{\rho_\varepsilon, \sqrt{b}\}$ as a test function and that $\rho_\varepsilon^2(a_\varepsilon - \rho_\varepsilon^2) \leq \sqrt{b}\rho_\varepsilon(a_\varepsilon - \rho_\varepsilon^2)$ whenever $\rho_\varepsilon \leq \sqrt{b}$.

The estimate on the gradient follows from the Gagliardo–Nirenberg type inequality for functions $u \in H^2(\Omega, \mathbb{C})$ such that $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$ (see [DS21, Lemma 3.2])

$$\|\nabla u\|^2_{L^\infty(\Omega,\mathbb{C})} \leq C \left(\|\Delta u\|_{L^\infty(\Omega,\mathbb{C})} + \|u\|_{L^\infty(\Omega,\mathbb{C})} \right) \|u\|_{L^\infty(\Omega,\mathbb{C})} . \tag{1.3.7}$$

Indeed, since $\|\rho_\varepsilon\|_{L^\infty(\Omega)}, \|a_\varepsilon\|_{L^\infty(\Omega)} \leq 1$, from (0.0.3) we obtain that $\|\Delta \rho_\varepsilon\|_{L^\infty(\Omega,\mathbb{C})} \leq \frac{C}{\varepsilon^2}$, which leads to

$$\|\nabla \rho_\varepsilon\|_{L^\infty(\Omega,\mathbb{R}^2)} \leq \frac{C}{\varepsilon}.$$

□

1.4 Critical points in the Coulomb gauge

Suppose (u, A) minimizes GL_ε. A consequence of the gauge invariance property of GL_ε is that estimates on A could vary considerably, since $(ue^{i\phi}, A + \nabla\phi)$ is also a minimizer for any ϕ. Hence, it is natural to fix a particular gauge for (u, A).

To obtain the Coulomb gauge, we choose ϕ as the solution of the following elliptic problem

$$\begin{cases} \Delta\phi &= \operatorname{div} A \quad \text{in } \Omega \\ \nabla\phi \cdot \nu &= A \cdot \nu \quad \text{on } \partial\Omega, \end{cases}$$

which has a unique solution modulo a constant (in particular, $\nabla\phi$ is uniquely determined) since it satisfies the compatibility condition $\int_\Omega \operatorname{div} A = \int_{\partial\Omega} A \cdot \nu$. The reason we choose to fix the Coulomb gauge is the following crucial regularity result

Proposition 1.4.1. *Suppose $A \in H^1(\Omega, \mathbb{R}^2)$ satisfies the Coulomb gauge condition*

(0.0.2). *Then, there exists* $C = C(\Omega) > 0$ *such that*

$$\|A\|_{H^1(\Omega,\mathbb{R}^2)} \leq \|\operatorname{curl} A\|_{L^2(\Omega)} . \tag{1.4.1}$$

Moreover, if $A \in H^2(\Omega, \mathbb{R}^2)$, *we also have*

$$\|A\|_{H^2(\Omega,\mathbb{R}^2)} \leq C \|\operatorname{curl} A\|_{H^1(\Omega)} . \tag{1.4.2}$$

Proof. Suppose $\operatorname{div} A = 0$ and $A \cdot \nu = 0$ on $\partial\Omega$. Since we assume Ω is simply connected, we deduce that there exists ζ such that $A = \nabla^{\perp}\zeta$. Moreover, the boundary condition means that $\nabla\zeta \cdot \tau = 0$. This implies that ζ is constant on $\partial\Omega$ and, without losing generality, take that constant as 0. Thus, ζ solves the following elliptic problem

$$\begin{cases} \Delta\zeta & = & \operatorname{curl} A & \text{in } \Omega \\ \zeta & = & 0 & \text{on } \partial\Omega, \end{cases}$$

Therefore, we obtain estimates (1.4.1) and (1.4.2) by elliptic regularity, since

$$\|A\|_{H^1(\Omega,\mathbb{R}^2)} = \|\nabla\zeta\|_{H^1(\Omega,\mathbb{R}^2)} \leq C \|\operatorname{curl} A\|_{L^2(\Omega)}$$

and

$$\|A\|_{H^2(\Omega,\mathbb{R}^2)} = \|\nabla\zeta\|_{H^2(\Omega,\mathbb{R}^2)} \leq C \|\operatorname{curl} A\|_{H^1(\Omega)}$$

□

The regularity results given by the Coulomb gauge, combined with the Ginzburg–Landau equations (1.2.1) allow us to deduce more favorable bounds for minimizing configurations.

Proposition 1.4.2. *Let* $(\mathbf{u}, A) = (\rho_\varepsilon u, A)$ *be a solution of* (1.2.1), *where* A *satisfies* (0.0.2). *Then*

$$\|A\|_{L^\infty(\Omega,\mathbb{R}^2)} \leq C(E_\varepsilon(\rho_\varepsilon) + F_{\varepsilon,\rho_\varepsilon}(u, A))^{1/2}, \tag{1.4.3}$$

where $C = C(\Omega) > 0$.

Proof. From the second equation in (1.2.1) and the Cauchy–Schwarz inequality, we deduce that

$$\|\nabla \operatorname{curl} A\|_{L^2(\Omega,\mathbb{R}^2)} = \|\langle i\mathbf{u}, \nabla_A \mathbf{u}\rangle\|_{L^2(\Omega,\mathbb{C})} \leq \|\mathbf{u}\|_{L^2(\Omega,\mathbb{C})} \|\nabla_A \mathbf{u}\|_{L^2(\Omega,\mathbb{C}^2)} .$$

Since $|\mathbf{u}| \leq 1$ (recall (1.2.3)), it follows that

$$\|\nabla \operatorname{curl} A\|^2_{L^2(\Omega,\mathbb{R}^2)} \leq \|\nabla_A \mathbf{u}\|^2_{L^2(\Omega,\mathbb{C}^2)} \leq C E_\varepsilon(\mathbf{u}, A).$$

The decoupling (1.3.4) then yields

$$\|\nabla \operatorname{curl} A\|^2_{L^2(\Omega,\mathbb{R}^2)} \leq C \left(E_\varepsilon(\rho_\varepsilon) + F_{\varepsilon,\rho_\varepsilon}(u, A)\right) .$$

Moreover, since $\|\operatorname{curl} A\|^2_{L^2(\Omega)} \leq 2F_{\varepsilon,\rho_\varepsilon}(u, A)$, we deduce that

$$\|\operatorname{curl} A\|^2_{H^1(\Omega)} \leq C(E_\varepsilon(\rho_\varepsilon) + F_{\varepsilon,\rho_\varepsilon}(u, A)).$$

Finally, by (1.4.2) and Sobolev embedding, we obtain (1.4.3). □

We use the preceding result in tandem with the following proposition

Proposition 1.4.3. *Let* $(\mathbf{u}, A) = (\rho_\varepsilon u, A)$ *be a solution of* (1.2.1)*, where* A *satisfies* (0.0.2) *and* $\|A\|_{L^\infty(\Omega,\mathbb{R}^2)} \leq \frac{C}{\varepsilon}$ *for some* $C > 0$ *not depending on* ε*. Then*

$$\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{C}^2)} \leq \frac{C}{\varepsilon} \quad \text{and} \quad \|\nabla u\|_{L^\infty(\Omega,\mathbb{C}^2)} \leq \frac{C}{\varepsilon}, \tag{1.4.4}$$

where $C > 0$ *does not depend on* ε*.*

Proof. By expanding the first equation in (1.2.1) and using $\operatorname{div} A = 0$ from (0.0.2), we get

$$-\Delta \mathbf{u} = \frac{\mathbf{u}(a_\varepsilon - |\mathbf{u}|^2)}{\varepsilon^2} - 2i(A \cdot \nabla)\mathbf{u} - |A|^2 \mathbf{u}.$$

Moreover, from the boundary conditions $\nabla_A \mathbf{u} \cdot \nu = 0$ and $A \cdot \nu = 0$ on $\partial\Omega$, we get

$$\nabla \mathbf{u} \cdot \nu = 0 \text{ on } \partial\Omega.$$

Therefore, $\mathbf{u}$ satisfies (1.3.7). Combining this with (1.2.3) and our bound on $\|A\|_{L^\infty(\Omega,\mathbb{R}^2)}$, we deduce that

$$\begin{aligned} \|\nabla \mathbf{u}\|^2_{L^\infty(\Omega,\mathbb{C}^2)} &\leq C\left(\frac{1}{\varepsilon^2} + \|A\|_{L^\infty(\Omega,\mathbb{R}^2)} \|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{C}^2)} + \|A\|^2_{L^\infty(\Omega,\mathbb{R}^2)}\right) \\ &\leq \frac{C}{\varepsilon}\left(\frac{1}{\varepsilon} + \|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{C}^2)}\right), \end{aligned}$$

from where it follows that

$$\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{C}^2)} \leq \frac{C}{\varepsilon}.$$

Finally, using (1.3.5), we get

$$\|\nabla u\|_{L^\infty(\Omega,\mathbb{C}^2)} = \left\|\nabla\left(\frac{\mathbf{u}}{\rho_\varepsilon}\right)\right\|_{L^\infty(\Omega,\mathbb{C}^2)} \leq C\left(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{C}^2)} + \|\nabla \rho_\varepsilon\|_{L^\infty(\Omega,\mathbb{R}^2)}\right) \leq \frac{C}{\varepsilon}.$$

$\square$

Chapter 2

Characterization of vortex configurations

In this chapter we prepare the necessary tools to estimate the first critical field. We develop on the previously introduced necessary tools for identifying vortices and estimating their energetic cost, the vortex ball construction and the vorticity estimate. We also introduce the Meissner configuration to help us characterize vortexless configurations.

2.1 Approximation of the vortexless state

We have previously stated that a good approximation of a vortexless configuration follows from minimizing on A the energy $GL_\varepsilon(\rho_\varepsilon, h_{\text{ex}}A)$. Using the energy decoupling (1.3.1), we can write the energy as

$$GL_\varepsilon(\rho_\varepsilon, h_{\text{ex}}A) = GL_\varepsilon(\rho_\varepsilon \cdot 1, h_{\text{ex}}A) = E_\varepsilon(\rho_\varepsilon) + \frac{1}{2}\int_\Omega \rho_\varepsilon^2 h_{\text{ex}}^2 |A|^2 + h_{\text{ex}}^2 |\operatorname{curl} A - 1|^2.$$

Thus, we only have to look for minimizers of

$$J(A) := \frac{1}{2} \int_\Omega \rho_\varepsilon^2 |A|^2 + |\operatorname{curl} A - 1|^2.$$

An adequate space for minimization is the space of vector fields which satisfies the Coulomb gauge condition (0.0.2) $H_\sigma := \{A \in H^1(\Omega, \mathbb{R}^2) \colon A \text{ satisfies } (0.0.2)\}$ with the inherited H^1 norm. We can immediately deduce that J is strictly convex and H^1-continuous. Moreover, by H^1 estimates of the Coulomb gauge (1.4.1), we also deduce that J is coercive in H_σ. Therefore, J admits a unique minimizer in H_σ which we will denote by A_ε^0.

The Euler–Lagrange equation solved by A_ε^0 corresponds to

$$\begin{cases} -\nabla^\perp \operatorname{curl} A_\varepsilon^0 + \rho_\varepsilon^2 A_\varepsilon^0 &= 0 \quad \text{in } \Omega \\ \operatorname{curl} A_\varepsilon^0 &= 1 \quad \text{on } \partial\Omega, \end{cases} \tag{2.1.1}$$

Let us define $h_\varepsilon^0 := \operatorname{curl} A_\varepsilon^0$. If we divide (2.1.1) by ρ_ε^2 and take curl on both sides, we see that h_ε^0 solves the following Dirichlet problem (here we use (1.1.1))

$$\begin{cases} -\operatorname{div} \dfrac{\nabla h_\varepsilon^0}{\rho_\varepsilon^2} + h_\varepsilon^0 &= 0 \quad \text{in } \Omega \\ h_\varepsilon^0 &= 1 \quad \text{on } \partial\Omega, \end{cases}$$

This leads us to the previously mentioned solution of (0.0.6) which appears in the energy splitting (0.0.5), since we have the relation $\xi_\varepsilon = 1 - h_\varepsilon^0$.

2.1.1 Properties of ξ_ε and h_ε^0

First of all, we deduce from the maximum principle for h_ε^0 that, in $\overline{\Omega}$

$$h_\varepsilon^0 \le \max_{\partial\Omega} \left(\max\{h_\varepsilon^0, 0\}\right) \le 1$$

and

$$h_\varepsilon^0 \geq -\max_{\partial\Omega}\left(-\min\{h_\varepsilon^0, 0\}\right) \geq 0.$$

Thus, we have the following bounds on $\overline{\Omega}$

$$0 \leq h_\varepsilon^0 \leq 1.$$

Analogously, we have the same bounds for ξ_ε in Ω.

$$0 \leq \xi_\varepsilon \leq 1.$$

We also naturally deduce H^1 bounds. By using ξ_ε as a test function in (0.0.6) and integrating by parts, we have

$$\int_\Omega \frac{|\nabla \xi_\varepsilon|^2}{\rho_\varepsilon^2} + |\xi_\varepsilon|^2 = \int_\Omega \xi_\varepsilon.$$

Therefore, since $\rho_\varepsilon \leq 1$, we have

$$\|\xi_\varepsilon\|_{H^1(\Omega)}^2 \leq |\Omega|^{1/2} \|\xi_\varepsilon\|_{L^2(\Omega)}.$$

This implies

$$\|\xi_\varepsilon\|_{H^1(\Omega)} \leq C, \tag{2.1.2}$$

where $C > 0$ does not depend on ε.

Furthermore, we have a rather surprising $C_0^{0,1}$ bound for ξ_ε.

Proposition 2.1.1. *We have that*

$$\|\nabla \xi_\varepsilon\|_{L^\infty(\Omega)} \leq C, \tag{2.1.3}$$

where $C > 0$ does not depend on ε.

Proof. Note that $\nabla^\perp \operatorname{curl} A_\varepsilon^0 = -\nabla^\perp \xi_\varepsilon$. Equation (2.1.1) implies that $A_\varepsilon^0 = -\frac{\nabla^\perp \xi_\varepsilon}{\rho_\varepsilon^2}$.

Therefore, using (1.3.5), we have

$$\|\nabla\xi_\varepsilon\|_{L^\infty(\Omega,\mathbb{R}^2)} = \|\rho_\varepsilon^2 A_\varepsilon^0\|_{L^\infty(\Omega,\mathbb{R}^2)} \leq \|A_\varepsilon^0\|_{L^\infty(\Omega,\mathbb{R}^2)} \tag{2.1.4}$$

On the other hand, since A_ε^0 satisfies the Coulomb gauge condition (0.0.2), (1.4.2) yields

$$\left\|A_\varepsilon^0\right\|_{H^2(\Omega,\mathbb{R}^2)} \leq C\left\|\operatorname{curl} A_\varepsilon^0\right\|_{H^1(\Omega)} = C\left\|h_\varepsilon^0\right\|_{H^1(\Omega)}.$$

Combining with (2.1.2), we deduce that

$$\left\|A_\varepsilon^0\right\|_{H^2(\Omega,\mathbb{R}^2)} \leq C,$$

which, by Sobolev embedding, yields

$$\left\|A_\varepsilon^0\right\|_{L^\infty(\Omega,\mathbb{R}^2)} \leq C.$$

Inserting this in (2.1.4) concludes the proof. □

Finally, we will prove that ξ_ε stays sufficiently far from 0 as $\varepsilon \to 0$.

Proposition 2.1.2. *We have that*

$$\liminf_{\varepsilon\to 0} \max_\Omega \xi_\varepsilon > 0.$$

Proof. Let us assume towards a contradiction that, passing to a subsequence if necessary, we have

$$\max_\Omega \xi_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0.$$

By using ξ_ε as a test function in (0.0.6) and integrating by parts, we find

$$\int_\Omega \frac{|\nabla\xi_\varepsilon|^2}{\rho_\varepsilon^2} + \int_\Omega \xi_\varepsilon^2 = \int_\Omega \xi_\varepsilon.$$

Since $\rho_\varepsilon^2 \leq 1$, we deduce that

$$\|\xi_\varepsilon\|_{H^1(\Omega)}^2 \leq \int_\Omega \xi_\varepsilon \leq |\Omega| \max_\Omega \xi_\varepsilon.$$

Thus, $\|\xi_\varepsilon\|_{H^1(\Omega)}^2 \to 0$ as $\varepsilon \to 0$. On the other hand, by using an arbitrary $v \in H^1(\Omega)$ as a test function and integrating by parts, we find

$$\int_\Omega \frac{\nabla \xi_\varepsilon \cdot \nabla v}{\rho_\varepsilon^2} + \int_\Omega \xi_\varepsilon v = \int_\Omega v. \tag{2.1.5}$$

Using $b \leq \rho_\varepsilon^2$ and the Cauchy–Scharwz inequality, we find

$$\left| \int_\Omega \frac{\nabla \xi_\varepsilon \cdot \nabla v}{\rho_\varepsilon^2} \right| \leq b^{-1} \left| \int_\Omega \nabla \xi_\varepsilon \cdot \nabla v \right| \leq b^{-1} \|\nabla \xi_\varepsilon\|_{L^2(\Omega,\mathbb{R}^2)} \|\nabla v\|_{L^2(\Omega,\mathbb{R}^2)} \to 0 \quad \text{as } \varepsilon \to 0.$$

Similarly,

$$\left| \int_\Omega \xi_\varepsilon v \right| \leq \|\xi_\varepsilon\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \to 0 \quad \text{as } \varepsilon \to 0.$$

Hence, passing to the limit $\varepsilon \to 0$ in (2.1.5), we find $\int_\Omega v = 0$ for any $v \in H^1(\Omega)$, which is a contradiction. $\square$

Remark 2.1.1. *This result immediately yields* $\liminf_{\varepsilon \to 0} \max \psi_\varepsilon = \liminf_{\varepsilon \to 0} \max \frac{\xi_\varepsilon}{\rho_\varepsilon^2} > 0$. *Moreover, since* $\xi_\varepsilon = 0$ *on* $\partial\Omega$, *we have* $\psi_\varepsilon = 0$ *on* $\partial\Omega$. *We then deduce that there exists* $d > 0$, *independently of* ε, *such that* $\operatorname{dist}(\operatorname{argmax}_\Omega(\psi_\varepsilon), \partial\Omega) > d$ *for any* $\varepsilon > 0$.

2.2 Energy splitting

Now we have the necessary tools to prove one of the most important tools for our work: The splitting of the Ginzburg–Landau energy (0.0.5).

Proof of Proposition 0.0.1. From (1.3.4), we have

$$
\begin{aligned}
GL_\varepsilon(\mathbf{u},\mathbf{A}) &= E_\varepsilon(\rho_\varepsilon u,\mathbf{A}) + \frac{1}{2}\int_\Omega |\operatorname{curl}\mathbf{A} - h_{\text{ex}}|^2 \qquad (2.2.1)\\
&= E_\varepsilon(\rho_\varepsilon) + \frac{1}{2}\int_\Omega \rho_\varepsilon^2|\nabla_{\mathbf{A}}u|^2 + \rho_\varepsilon^4\frac{(1-|u|^2)^2}{2\varepsilon^2} + |\operatorname{curl}\mathbf{A} - h_{\text{ex}}|^2.
\end{aligned}
$$

By expanding the square $|\nabla_{\mathbf{A}}u|^2$ and integrating by parts (recall from (0.0.6) that $\xi_\varepsilon = 0$ on $\partial\Omega$), we find

$$
\begin{aligned}
\int_\Omega \rho_\varepsilon^2|\nabla_{\mathbf{A}}u|^2 &= \int_\Omega \rho_\varepsilon^2\left|\nabla_A u + ih_{\text{ex}}\frac{\nabla^\perp\xi_\varepsilon}{\rho_\varepsilon^2}u\right|^2 \qquad (2.2.2)\\
&= \int_\Omega \rho_\varepsilon^2\left(|\nabla_A u|^2 + h_{\text{ex}}^2\frac{|\nabla\xi_\varepsilon|^2}{\rho_\varepsilon^4}|u|^2 + 2\frac{h_{\text{ex}}}{\rho_\varepsilon^2}\langle\nabla_A u, iu\rangle\cdot\nabla^\perp\xi_\varepsilon\right)\\
&= \int_\Omega \rho_\varepsilon^2|\nabla_A u|^2 + h_{\text{ex}}^2\frac{|\nabla\xi_\varepsilon|^2}{\rho_\varepsilon^2}|u|^2 - 2h_{\text{ex}}\operatorname{curl}(\langle iu, \nabla_A u\rangle)\xi_\varepsilon.
\end{aligned}
$$

We now expand the square $|\operatorname{curl}\mathbf{A} - h_{\text{ex}}|^2$, which yields

$$
\begin{aligned}
\int_\Omega |\operatorname{curl}\mathbf{A} - h_{\text{ex}}|^2 &= \int_\Omega \left|\operatorname{curl}A + h_{\text{ex}}\operatorname{curl}A_\varepsilon^0 - h_{\text{ex}}\right|^2 \qquad (2.2.3)\\
&= \int_\Omega \left|\operatorname{curl}A + h_{\text{ex}}h_\varepsilon^0 - h_{\text{ex}}\right|^2\\
&= \int_\Omega |\operatorname{curl}A + h_{\text{ex}}(1-\xi_\varepsilon) - h_{\text{ex}}|^2\\
&= \int_\Omega |\operatorname{curl}A - h_{\text{ex}}\xi_\varepsilon|^2\\
&= \int_\Omega |\operatorname{curl}A|^2 + h_{\text{ex}}^2|\xi_\varepsilon|^2 - 2h_{\text{ex}}\xi_\varepsilon\operatorname{curl}A.
\end{aligned}
$$

Inserting (2.2.2) and (2.2.3) into (2.2.1), we deduce that

$$
GL_\varepsilon(\mathbf{u},\mathbf{A}) = E_\varepsilon(\rho_\varepsilon) + F_{\varepsilon,\rho_\varepsilon}(u,A) - h_{\text{ex}}\int_\Omega \mu(u,A)\xi_\varepsilon + \frac{h_{\text{ex}}^2}{2}\left(\frac{|\nabla\xi_\varepsilon|^2}{\rho_\varepsilon^2}|u|^2 + |\xi_\varepsilon|^2\right). \qquad (2.2.4)
$$

Let us now write $GL_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}}A^0_\varepsilon)$ in terms of the energies of ρ_ε and ξ_ε. We have

$$\begin{aligned} GL_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}}A^0_\varepsilon) &= E_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}}A^0_\varepsilon) + \frac{h_{\mathrm{ex}}^2}{2}\int_\Omega |\operatorname{curl} A^0_\varepsilon - 1|^2 \\ &\overset{(1.3.4)}{=} E_\varepsilon(\rho_\varepsilon) + \frac{h_{\mathrm{ex}}^2}{2}\int_\Omega \rho_\varepsilon^2 |A^0_\varepsilon|^2 + |h^0_\varepsilon - 1|^2 \\ &= E_\varepsilon(\rho_\varepsilon) + \frac{h_{\mathrm{ex}}^2}{2}\int_\Omega \rho_\varepsilon^2 \frac{|\nabla \xi_\varepsilon|^2}{\rho_\varepsilon^4} + |\xi_\varepsilon|^2 = E_\varepsilon(\rho_\varepsilon) + \frac{h_{\mathrm{ex}}^2}{2}\int_\Omega \frac{|\nabla \xi_\varepsilon|^2}{\rho_\varepsilon^2} + |\xi_\varepsilon|^2. \end{aligned} \tag{2.2.5}$$

Therefore, by writing $|u|^2$ as $1 + (|u|^2 - 1)$, we have

$$\begin{aligned} \frac{h_{\mathrm{ex}}^2}{2}\int_\Omega \left(\frac{|\nabla \xi_\varepsilon|^2}{\rho_\varepsilon^2}|u|^2 + |\xi_\varepsilon|^2\right) &= \frac{h_{\mathrm{ex}}^2}{2}\int_\Omega \left(\frac{|\nabla \xi_\varepsilon|^2}{\rho_\varepsilon^2} + |\xi_\varepsilon|^2\right) + \frac{h_{\mathrm{ex}}^2}{2}\int_\Omega \frac{|\nabla \xi_\varepsilon|^2}{\rho_\varepsilon^2}(|u|^2 - 1) \\ &\overset{(2.2.5)\&(0.0.7)}{=} GL_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}}A^0_\varepsilon) - E_\varepsilon(\rho_\varepsilon) + R_0. \end{aligned}$$

By inserting this into (2.2.4), we obtain (0.0.5). □

Note that from (2.2.5) in the preceding proof, we deduce the following bound for the energy of the Meissner configuration.

$$GL_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}}A^0_\varepsilon) \overset{(2.1.2)}{\leq} E_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}}A^0_\varepsilon) + Ch_{\mathrm{ex}}^2$$

2.3 Vortex ball construction

Before we obtain the first critical field, we need to quantify the energetic cost of vortices. To this end, we introduce the aforementioned vortex ball construction, which gives us a lower bound on the free energy of the configuration, which is the Ginzburg–Landau energy without the external field. In line with the decoupling (1.3.1), we consider the free energy in $B \subseteq \Omega$ with a weight $\eta\colon \Omega \to [\sqrt{b}, 1]$

$$F_{\varepsilon,\eta}(u, A; B) := \frac{1}{2}\int_B \eta^2 |\nabla_A u|^2 + \eta^4 \frac{(1 - |u|^2)^2}{2\varepsilon^2} + |\operatorname{curl} A|^2$$

Our plan of action is the following:

Step 1 State a lower bound for $F_{\varepsilon,\eta}(u,A;B)$ for an arbitrary ball $B\subset\Omega_\varepsilon$, where $\Omega_\varepsilon := \{x\in\Omega\colon \operatorname{dist}(x,\partial\Omega)>\varepsilon\}$. This is also the step where the presence of the weight makes the most difference.

Step 2 Construct an initial ball configuration $B_i = B(x_i,r_i)$ which covers the *essentially null set* S_E, which is the union of the connected components of $\left\{u\le\frac{1}{2}\right\}$ with non-zero boundary degree and such that $F_{\varepsilon,\eta}(u,A;B_i)$ satisfies a particular lower bound. We also obtain a lower bound for a function Λ_ε, which will be used in the ball growth process.

Step 3 Increase the radii of the balls in such a way that the lower bound of the collection is preserved. If two balls overlap during the process, they are replaced by a singular ball containing both of them which also preserves the lower bound.

By following these steps, we will obtain the main result from this section.

Proposition 2.3.1. *There exist* $\varepsilon_0, C>0$ *such that for any* $\varepsilon<\varepsilon_0$ *and* (u,A) *such that*

$$F_{\varepsilon,\eta}(u,A;\Omega)\le\varepsilon^{-\beta},$$

where $\beta\in(0,1)$*, the following holds. For every* $r\in(C\varepsilon^{1-\beta},\frac{1}{2})$ *there exists a collection of disjoint closed balls* $\mathcal{B}=\{B_i\}_i=\{B(a_i,r_i)\}$ *such that*

(1) $\{x\in\Omega_\varepsilon\colon ||u|-1|\ge\frac{1}{2}\}\subseteq\cup_i B_i$.

(2) $\sum_i r_i\le r$.

(3) *For any* $2b\le\overline{C}\le\left(\frac{r}{\varepsilon}\right)^{\frac{1}{2}}$ *it holds that either*

$$F_{\varepsilon,\eta}\left(u,A;\Omega\cap\bigcup_{B\in\mathcal{B}}B\right)\ge\overline{C}\log\frac{r}{\varepsilon},$$

or, for each $B \in \mathcal{B}$ such that $B \subset \Omega_\varepsilon$,

$$F_{\varepsilon,\eta}(u, A; B) \geq \pi \underline{\eta}^2(B)|d_B| \left(\log \frac{r}{\varepsilon \overline{C}} - C \right),$$

where $\underline{\eta}^2(B) = \min_B \eta^2$ and $d_B = \deg(u, \partial B)$.

The proof of Proposition 2.3.1 is an adaptation of [SS11, Section 5], which is a more up-to-date version of Jerrard's construction in [Jer99]. The proof is mostly identical, so we will refer to [SS11, Section 5] for details and indicate the differences caused by the presence of the weight η. We will also use the same notation for the constants as in the referred papers for an easier read.

Step 1 We start by obtaining a lower bound for an energy defined on a circle, which actually is the cornerstone of this new version of the ball construction method. In the following, we use the notation $\underline{\eta}^2(\Theta) := \min_\Theta \eta^2$, for any closed subset Θ of Ω.

Lemma 2.3.1. *Let $r > 0$ and $a \in \Omega$ such that $B = B(a, r) \in \Omega$. Define $m = \min_{\partial B} |u|$. Then, for any ε such that $0 < \frac{\varepsilon}{\underline{\eta}^2(B(a,r))} \leq r$, we have*

$$\frac{1}{2} \int_{\partial B(a,r)} \eta^2 |\nabla |u||^2 + \eta^4 \frac{(1 - |u|^2)^2}{2\varepsilon^2} \geq c_0 \underline{\eta}^2(B) \frac{(1 - m)^2}{\varepsilon}, \tag{2.3.1}$$

where c_0 is a universal constant.

Proof. We follow the proof of [Jer99, Lemma 2.3]. Within this proof, C denotes a positive constant that does not depend on r and that may change from line to line.

Let $x_m \in \partial B(a, r)$ such that $|u(x_m)| = m$ and define

$$\gamma := \frac{1}{2} \int_{\partial B(a,r)} |\nabla |u||^2.$$

Note that if $\gamma = 0$, then $|u|$ is constant on $\partial B(a, r)$ and therefore (2.3.1) follows immediately from the hypothesis on ε. Thus, we assume from now on that $\gamma > 0$.

From Morrey's inequality, we have, for any $x, y \in \partial B(a,r)$, that

$$||u(x)| - |u(y)|| \leq C \left\| \nabla |u| \right\|_{L^2(\partial B(a,r))} |x-y|^{\frac{1}{2}} = C\gamma^{\frac{1}{2}} |x-y|^{\frac{1}{2}}.$$

Therefore, for any $x \in \partial B(a,r)$ such that $|x - x_m|^{\frac{1}{2}} \leq \dfrac{|1-m|}{C\gamma^{\frac{1}{2}}}$, we have

$$|u(x)| \leq |u(x_m)| + C\gamma^{\frac{1}{2}} |x - x_m|^{\frac{1}{2}} \leq \frac{1+m}{2}.$$

Since $r \geq \frac{\varepsilon}{\underline{\eta}^2(B)}$, for any $\sigma > 0$, the arclength of $\partial B(x,r) \cap B(x_m, \sigma)$ must be greater than $C \min\{\sigma, \frac{\varepsilon}{\underline{\eta}^2(B)}\}$. Moreover, since $(1-|u|^2)^2 \geq \frac{(1-m)^2}{C}$ whenever $|u| \leq \frac{1+m}{2}$, by choosing $\sigma = \frac{(1-m)^2}{\underline{\eta}^2(B)\gamma}$, we find

$$\begin{aligned}
\frac{1}{2} \int_{\partial B(x,r)} \eta^2 |\nabla |u||^2 + &\eta^4 \frac{(1-|u|^2)^2}{2\varepsilon^2} \\
&\geq \underline{\eta}^2(B)\gamma + \underline{\eta}^2(B)^2 \frac{(1-m)^2}{C\varepsilon^2} \min\left\{ \frac{\varepsilon}{\underline{\eta}^2(B)}, \frac{(1-m)^2}{\underline{\eta}^2(B)\gamma} \right\} \\
&= \underline{\eta}^2(B) \left(\gamma + \frac{(1-m)^2}{C\varepsilon^2} \min\left\{ \varepsilon, \frac{(1-m^2)}{\gamma} \right\} \right).
\end{aligned}$$

If $\varepsilon \leq \frac{(1-m)^2}{\gamma}$, we obtain (2.3.1). Otherwise, we can minimize $\gamma + \frac{K^2}{\gamma}$ with respect to γ, where $K = \frac{(1-m)^2}{C\varepsilon}$. Since $\gamma = K$ is a stationary point and $\gamma + \frac{K^2}{\gamma}$ is convex, we conclude that $2K$ is the minimum, which means $\gamma \geq 2K$. Therefore

$$\frac{1}{2} \int_{\partial B(x,r)} \eta^2 |\nabla |u||^2 + \eta^4 \frac{(1-|u|^2)^2}{2\varepsilon^2} \geq C \underline{\eta}^2(B) \frac{(1-m)^2}{\varepsilon},$$

which means (2.3.1) holds in all cases. □

Step 2 For a compact set $K \subseteq \Omega$ such that $\partial K \cap S_E \neq \emptyset$, we define

$$\deg_E(u, \partial K) := \sum_i \deg(u, S_i),$$

where S_i are the connected components of S_E compactly contained in K. Applying the previous lemma, we obtain the following result.

Lemma 2.3.2. *There exists a (finite) collection of disjoint closed balls* $\{B_i\}_i = \{B(a_i, r_i)\}_i$ *such that*

(1) *For each* i, $r_i \geq \frac{\varepsilon}{\underline{\eta}^2(B_i)}$.

(2) $S_E \cap \Omega_\varepsilon \subseteq \cup_i B_i$.

(3) *There exists a universal constant* $c_1 > 0$ *such that, for each* i, *we have*

$$F_{\varepsilon,\eta}(u, A; , \Omega \cap B_i) \geq c_1 \underline{\eta}^2(B) \frac{r_i}{\varepsilon}.$$

Proof. The proof is a slight modification of the proof of [Jer99, Proposition 3.3]. Indeed, by noting that from [Jer99, Lemma 3.2], we have

$$\int_{S_i} \eta^2 |\nabla u|^2 \geq \underline{\eta}^2(S_i) \int_{S_i} |\nabla u|^2 \geq \frac{\underline{\eta}^2(S_i)}{C} |\deg(u, \partial S_i)|,$$

the proof is exactly as the proof of [Jer99, Lemma 3.3], using of course (2.3.1) instead of the lower bound in [Jer99, Lemma 2.5] and the fact that $\underline{\eta}^2(\Theta_1) \geq \underline{\eta}^2(\Theta_2)$ for any closed sets such that $\Theta_1 \subseteq \Theta_2$. The constant c_1 is the same as the constant c_0 in Jerrard's proof. □

From now on, we closely follow [SS11, Section 5].

Proposition 2.3.2. *For a small enough* $c_2 \in (0, c_1)$, *let*

$$\lambda_\varepsilon(x) = \min\left(\frac{c_2}{\varepsilon}, \frac{\pi}{x} \frac{1}{1 + \frac{x}{2} + \frac{\pi\varepsilon}{c_0 x}}\right).$$

Then, for any closed ball $B = B(a, r)$ *such that* $B \subset \Omega_\varepsilon$, $\partial B \cap S_E = \emptyset$, *and* $\frac{\varepsilon}{\underline{\eta}^2(B)} \leq r \leq \frac{|d|}{2}$, *where* $d = \deg_E(u, \partial B) \neq 0$, *we have*

$$\frac{1}{2} \int_{\partial B} \eta^2 |\nabla_A u|^2 + \eta^4 \frac{(1 - |u|^2)^2}{2\varepsilon^2} + \frac{1}{2} \int_B |\operatorname{curl} A|^2 \geq \underline{\eta}^2(B) \lambda_\varepsilon\left(\frac{r}{|d|}\right). \tag{2.3.2}$$

Moreover, $\Lambda_\varepsilon(s) := \int_0^s \lambda_\varepsilon$ *is increasing, the function* $s \to \frac{\Lambda_\varepsilon(s)}{s}$ *is decreasing and it satisfies*

$$\lim_{s\to 0}\frac{\Lambda_\varepsilon(s)}{s} < \frac{c_1}{\varepsilon}, \qquad \frac{\Lambda_\varepsilon(\varepsilon)}{\varepsilon} > \frac{c_3}{\varepsilon},$$

for some sufficiently small constant c_3. *Finally, for any* $s \in \left(\varepsilon, \frac{1}{2}\right)$ *and some* $C_0 > 0$ *we have*

$$\Lambda_\varepsilon(s) \geq \pi \log\frac{s}{\varepsilon} - C_0.$$

Proof. The proof is almost exactly as the proof of [SS11, Proposition 5.3]. In fact, the functions $\lambda_\varepsilon, \Lambda_\varepsilon$ are the same as in this proof, and since $\eta \leq 1$, we have $|\operatorname{curl} A|^2 \geq \underline{\eta}^2(B)|\operatorname{curl} A|^2$. Hence, we only need to carry around the weight $\underline{\eta}^2(B)$ and mimic the proof of [SS11, Proposition 5.3]. ☐

Step 3 With these estimates at hand, the ball construction procedure of growing and merging balls yields the following result.

Proposition 2.3.3. *For any* $s \in \left(0, \frac{1}{2}\right)$, *there exists a collection of disjoint closed balls* $\mathcal{B}(s)$, *depending only on* u, *such that*

1. $\mathcal{B}(s) \subset \mathcal{B}(t)$ *for* $s < t$ *and the total radius of the collection is continuous with respect to* s.
2. $S_E \subseteq \mathcal{B}(s)$, *for any* s.
3. *For any* $B = B(a,r) \in \mathcal{B}(s)$,

$$F_{\varepsilon,\eta}(u, A; B) \geq \underline{\eta}^2(B) r \frac{\Lambda_\varepsilon(s)}{s}.$$

4. *For any* $B = B(a,r) \in \mathcal{B}(s)$ *such that* $B \subset \Omega_\varepsilon$, *we have* $r \geq s|d_B|$, *where* $d_B = \deg_E(u, \partial B)$.

Proof. The proof follows the process of growing and merging balls described in [Jer99, Proposition 4.1] and [SS11, Proposition 5.4]. Let $\mathcal{B} = \{B_i\}_i = \{B(a_i, r_i)\}$ be the collection given by Lemma 2.3.2. We start by choosing $s_0 < \frac{1}{2}$ small enough so that the balls in $\mathcal{B}$ satisfy items 3 and 4 (item 2 is obviously also satisfied). In particular, for each $B = B(a,r) \in \mathcal{B}$ we have

$$F_{\varepsilon,\eta}(u,A;B) \geq c_1 \underline{\eta}^2(B)\frac{r}{\varepsilon} \geq \underline{\eta}^2(B) r \frac{\Lambda_\varepsilon(s_0)}{s_0}.$$

We construct the collection $\mathcal{B}(s)$ as follows. For $s \leq s_0$, we let $\mathcal{B}(s) = \mathcal{B}$. Then, as s increases, we let the radius of each ball grow so that $r_i = s|d_{B_i}|$. Observe that the bound of item 3 is preserved during the growth process, which follows from (2.3.2) and the fact that $\underline{\eta}^2(B_i(s)) \geq \underline{\eta}^2(B_i(t))$ for $s < t$ (since $B_i(s) \subset B_i(t)$). If at a moment two balls $B_1 = B(a_1, r_1)$ and $B_2 = B(a_2, r_2)$ intersect each other, we merge these balls into a larger ball that contains them with a radius equal to the sum of the radii of the merged balls. This ball can be explicitly written as $B = B\left(\frac{a_1 r_1 + a_2 r_2}{r_1 + r_2}, r_1 + r_2\right)$. The bound of item 3 still holds after the merging process, since we have $d_B = d_{B_1} + d_{B_2}$ and therefore $|d_B| \leq |d_{B_1}| + |d_{B_2}|$, and $\underline{\eta}^2(B) \leq \underline{\eta}^2(B_1) + \underline{\eta}^2(B_2)$. This process of growing and merging continues as long as (2.3.2) can be satisfied, that is, until $s = \frac{1}{2}$. □

With these tools at hand, the proof of Proposition 2.3.1 is exactly as the proof of [SS11, Proposition 2.1]. We only need to carry around the weight $\underline{\eta}^2(B)$ throughout the argument.

Some important remarks to mention are the following.

Remark 2.3.1. *Let us remark that* [SS11, Proposition 2.1] *states that the ball collection covers the set* $\{x \in \Omega_\varepsilon \colon |u(x)| < \frac{1}{2}\}$*, contrary to what we have written here. However, a careful inspection of the proof reveals that the ball collection is obtained by merging with a cover of the set* $\{x \in \Omega_\varepsilon \colon |1 - |u|| \geq \frac{1}{2}\}$ *given by* [SS07, Proposition 4.8]. *This proposition also holds in the inhomogeneous case, since* $b \leq \eta^2 \leq 1$*, which in turn gives* $F_{\varepsilon,\eta}(u,A;\Omega) \leq F_\varepsilon(u,A;\Omega) \leq b^{-1} F_{\varepsilon,\eta}(u,A;\Omega)$*, where* $F_\varepsilon(u,A)$ *is the particular case for* $F_{\varepsilon,\eta}(u,A)$ *with* $\eta \equiv 1$.

Remark 2.3.2. *In the situation where* $d_B \neq 0$ *for some* $B \subset \Omega_\varepsilon$*, a natural choice* *for* $\overline{C}$ *is* $\pi\tilde{D}$*, where* $\tilde{D} := \sum_{B\in\mathcal{B}\cap\Omega_\varepsilon} \underline{\eta}^2(B)|d_B|$*. With this choice, in all cases we have*

$$F_{\varepsilon,\eta}\left(u, A; \Omega \cap \bigcup_{B\in\mathcal{B}} B\right) \geq \pi\tilde{D}\left(\log\frac{r}{\varepsilon\tilde{D}} - C\right). \tag{2.3.3}$$

Notice that this choice is possible since in this case $\bar{C} \geq \pi b > 2b$*. Moreover, if* $d_B = 0$ *for every* $B \subset \Omega_\varepsilon$*, then* (2.3.3) *still holds, since the RHS vanishes. Moreover, under the assumptions of Proposition 2.3.1, we deduce from* (2.3.3) *and* $r > C\varepsilon^{1-\beta}$*, that*

$$\sum_i |d_{B_i}| \leq C\frac{F_{\varepsilon,\rho_\varepsilon}(u,A)}{\beta|\log\varepsilon|}, \tag{2.3.4}$$

where $C > 0$ *is a constant that does not depend on* ε*.*

Remark 2.3.3. *In* [SS11, Proposition 2.1], $\overline{C}$ *must be larger than or equal to* 2*. However, a careful inspection reveals that one can replace* 2 *by any universal constant in* $(0,\pi)$ *and the argument of proof holds exactly the same. Notice that when* $\eta \equiv 1$*,* $\pi\tilde{D} \geq \pi$*, and therefore we need to be able to choose* $\overline{C} \geq \pi$ *in order to obtain* (2.3.3)*. Of course, the condition* $\overline{C} \geq 2$ *makes this choice possible, but the same holds for any constant in* $(0,\pi)$*.*

2.4 Vorticity estimate

We follow the preceding construction with a vorticity estimate for the constructed balls. This estimate is also referred to as the Jacobian estimate, since the measure $\mu(u,A)$ is considered a gauge-invariant version of the Jacobian determinant.

This result gives us an approximation for $\mu(u,A)$ in the dual norm $(C_0^{0,1})^*$ depending on the free energy $F_{\varepsilon,\eta}(u,A)$.

Proposition 2.4.1. *Let* $\mathcal{B} = \{B_i\}_i = \{B(a_i,r_i)\}_i$ *be a finite collection of disjoint* *closed balls and* $\varepsilon > 0$ *such that*

$$\left\{x \in \Omega_\varepsilon \colon ||u(x)| - 1| \geq \frac{1}{2}\right\} \subseteq \bigcup_i B_i, \tag{2.4.1}$$

where $\Omega_\varepsilon := \{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$*. Then, for any* $r = \sum_i r_i \leq 1, \varepsilon \leq 1$*, there exists* $C = C(\Omega) > 0$ *such that*

$$\left\| \mu - 2\pi \sum_i d_{B_i} \delta_{a_i} \right\|_{\left(C_0^{0,1}(\Omega)\right)^*} \leq C \max\{\varepsilon, r\} \left(1 + \frac{M}{b^2}\right), \tag{2.4.2}$$

where $d_{B_i} = \deg(u, \partial B_i)$ *if* $B_i \subset \Omega_\varepsilon$ *and 0 otherwise,* $M = F_{\varepsilon,\eta}(u, A)$*,* $\eta \colon \Omega \to [\sqrt{b}, 1]$*, and* $\left(C_0^{0,1}(\Omega)\right)^*$ *is the dual space of* $C_0^{0,1}(\Omega)$*.*

Proof. First, we state the proof in the particular case $\eta \equiv 1$ and $b = 1$. Once we prove the estimate in this particular case, we deduce (2.4.2) by noting that

$$\int_\Omega |\nabla_A u|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2} \leq \frac{1}{b^2} \int_\Omega \eta^2 |\nabla_A u|^2 + \eta^4 \frac{(1 - |u|^2)^2}{2\varepsilon^2}.$$

We denote the free energy in this case by $F_\varepsilon(u, A)$ and follow the proof of [SS07, Theorem 6.1]. We let $\chi \colon \mathbb{R}^+ \to \mathbb{R}^+$ be the continuous piece-wise affine function, defined via

$$\chi(t) = \begin{cases} 2t & \text{if } t \in \left[0, \frac{1}{2}\right]. \\ 1 & \text{if } t \in \left[\frac{1}{2}, \frac{3}{2}\right]. \\ 1 + 2\left(t - \frac{3}{2}\right) & \text{if } t \in \left[\frac{3}{2}, 2\right] \\ t & \text{if } t \in [2, \infty). \end{cases}$$

This function is such that, for any $t > 0$, $\chi(t) \leq 2t$, $\chi'(t) \leq 2$, $|\chi(t) - t| \leq |1 - t|$, and

$$|\chi(t)^2 - t^2| = |\chi(t) + t||\chi(t) - t| \leq 3t|1 - t|.$$

We now define $\rho = |u|$, $\tilde{u} = \frac{\chi(\rho)}{\rho} u$, and $\tilde{\mu} = \mu(\tilde{u}, A) = \operatorname{curl}(\langle i\tilde{u}, \nabla_A \tilde{u}\rangle + A)$. Note that $|\tilde{u}| = 1$ when $||u| - 1| < 1/2$ (in particular, in $\Omega_\varepsilon \setminus \cup_i B_i$). We proceed in several steps.

Step 1 *(Closeness of μ and $\tilde{\mu}$).* We claim that there exists a universal constant $C > 0$ such that

$$\|\mu - \tilde{\mu}\|_{(C_0^{0,1})^*(\Omega)} \leq C\varepsilon M. \tag{2.4.3}$$

In fact, let $\zeta \in C_0^{0,1}(\Omega)$ such that $\|\zeta\|_{C_0^{0,1}(\Omega)} = \|\nabla\zeta\|_{L^\infty(\Omega,\mathbb{R}^2)} \leq 1$. Integration by parts yields

$$\left|\int_\Omega (\mu - \tilde{\mu})\zeta\right| = \left|\int_\Omega (\langle iu, \nabla_A u\rangle - \langle i\tilde{u}, \nabla_A \tilde{u}\rangle) \cdot \nabla^\perp \zeta\right|.$$

Arguing as in the proof of [SS07, Lemma 6.2], we find

$$|\langle iu, \nabla_A u\rangle - \langle i\tilde{u}, \nabla_A \tilde{u}\rangle| \leq 3|1-\rho||\nabla_A u|.$$

Therefore, using the definition of $F_\varepsilon(u, A)$ and the general inequality $|1 - |x|| \leq |1 - |x|^2|$

$$\begin{aligned}\left|\int_\Omega (\mu - \tilde{\mu})\zeta\right| &\leq C \|1-\rho\|_{L^2(\Omega)} \|\nabla_A u\|_{L^2(\Omega,\mathbb{C}^2)} \|\nabla\zeta\|_{L^\infty(\Omega,\mathbb{R}^2)} \\ &\leq C \left\|1-\rho^2\right\|_{L^2(\Omega)} M^{\frac{1}{2}} \|\zeta\|_{C_0^{0,1}(\Omega)} \\ &\leq C\varepsilon M.\end{aligned}$$

Step 2 *(The advantage of working with $\tilde{\mu}$).* A direct computation shows that

$$\tilde{\mu} = 0 \quad \text{in } \Omega_\varepsilon \setminus \cup_i B_i \quad \text{and} \quad \int_{B_i} \tilde{\mu} = 2\pi \deg(u, \partial B_i) \quad \text{if } B_i \subset \Omega_\varepsilon. \tag{2.4.4}$$

Since we have that $\tilde{\mu} = 0$ whenever $|\tilde{u}| = 1$, i.e. when $||u| - 1| \leq 1/2$, using (2.4.1), we deduce that

$$\int_\Omega \tilde{\mu}\zeta = \sum_i \int_{B_i \cap \Omega} \tilde{\mu}\zeta + \int_U \tilde{\mu}\zeta, \tag{2.4.5}$$

where $U = \{x \in \Omega \setminus \Omega_\varepsilon \colon ||u| - 1| \geq 1/2\}$.

To estimate the vorticity in U, we need an estimate of $|\zeta|$ near the boundary. Since $\zeta = 0$ on $\partial\Omega$, for $x \in \Omega \setminus \Omega_\varepsilon$ we have

$$|\zeta(x)| \leq \|\nabla\zeta\|_{L^\infty(\Omega,\mathbb{R}^2)}\, \varepsilon \leq C\, \|\zeta\|_{C_0^{0,1}(\Omega)}\, \varepsilon.$$

Hence

$$\left|\int_U \tilde{\mu}\zeta\right| \leq C\varepsilon\, \|\zeta\|_{C_0^{0,1}(\Omega)} \int_U |\tilde{\mu}|. \tag{2.4.6}$$

Analogously, if B_i is such that $B_i \cap \Omega \setminus \Omega_\varepsilon \neq \emptyset$, we have that

$$|\zeta(x)| \leq \|\zeta\|_{C_0^{0,1}(\Omega)}\, (\varepsilon + 2r_i) \leq C\, \|\zeta\|_{C_0^{0,1}(\Omega)}\, (\varepsilon + r_i),$$

for any $x \in B_i \cap \Omega$. Therefore, for any such ball B_i, we have

$$\left|\int_{B_i\cap\Omega} \tilde{\mu}\zeta\right| \leq C(\varepsilon + r_i)\, \|\zeta\|_{C_0^{0,1}(\Omega)} \int_{B_i\cap\Omega} |\tilde{\mu}|, \tag{2.4.7}$$

In particular, by combining (2.4.5) with (2.4.6) and (2.4.7), we deduce that

$$\left|\int_\Omega \tilde{\mu}\zeta - \sum_{j\in J} \int_{B_j} \tilde{\mu}\zeta\right| \leq C(\max\{\varepsilon, r\})\, \|\zeta\|_{C^{0,1}(\Omega)} \int_\Omega |\tilde{\mu}|, \tag{2.4.8}$$

where J is the set of indices j such that $B_j \subseteq \Omega_\varepsilon$.

Since $|\tilde{u}| = 1$ on ∂B_j for each $j \in J$, in view of (2.4.4), we have that $\int_{B_j} \tilde{\mu} = 2\pi d_j$. Thus

$$\begin{aligned}\left|\int_{B_j} \tilde{\mu}\zeta - 2\pi d_j \zeta(a_j)\right| &= \left|\int_{B_j} \tilde{\mu}(\zeta - \zeta(a_j))\right| \\ &\leq C r_j^\beta \|\zeta\|_{C_0^{0,1}(B_j)} \int_{B_j} |\tilde{\mu}| \leq C r_j\, \|\zeta\|_{C_0^{0,1}(\Omega)} \int_{B_j} |\tilde{\mu}|. \end{aligned}\tag{2.4.9}$$

Finally, we need a bound for $\|\tilde{\mu}\|_{L^1(\Omega)}$. An explicit calculation shows that $\tilde{\mu} =$

$2(\partial_1 - iA_1)\tilde{u} \times (\partial_2 - iA_2)\tilde{u} + \operatorname{curl} A$. In particular

$$\int_\Omega |\tilde{\mu}| \leq \int_\Omega 2|\nabla_A \tilde{u}|^2 + |\operatorname{curl} A|.$$

Using the Cauchy-Schwarz inequality and the fact that

$$|\nabla_A u|^2 = \chi(\rho)^2 \left|\nabla \frac{u}{\rho} - A\right|^2 + \chi'(\rho)^2 |\nabla\rho|^2 \leq 4\rho^2 \left|\nabla \frac{u}{\rho} - A\right|^2 + 4|\nabla\rho|^2 = 4|\nabla_A u|^2,$$

we deduce

$$\int_\Omega |\tilde{\mu}| \leq CM.$$

By inserting this into (2.4.8) and (2.4.9), we deduce that

$$\left|\int_\Omega \tilde{\mu}\zeta - 2\pi \sum_i d_i \zeta(a_i)\right| \leq C(\max\{\varepsilon, r\}) \, \|\zeta\|_{C_0^{0,1}(\Omega)} \, M \tag{2.4.10}$$

Step 3 *(Conclusion)* Finally, by combining (2.4.3) and (2.4.10), we deduce that

$$\begin{aligned}\left|\int_\Omega \mu\zeta - 2\pi \sum_i d_i \zeta(a_i)\right| &\leq \left|\int_\Omega (\mu - \tilde{\mu})\zeta\right| + \left|\int_\Omega \tilde{\mu}\zeta - 2\pi \sum_i d_i \zeta(a_i)\right| \\ &\leq C\varepsilon M \, \|\zeta\|_{C_0^{0,1}(\Omega)} + C(\max\{\varepsilon, r\}) M \, \|\zeta\|_{C_0^{0,1}(\Omega)} \\ &\leq C \max\{\varepsilon, r\} M \, \|\zeta\|_{C_0^{0,1}(\Omega)} .\end{aligned}$$

This concludes the proof.

□

Chapter 3

Estimation of the first critical field

We have developed the necessary tools to start proving our main results. In this chapter we prove Theorem 1 and Theorem 2, related to the characterization of the first critical field.

3.1 Lower bound

We begin with the proof of Theorem 1, which characterizes minimizers below $H_{c_1}^{\varepsilon}$. First, we need a tool to characterize strongly vortexless minimizers. This is referred to in the literature as a *clearing out* result.

Proposition 3.1.1. *Let* $(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ *be a configuration such that*

$$\|\nabla |u|\|_{L^\infty(\Omega,\mathbb{R}^2)} \leq \frac{C}{\varepsilon} \quad \text{and} \quad F_{\varepsilon,\rho_\varepsilon}(u, A) = o(1).$$

Then $\|1 - |u|\|_{L^\infty(\Omega)} = o(1)$.

Proof. The proof follows [BBH94, Theorem III.3]. Suppose by contradiction that we have some $c > 0$ such that $||u(x_0)| - 1| > c$ for some $x_0 \in \Omega$. Then, for every x in a closed ball $B = B(x_0, r_\varepsilon) \subset \Omega$, where $r_\varepsilon = o(1)$ will be chosen later, we have

$$||u(x)| - |u(x_0)|| \leq \|\nabla |u|\|_{L^\infty(\Omega)} |x - x_0| \leq \frac{r_\varepsilon}{\varepsilon}.$$

Thus

$$||1 - |u(x)|| - |1 - |u(x_0)||| \leq \frac{r_\varepsilon}{\varepsilon}.$$

From this we deduce that

$$|1 - |u(x)|| \geq |1 - |u(x_0)|| - \frac{r_\varepsilon}{\varepsilon} > c - \frac{r_\varepsilon}{\varepsilon}.$$

If we choose $r_\varepsilon = \frac{c}{2}\varepsilon$ we have

$$|1 - |u(x)|^2| \geq |1 - |u(x)|| \geq \frac{c}{2}.$$

Squaring both sides and multiplying by ρ_ε^4 yields

$$\rho_\varepsilon^4 (1 - |u|^2)^2 \geq \rho_\varepsilon^4 \frac{c^2}{4}.$$

Integrating over B and using that $\rho_\varepsilon \geq b$, we obtain

$$\int_B \rho_\varepsilon^4 (1 - |u|^2)^2 \geq \pi r_\varepsilon^2 b^4 \frac{c^2}{4} = \pi b^4 \frac{c^4}{16} \varepsilon^2.$$

Finally, we divide by $4\varepsilon^2$, to get

$$\pi b^4 \frac{c^4}{64} \leq F_{\varepsilon, \rho_\varepsilon}(u, A) = o(1),$$

which is a contradiction since the left hand side is a positive constant that does not depend on ε. $\square$

Proof of Theorem 1. Note that all the results in this theorem are gauge-invariant. Therefore, we may assume without loss of generality that $(\mathbf{u}, \mathbf{A})$ is in the Coulomb gauge, that is, $\mathbf{A}$ satisfies (0.0.2). Also, in this proof, $C > 0$ denotes a constant independent of ε that might change from line to line.

Step 1 *(Proving that $F_{\varepsilon,\rho_\varepsilon}(u,A) \le Ch_{\rm ex}^2$ for some $C>0$ independent of ε).* Since $(\mathbf{u},\mathbf{A})$ is a global minimizer, we have $GL_\varepsilon(\mathbf{u},\mathbf{A}) \le GL_\varepsilon(\rho_\varepsilon, h_{\rm ex}A_\varepsilon^0)$. By integrating by parts the third term in the RHS of (0.0.5) (recall that $\xi_\varepsilon = 0$ on $\partial\Omega$) and inserting the previous inequality, we deduce that

$$\begin{aligned} F_{\varepsilon,\rho_\varepsilon}(u,A) &= (GL_\varepsilon(\mathbf{u},\mathbf{A}) - GL_\varepsilon(\rho_\varepsilon, h_{\rm ex}A_\varepsilon^0)) + h_{\rm ex}\int_\Omega \mu(u,A)\xi_\varepsilon - R_0 \\ &\le h_{\rm ex}\int_\Omega (\langle iu, \nabla_A u\rangle + A)\cdot\nabla^\perp\xi_\varepsilon + |R_0|. \end{aligned}$$

By using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} F_{\varepsilon,\rho_\varepsilon}(u,A) \le h_{\rm ex}(&\|u\|_{L^2(\Omega,\mathbb{C})}\|\nabla_A u\|_{L^2(\Omega,\mathbb{C}^2)}\|\nabla\xi_\varepsilon\|_{L^\infty(\Omega,\mathbb{R}^2)} + \\ &\|A\|_{L^2(\Omega,\mathbb{R}^2)}\|\nabla\xi_\varepsilon\|_{L^2(\Omega,\mathbb{R}^2)}) + |R_0|. \end{aligned} \tag{3.1.1}$$

Since $(\mathbf{u},\mathbf{A})$ solves (1.2.1), we have (1.2.3). Combining this with $\|\nabla_A u\|_{L^2(\Omega,\mathbb{C}^2)} \le F_{\varepsilon,\rho_\varepsilon}(u,A)^{\frac12}$ and (2.1.3), yields that

$$\|u\|_{L^2(\Omega,\mathbb{C})}\|\nabla_A u\|_{L^2(\Omega,\mathbb{C}^2)}\|\nabla\xi_\varepsilon\|_{L^\infty(\Omega,\mathbb{R}^2)} \le CF_{\varepsilon,\rho_\varepsilon}(u,A)^{\frac12}. \tag{3.1.2}$$

Moreover, since both $\mathbf{A}$ and A_ε^0 are in the Coulomb gauge, we deduce that A is as well, that is, it satisfies (0.0.2). Hence, using (1.4.1), we get that

$$\|A\|_{L^2(\Omega,\mathbb{R}^2)} \le \|A\|_{H^1(\Omega,\mathbb{R}^2)} \le C\|\mathrm{curl}\,A\|_{L^2(\Omega)} \le CF_{\varepsilon,\rho_\varepsilon}(u,A)^{\frac12},$$

which combined with (2.1.2) yields

$$\|A\|_{L^2(\Omega,\mathbb{R}^2)}\|\nabla\xi_\varepsilon\|_{L^2(\Omega,\mathbb{R}^2)} \le CF_{\varepsilon,\rho_\varepsilon}(u,A)^{\frac12}. \tag{3.1.3}$$

Meanwhile, for R_0

$$|R_0| \leq Ch_{\text{ex}}^2 \left\| |u|^2 - 1 \right\|_{L^2(\Omega)} \leq Ch_{\text{ex}}^2 \varepsilon F_{\varepsilon,\rho_\varepsilon}(u,A)^{\frac{1}{2}}. \tag{3.1.4}$$

Finally, by combining (3.1.1) with (3.1.2), (3.1.3), and (3.1.4), we obtain

$$\begin{aligned} F_{\varepsilon,\rho_\varepsilon}(u,A) &\leq C\left(h_{\text{ex}} F_{\varepsilon,\rho_\varepsilon}(u,A)^{\frac{1}{2}} + h_{\text{ex}}^2 \varepsilon F_{\varepsilon,\rho_\varepsilon}(u,A)^{\frac{1}{2}}\right) \\ &\leq C\left(h_{\text{ex}}^2 + h_{\text{ex}} F_{\varepsilon,\rho_\varepsilon}(u,A)^{\frac{1}{2}}\right). \end{aligned}$$

It follows that

$$F_{\varepsilon,\rho_\varepsilon}(u,A) \leq Ch_{\text{ex}}^2. \tag{3.1.5}$$

Step 2 *(Estimates for $F_{\varepsilon,\rho_\varepsilon}(u,A)$ and $\|\mu(u,A)\|_{\left(C_0^{0,1}\right)^*}$. Proof of weak vortexlessness.* From (3.1.5) and $h_{\text{ex}} = O(|\log\varepsilon|)$, we have

$$F_{\varepsilon,\rho_\varepsilon}(u,A) \leq C|\log\varepsilon|^2. \tag{3.1.6}$$

We can therefore apply Proposition 2.3.1 considering Remark 2.3.2 to obtain a finite collection of disjoint balls $\{B_i\}_i = \{B(a_i,r_i)\}_i$ with $\sum_i r_i \leq r = |\log\varepsilon|^{-\beta}$, where $\beta > 0$ will be chosen later, containing $\left\{||u|-1| \geq \frac{1}{2}\right\}$ such that

$$\begin{aligned} F_{\varepsilon,\rho_\varepsilon}(u,A) &\geq \pi \sum_i \rho_\varepsilon^2(\underline{a_i})|d_{B_i}|\left(\log\frac{|\log\varepsilon|^{-\beta}}{\tilde{D}\varepsilon} - C\right) \\ &= \pi \sum_i \rho_\varepsilon^2(\underline{a_i})|d_{B_i}|(|\log\varepsilon| - \beta\log|\log\varepsilon| - \log\tilde{D} - C) \\ &\overset{(2.3.4)}{\geq} \pi \sum_i \rho_\varepsilon^2(\underline{a_i})|d_{B_i}|\left(|\log\varepsilon| - \beta\log|\log\varepsilon| - C\log\frac{F_{\varepsilon,\rho_\varepsilon}(u,A)}{|\log\varepsilon|} - C\right) \\ &\overset{(3.1.6)}{\geq} \pi \sum_i \rho_\varepsilon^2(\underline{a_i})|d_{B_i}|(|\log\varepsilon| - C\log|\log\varepsilon|), \end{aligned} \tag{3.1.7}$$

where $\underline{a_i} \in B_i$ is such that $\rho_\varepsilon^2(\underline{a_i}) = \min_{B_i} \rho_\varepsilon^2$.

On the other hand, applying Proposition 2.4.1, we have

$$\begin{aligned}\left|h_{\mathrm{ex}}\int_\Omega \mu(u,A)\xi_\varepsilon\right| &\overset{(2.4.2)}{\leq} 2\pi h_{\mathrm{ex}}\sum_i |d_i|\xi_\varepsilon(a_i) + Ch_{\mathrm{ex}}r(1+F_{\varepsilon,\rho_\varepsilon}(u,A))\left\|\nabla\xi_\varepsilon\right\|_{L^\infty(\Omega)}\\ &\overset{(2.1.3)\&(3.1.6)}{\leq} 2\pi h_{\mathrm{ex}}\sum_i |d_i|\xi_\varepsilon(a_i) + O(|\log\varepsilon|^{3-\beta})\end{aligned}$$

It also follows from (2.1.3) that

$$|\xi_\varepsilon(a_i)-\xi_\varepsilon(\underline{a_i})| \leq \left\|\nabla\xi_\varepsilon\right\|_{L^\infty(\Omega)}|a_i-\underline{a_i}| \leq Cr_i \leq C|\log\varepsilon|^{-\beta}.$$

Therefore, we have

$$\begin{aligned}\left|h_{\mathrm{ex}}\int_\Omega \mu(u,A)\xi_\varepsilon\right| &\leq 2\pi h_{\mathrm{ex}}\sum_i |d_i|\xi_\varepsilon(\underline{a_i}) + C|\log\varepsilon|^{-\beta}h_{\mathrm{ex}}\sum_i|d_i| + O(|\log\varepsilon|^{3-\beta})\\ &\overset{(2.3.4)}{\leq} 2\pi h_{\mathrm{ex}}\sum_i |d_i|\xi_\varepsilon(\underline{a_i}) + C|\log\varepsilon|^{-\beta}h_{\mathrm{ex}}\frac{F_{\varepsilon,\rho_\varepsilon}(u,A)}{|\log\varepsilon|} + O(|\log\varepsilon|^{3-\beta})\\ &\overset{(3.1.6)}{\leq} 2\pi h_{\mathrm{ex}}\sum_i |d_i|\xi_\varepsilon(\underline{a_i}) + O(|\log\varepsilon|^{3-\beta}).\end{aligned}$$

Thus, by choosing $\beta>3$, we get

$$\left|h_{\mathrm{ex}}\int_\Omega \mu(u,A)\xi_\varepsilon\right| \leq 2\pi h_{\mathrm{ex}}\sum_i |d_i|\xi_\varepsilon(\underline{a_i}) + o(1). \tag{3.1.8}$$

Combining (3.1.7) and (3.1.8), we deduce that

$$\begin{aligned}&F_{\varepsilon,\rho_\varepsilon}(u,A) - h_{\mathrm{ex}}\int_\Omega \mu(u,A)\xi_\varepsilon \geq\\ &\qquad\pi\sum_i \rho_\varepsilon^2(\underline{a_i})|d_i|\left(|\log\varepsilon| - C\log|\log\varepsilon| - 2h_{\mathrm{ex}}\psi_\varepsilon(\underline{a_i})\right) + o(1),\end{aligned}\tag{3.1.9}$$

where we recall that ψ_ε was defined as $\frac{\xi_\varepsilon}{\rho_\varepsilon^2}$. Therefore, since $h_{\mathrm{ex}} \leq H_{c_1}^\varepsilon - K_0\log|\log\varepsilon|$, we have

$$\begin{aligned}|\log\varepsilon| - C\log|\log\varepsilon| - 2h_{\text{ex}}\psi_\varepsilon(\underline{a_i}) &\geq |\log\varepsilon| - C\log|\log\varepsilon| - 2h_{\text{ex}}\max_\Omega \psi_\varepsilon \\ &\geq \log|\log\varepsilon|\left(2\max_\Omega \psi_\varepsilon K_0 - C\right).\end{aligned}$$

Proposition 2.1.2 allows us to choose $K_0 > 0$, independently of ε, so that

$$2\max_\Omega \psi_\varepsilon K_0 - C > 1.$$

Inserting this into (3.1.9), we find

$$F_{\varepsilon,\rho_\varepsilon}(u,A) - h_{\text{ex}}\int_\Omega \mu(u,A)\xi_\varepsilon \geq \pi\sum_i \rho_\varepsilon^2(\underline{a_i})|d_i|\log|\log\varepsilon| + o(1). \tag{3.1.10}$$

Moreover, since $GL_\varepsilon(\mathbf{u},\mathbf{A}) \leq GL_\varepsilon(\rho_\varepsilon, h_{\text{ex}}A_\varepsilon^0)$, it follows from (0.0.5) that

$$F_{\varepsilon,\rho_\varepsilon}(u,A) - h_{\text{ex}}\int_\Omega \mu(u,A)\xi_\varepsilon + R_0 \leq 0.$$

In addition,

$$|R_0| \leq Ch_{\text{ex}}^2\varepsilon F_{\varepsilon,\rho_\varepsilon}(u,A)^{\frac{1}{2}} \overset{(3.1.6)}{\leq} C\varepsilon|\log\varepsilon|^3 = o(1). \tag{3.1.11}$$

Hence,

$$F_{\varepsilon,\rho_\varepsilon}(u,A) - h_{\text{ex}}\int_\Omega \mu(u,A)\xi_\varepsilon \leq o(1). \tag{3.1.12}$$

By combining (3.1.10) and (3.1.12), using also $\rho_\varepsilon^2 \geq b$, we deduce that $\sum_i |d_i| = 0$ and thus $d_i = 0$ for all i. In turn, from (2.4.2) it follows that

$$h_{\text{ex}}\|\mu(u,A)\|_{\left(C_0^{0,1}(\Omega)\right)^*} \leq Ch_{\text{ex}}r(1 + F_{\varepsilon,\rho_\varepsilon}(u,A)) \leq C|\log\varepsilon|^{3-\beta} = o(1). \tag{3.1.13}$$

Therefore, we deduce that $(\mathbf{u},\mathbf{A})$ is weakly vortexless.

Step 3 *(Clearing out. Proof of strong vortexlessness and Meissner energy approxi-*

mation.) Since $(\mathbf{u}, \mathbf{A})$ is in the Coulomb gauge, we have

$$\begin{aligned}\|A\|_{L^\infty(\Omega,\mathbb{R}^2)} &\overset{(1.4.3)}{\leq} C(E_\varepsilon(\rho_\varepsilon) + F_{\varepsilon,\rho_\varepsilon}(u, A))^{\frac{1}{2}} \\ &\overset{(3.1.6)}{\leq} C\left(\frac{1}{\varepsilon^2} + |\log \varepsilon|^2\right)^{\frac{1}{2}} \\ &\leq \frac{C}{\varepsilon}.\end{aligned}$$

Then, it follows from (1.4.4) that

$$\|\nabla|u|\|_{L^\infty(\Omega,\mathbb{R}^2)} \leq \|\nabla u\|_{L^\infty(\Omega,\mathbb{C}^2)} \leq \frac{C}{\varepsilon}.$$

On the other hand, by combining (3.1.12) with (3.1.13), we find

$$F_{\varepsilon,\rho_\varepsilon}(u, A) \leq h_{\mathrm{ex}} \int_\Omega \mu(u, A)\xi_\varepsilon + o(1) \overset{(2.1.3)\&(3.1.13)}{=} o(1). \tag{3.1.14}$$

Hence, Proposition 3.1.1 yields strong vortexlessness for $(\mathbf{u}, \mathbf{A})$.

Finally, we can directly deduce the Meissner energy approximation result.

$$\begin{aligned}GL_\varepsilon(\mathbf{u}, \mathbf{A}) &= GL_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}} A^0_\varepsilon) + F_{\varepsilon,\rho_\varepsilon}(u, A) - h_{\mathrm{ex}} \int_\Omega \mu(u, A)\xi_\varepsilon + R_0 \\ &\overset{(3.1.11)\&(3.1.13)\&(3.1.14)}{=} GL_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}} A^0_\varepsilon) + o(1)\end{aligned}$$

Thus, we have concluded the proof of Theorem 1.

□

3.2 Upper bound

In this proof, we will construct a configuration of the form $(\mathbf{u}, \mathbf{A}) = (\rho_\varepsilon u, 0 + h_{\mathrm{ex}} A_\varepsilon^0)$, with a vortex of degree 1 centered at $x_\varepsilon^0 \in \Omega$, where x_ε^0 is such that

$$\psi_\varepsilon(x_\varepsilon^0) = \max_\Omega \psi_\varepsilon.$$

We will prove that the energy of this particular configuration is much lower than the energy of the Meissner configuration, which in turn guarantees that global minimizers of GL_ε in this regime have vortices.

Proof of Theorem 2. We follow several steps for the proof.

Step 1 *(Constructing the configuration).* Let Φ be a multiple of the fundamental solution of the Laplace's equation centered at x_ε^0, that is,

$$\Phi(x) = \log \frac{1}{|x - x_\varepsilon^0|}.$$

We begin by constructing a phase φ in $\Omega \setminus \{x_\varepsilon^0\}$ as follows. Let Θ be the phase of

$$\frac{z - x_\varepsilon^0}{|z - x_\varepsilon^0|}.$$

Since

$$-\Delta\Phi = 2\pi\delta_{x_\varepsilon^0} = \operatorname{curl} \nabla\Theta \quad \text{in } \Omega,$$

we have that, in the sense of distributions,

$$\operatorname{curl}(-\nabla^\perp\Phi - \nabla\Theta) = 0 \quad \text{in } \Omega.$$

Therefore, there exists g such that $\nabla g = -\nabla^\perp\Phi - \nabla\Theta$. We let $\varphi = \Theta + g$. Observe that φ is well defined modulo 2π in $\Omega \setminus \{x_\varepsilon^0\}$ and satisfies the following relation

$$\nabla\varphi = -\nabla^\perp\Phi. \tag{3.2.1}$$

Let $r_\varepsilon = |\log \varepsilon|^{-M}$, where $M > 0$ will be chosen later on, and consider the ball $B_\varepsilon = B(x_\varepsilon^0, r_\varepsilon) \subset \Omega$. Notice that this condition holds for any ε sufficiently small in view of Remark 2.1.1.

We can now define u. For $x \in \Omega \setminus B_\varepsilon$, we let $u(x) = e^{i\varphi(x)}$ and, for $x \in B_\varepsilon$, we define

$$u(x) = \frac{1}{f(R_\varepsilon)} f\left(\frac{|x - x_\varepsilon^0|}{\varepsilon}\right) e^{i\varphi(x)},$$

where R_ε is such that $r_\varepsilon = \varepsilon R_\varepsilon$ and $f \colon \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $f(0) = 0$, $f(r) \to 1$ as $r \to \infty$ and satisfies the following asymptotic estimate

$$\frac{1}{2}\int_0^R \left(f'(r)^2 + \frac{f(r)^2}{r^2} + \frac{(1 - f(r)^2)^2}{2}\right) 2\pi r dr = \pi \log R + O(1) \quad \text{as } R \to \infty. \tag{3.2.2}$$

The function f is the modulus of what is referred to as *the degree-one radial solution* [SS07, Definition 3.6], and its existence and properties are given by [SS07, Proposition 3.11].

Step 2 *(Estimating the energy inside B_ε).* Let $k_\varepsilon = \sup_{x \in B_\varepsilon} |\rho_\varepsilon^2(x) - \rho_\varepsilon^2(x_\varepsilon^0)|$. Using that $\rho_\varepsilon^2 \leq 1$, We have

$$\begin{aligned} F_{\varepsilon, \rho_\varepsilon, B_\varepsilon}(u, 0) = \frac{1}{2}\int_{B_\varepsilon} \rho_\varepsilon^2 |\nabla u|^2 + \rho_\varepsilon^4 \frac{(1 - |u|^2)^2}{2\varepsilon^2} &\leq \frac{1}{2}\int_{B_\varepsilon} \rho_\varepsilon^2 \left(|\nabla u|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2}\right) \\ &\leq \frac{1}{2}(\rho_\varepsilon^2(x_\varepsilon^0) + k_\varepsilon) \int_{B_\varepsilon} |\nabla u|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2}. \end{aligned}$$

We now estimate the integral that appears in the RHS of the last inequality. Since $|\nabla u|^2 = |\nabla |u||^2 + |u|^2 |\nabla \varphi|^2$, it follows by letting $r = \frac{|x - x_\varepsilon^0|}{\varepsilon}$ and performing a direct calculation that

$$\frac{1}{2}\int_{B_\varepsilon} |\nabla u|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2} = \frac{1}{2}\int_{B_\varepsilon} \left(\frac{f'(r)^2}{\varepsilon^2 f(R_\varepsilon)^2} + \frac{f(r)^2}{f(R_\varepsilon)^2}|\nabla \Phi(x)|^2 + \frac{1}{2\varepsilon^2}\left(1 - \frac{f(r)^2}{f(R_\varepsilon)^2}\right)^2\right) dx.$$

Note that $|\nabla \Phi(x)| = \frac{1}{|x - x_\varepsilon^0|} = \frac{1}{\varepsilon r}$. By changing the variable of integration to r, we obtain

$$\frac{1}{2}\int_{B_\varepsilon} |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} = \frac{1}{2}\int_0^{R_\varepsilon} \left(\frac{f'(r)^2}{f(R_\varepsilon)^2} + \frac{f(r)^2}{f(R_\varepsilon)^2}\frac{1}{r^2} + \frac{1}{2}\left(1 - \frac{f(r)^2}{f(R_\varepsilon)^2}\right)^2 \right) 2\pi r dr.$$

Since $R_\varepsilon \to \infty$ as $\varepsilon \to 0$, we have $f(R_\varepsilon) \to 1$ as $\varepsilon \to 0$. Therefore, it follows from (3.2.2) that

$$\begin{aligned} \frac{1}{2}\int_{B_\varepsilon} \rho_\varepsilon^2 |\nabla u|^2 + \rho_\varepsilon^4 \frac{(1-|u|^2)^2}{2\varepsilon^2} &\leq (\rho_\varepsilon^2(x_\varepsilon^0) + k_\varepsilon)(\pi \log R_\varepsilon + O(1)) \\ &= (\rho_\varepsilon^2(x_\varepsilon^0) + k_\varepsilon)(\pi \log r_\varepsilon - \pi \log \varepsilon + O(1)) \\ &= (\rho_\varepsilon^2(x_\varepsilon^0) + k_\varepsilon)(\pi |\log \varepsilon| - \pi M \log|\log \varepsilon|). \end{aligned}$$

From the hypothesis on ρ_ε we have $k_\varepsilon \leq [\rho_\varepsilon^2]_{C^{0,\alpha}(\Omega)} r_\varepsilon^\alpha \leq |\log \varepsilon|^{m-\alpha M}$. Therefore, by choosing a sufficiently large M, we have

$$\frac{1}{2}\int_{B_\varepsilon} \rho_\varepsilon^2 |\nabla u|^2 + \rho_\varepsilon^4 \frac{(1-|u|^2)^2}{2\varepsilon^2} \leq \rho_\varepsilon^2(x_\varepsilon^0)\left(\pi|\log\varepsilon| - \pi M \log|\log\varepsilon|\right).$$

Step 3 *(Estimating the energy outside B_ε).* Let $C(\Omega) = \operatorname{diam}(\Omega)$. Since $|u| = 1$ outside B_ε, we have $\nabla |u| = 0$ and thus

$$\begin{aligned} \int_{\Omega \setminus B_\varepsilon} |\nabla u|^2 &= \int_{\Omega \setminus B_\varepsilon} |\nabla |u||^2 + |u|^2 |\nabla \varphi|^2 \\ &\overset{(3.2.1)}{=} \int_{\Omega \setminus B_\varepsilon} |\nabla \Phi|^2. \end{aligned}$$

Therefore, using once again that $\rho_\varepsilon^2 \leq 1$, we have

$$\begin{aligned}
F_{\varepsilon,\rho_\varepsilon,\Omega\setminus B_\varepsilon}(u,0) = \frac{1}{2}\int_{\Omega\setminus B_\varepsilon} \rho_\varepsilon^2|\nabla u|^2 + \rho_\varepsilon^4\frac{(1-|u|^2)^2}{2\varepsilon^2} &\leq \frac{1}{2}\int_{\Omega\setminus B_\varepsilon} |\nabla\Phi|^2 \\
&= \frac{1}{2}\int_{\Omega\setminus B_\varepsilon} \frac{1}{|x-x_\varepsilon^0|^2}dx \\
&\leq \frac{1}{2}\int_{r_\varepsilon}^{C(\Omega)} \frac{1}{r^2}2\pi r dr \\
&= -\pi\log r_\varepsilon + O(1) \\
&= \pi M\log|\log\varepsilon| + O(1).
\end{aligned}$$

Hence, by combining the estimates obtained in **Step 2** and **Step 3**, we obtain the following upper bound for the free energy

$$F_{\varepsilon,\rho_\varepsilon}(u,0) \leq \pi\left(\rho_\varepsilon^2(x_\varepsilon^0)|\log\varepsilon| + (1-\rho_\varepsilon^2(x_\varepsilon^0))M\log|\log\varepsilon|\right) + O(1). \tag{3.2.3}$$

Step 4 *(Computation of the full Ginzburg–Landau energy of the constructed configuration).* Consider the configuration $(\mathbf{u},\mathbf{A}) = (\rho_\varepsilon u, 0 - h_{\mathrm{ex}}A_\varepsilon^0)$. We split $GL_\varepsilon(\mathbf{u},\mathbf{A})$ using (0.0.5), to obtain

$$\begin{aligned}
GL_\varepsilon(\mathbf{u},\mathbf{A}) - GL_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}}A_\varepsilon^0) &= F_{\varepsilon,\rho_\varepsilon}(u,0) - h_{\mathrm{ex}}\int_\Omega \mu(u,0)\xi_\varepsilon + R_0 \\
&\overset{(3.2.3)}{\leq} \pi\rho_\varepsilon^2(x_\varepsilon^0)|\log\varepsilon| + \left(1-\rho_\varepsilon^2(x_\varepsilon^0)\right)\pi M\log|\log\varepsilon| \\
&\qquad - h_{\mathrm{ex}}\int_\Omega \mu(u,0)\xi_\varepsilon + R_0.
\end{aligned} \tag{3.2.4}$$

Here is where the hypothesis on h_{ex}

$$H_{c_1}^\varepsilon + K^0\log|\log\varepsilon| \leq h_{\mathrm{ex}} \leq |\log\varepsilon|^N \tag{3.2.5}$$

plays its role. First, we have

$$|R_0| \overset{(3.1.4)}{\leq} Ch_{\text{ex}}^2 \varepsilon F_{\varepsilon,\rho_\varepsilon}(u,0)^{\frac{1}{2}} \overset{(3.2.3)\&(3.2.5)}{\leq} C\varepsilon|\log\varepsilon|^{\frac{1}{2}+2N} = o(1). \tag{3.2.6}$$

On the other hand, since $|u| = 1$ in $\Omega \setminus B_\varepsilon$, from (2.4.2) it follows that (recall $r_\varepsilon = |\log\varepsilon|^{-M}$)

$$\begin{aligned}\int_\Omega \mu(u,0)\xi_\varepsilon &\geq 2\pi\xi_\varepsilon(x_\varepsilon^0) - Cr_\varepsilon F_{\varepsilon,\rho_\varepsilon}(u,0)\,\|\nabla\xi_\varepsilon\|_{L^\infty(\Omega)} \\ &\overset{(3.2.3)}{\geq} 2\pi\xi_\varepsilon(x_\varepsilon^0) - C|\log\varepsilon|^{-M}|\log\varepsilon|.\end{aligned}$$

Therefore, we have

$$h_{\text{ex}}\int_\Omega \mu(u,0)\xi_\varepsilon \geq 2\pi h_{\text{ex}}\xi_\varepsilon(x_\varepsilon^0) - C|\log\varepsilon|^{N-M+1}.$$

By choosing a larger M if necessary, we get

$$h_{\text{ex}}\int_\Omega \mu(u,0)\xi_\varepsilon \geq 2\pi h_{\text{ex}}\xi_\varepsilon(x_\varepsilon^0) + o(1). \tag{3.2.7}$$

Finally, by combining (3.2.4), (3.2.6), (3.2.7), and $-\rho_\varepsilon^2(x_\varepsilon^0) \leq -b$, we are led to

$$\begin{aligned}&GL_\varepsilon(\mathbf{u},\mathbf{A}) - GL_\varepsilon(\rho_\varepsilon, h_{\text{ex}}A_\varepsilon^0) \\ &\quad\leq \pi\left(\rho_\varepsilon^2(x_\varepsilon^0)|\log\varepsilon| + (1-\rho_\varepsilon^2(x_\varepsilon^0))M\log|\log\varepsilon| - 2h_{\text{ex}}\xi_\varepsilon(x_\varepsilon^0)\right) + o(1) \\ &\quad\leq \pi|\log\varepsilon|\left(\rho_\varepsilon^2(x_\varepsilon^0) - \frac{\xi_\varepsilon(x_\varepsilon^0)}{\max_\Omega\frac{\xi_\varepsilon}{\rho_\varepsilon^2}}\right) + \pi\log|\log\varepsilon|\left((1-b)M - 2K^0\xi_\varepsilon(x_\varepsilon^0)\right) + o(1).\end{aligned}$$

Since $\psi_\varepsilon = \frac{\xi_\varepsilon}{\rho_\varepsilon^2}$ achieves its maximum at x_ε^0, the term of order $|\log\varepsilon|$ in the RHS of the last inequality is equal to 0. Therefore, since $\liminf_{\varepsilon\to 0}\max_\Omega\xi_\varepsilon > 0$ (see Proposition 2.1.2), we may choose K_0, independently of ε, such that we have

$$GL_\varepsilon(\mathbf{u},\mathbf{A}) - GL_\varepsilon(\rho_\varepsilon, h_{\text{ex}}A_\varepsilon^0) < -\log|\log\varepsilon|.$$

Step 5 *(Conclusion).* Let $(\mathbf{u}_0, \mathbf{A}_0) = (\rho_\varepsilon u_0, A_0 + h_{\mathrm{ex}} A_\varepsilon^0)$ be a vortexless configuration, that is, $|u_0| > c$ for some $c > 0$ independent of ε, such that $GL_\varepsilon(\mathbf{u}_0, \mathbf{A}_0) \leq GL_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}} A_\varepsilon^0)$. We split its Ginzburg–Landau energy with (0.0.5) to obtain

$$0 > GL_\varepsilon(\mathbf{u}_0, \mathbf{A}_0) - GL_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}} A_\varepsilon^0) = F_{\varepsilon,\rho_\varepsilon}(u_0, A_0) - h_{\mathrm{ex}} \int_\Omega \mu(u_0, A_0)\xi_\varepsilon + R_0.$$

By integration by parts, we have (recall $\xi_\varepsilon = 0$ on $\partial\Omega$)

$$\int_\Omega \mu(u_0, A_0)\xi_\varepsilon = \int_\Omega (\langle iu_0\,, \nabla_{A_0} u_0\rangle + A_0) \cdot \nabla^\perp \xi_\varepsilon.$$

Since $|u_0| > c$, we can write $u_0 = |u_0| e^{i\varphi_0}$. A direct calculation shows that

$$\langle iu_0\,, \nabla_{A_0} u_0\rangle + A_0 = (1 - |u_0|^2)(\nabla\varphi_0 - A_0) + \nabla\varphi_0.$$

Integration by parts then yields

$$\int_\Omega \nabla\varphi_0 \cdot \nabla^\perp \xi_\varepsilon = -\int_\Omega \xi_\varepsilon \operatorname{curl} \nabla\varphi_0 = 0.$$

Hence, from the Cauchy–Schwarz inequality it follows that

$$\begin{aligned} h_{\mathrm{ex}} \left| \int_\Omega \mu(u_0, A_0)\xi_\varepsilon \right| &= h_{\mathrm{ex}} \left| \int_\Omega (1 - |u_0|^2)(\nabla\varphi_0 - A_0) \right| \\ &\leq C h_{\mathrm{ex}} \left\| 1 - |u_0|^2 \right\|_{L^2(\Omega)} \|\nabla\varphi_0 - A_0\|_{L^2(\Omega,\mathbb{R}^2)} \\ &\overset{(3.2.5)}{\leq} C|\log\varepsilon|^N \varepsilon F_{\varepsilon,\rho_\varepsilon}(u_0, A_0) = o(1) F_{\varepsilon,\rho_\varepsilon}(u_0, A_0). \end{aligned}$$

On the other hand,

$$|R_0| \overset{(3.1.4)}{\leq} C h_{\mathrm{ex}}^2 \varepsilon F_{\varepsilon,\rho_\varepsilon}(u_0, A_0)^{\frac{1}{2}} \overset{(3.2.5)}{\leq} C\varepsilon |\log\varepsilon|^{2N} F_{\varepsilon,\rho_\varepsilon}(u_0, A_0)^{\frac{1}{2}}.$$

Therefore, we have

$$\begin{aligned} 0 &> GL_\varepsilon(\mathbf{u}_0, \mathbf{A}_0) - GL_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}} A_\varepsilon^0) \\ &> F_{\varepsilon,\rho_\varepsilon}(u_0, A_0)(1 - o(1)) - \varepsilon |\log \varepsilon|^{2N} F_{\varepsilon,\rho_\varepsilon}(u_0, A_0)^{\frac{1}{2}}. \end{aligned}$$

This implies that $F_{\varepsilon,\rho_\varepsilon}(u_0, A_0)^{\frac{1}{2}} \leq C\varepsilon |\log \varepsilon|^{2N} = o(1)$ and therefore

$$GL_\varepsilon(\mathbf{u}_0, \mathbf{A}_0) - GL_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}} A_\varepsilon^0) = o(1).$$

This means that $GL_\varepsilon(\mathbf{u}, \mathbf{A}) \ll GL_\varepsilon(\mathbf{u}_0, \mathbf{A}_0)$ for every vortexless configuration $(\mathbf{u}_0, \mathbf{A}_0)$. Hence, global minimizers of GL_ε in the regime (3.2.5) do have vortices. This concludes the proof of the theorem.

$\square$

Remark 3.2.1. *Notice that the hypothesis on ρ_ε only plays a role at the end of* **Step 2***. Moreover, we can replace $[\rho_\varepsilon^2]_{C^{0,\alpha}(\Omega)} \leq |\log \varepsilon|^m$ by $[\rho_\varepsilon^2]_{C^{0,\alpha}(B_\varepsilon)} \leq |\log \varepsilon|^m$, that is, we only need a control over the Hölder seminorm around the points where the function ψ achieves its maximum in $\overline{\Omega}$.*

Chapter 4

Stability of the Meissner configuration

The goal of this chapter is to see that the Meissner configuration $(\rho_\varepsilon, h_{\text{ex}} A^0_\varepsilon)$ still characterizes vortexless configurations for some regimes of h_{ex} that could go above the first critical field. Although vortexless configurations can no longer minimize GL_ε, they can still minimize locally, which means that they solve the Ginzburg–Landau equations (1.2.1). These proofs follow the steps described in Sections 2 and 3 from [Ser99].

4.1 Existence of a vortexless local minimizer

Proof of Theorem 3. In this proof, we use C to denote a generic positive constant independent of ε that might change in each line.

Step 1 *(Construction of the locally minimizing vortexless configuration. Proof vortexlessness and Item (1).)* Fix $\beta \in (0, 2-4\alpha)$ and let

$$U = \left\{ (\mathbf{u}, \mathbf{A}) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2) \colon \operatorname{div} A = 0 \text{ in } \Omega,\ A \cdot \nu = 0 \text{ on } \partial\Omega,\ F_{\varepsilon,\rho_\varepsilon}(u, A) < \varepsilon^\beta \right\},$$

which is a relatively open set of the configurations in the Coulomb Gauge in $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$. First, let us prove that there exists a configuration $(\mathbf{u}_\varepsilon, \mathbf{A}_\varepsilon)$ that minimizes GL_ε over $\overline{U}$. Note that if $(\mathbf{u}, \mathbf{A}) = (\rho_\varepsilon, h_{\mathrm{ex}} A^0_\varepsilon)$, then $(u, A) = (1, 0)$. This means that $F_{\varepsilon,\rho_\varepsilon}(1, 0) = 0$ and $A = 0$ (trivially) satisfies (0.0.2). It follows that $(\rho_\varepsilon, h_{\mathrm{ex}} A^0_\varepsilon) \in U$ and therefore, $U \neq \emptyset$.

On one hand, using Sobolev embedding and the Cauchy–Schwarz inequality, we find that each $(\mathbf{u}, \mathbf{A}) = (\rho_\varepsilon u, A + h_{\mathrm{ex}} A^0_\varepsilon) \in U$ satisfies

$$\|A\|^2_{H^1(\Omega,\mathbb{R}^2)} \overset{(1.4.1)}{\leq} C \|\operatorname{curl} A\|^2_{L^2(\Omega)} \leq C F_{\varepsilon,\rho_\varepsilon}(u, A) < C\varepsilon^\beta,$$

$$\|u\|^2_{L^4(\Omega,\mathbb{C})} = \left\|u^2\right\|_{L^2(\Omega,\mathbb{C})} \leq C + \|1 - |u|^2\|_{L^2(\Omega)} \leq C(1 + \varepsilon F_{\varepsilon,\rho_\varepsilon}(u, A)^{\frac{1}{2}}) < C(1 + \varepsilon^{1+\frac{\beta}{2}}),$$

$$\begin{aligned}\|\nabla u\|_{L^2(\Omega,\mathbb{C}^2)} &\leq C \|\nabla_A u\|_{L^2(\Omega,\mathbb{C}^2)} + \|Au\|_{L^2(\Omega,\mathbb{C})} \leq F_{\varepsilon,\rho_\varepsilon}(u, A)^{\frac{1}{2}} + \|A\|_{L^4(\Omega,\mathbb{R}^2)} \|u\|_{L^4(\Omega,\mathbb{C})} \\ &\leq F_{\varepsilon,\rho_\varepsilon}(u, A)^{\frac{1}{2}} + C \|A\|_{H^1(\Omega,\mathbb{R}^2)} (1 + \varepsilon^{1+\frac{\beta}{2}}) \leq C\varepsilon^{\frac{\beta}{2}} + \varepsilon^{1+\beta}.\end{aligned}$$

Hence, U is bounded.

On the other hand, by writing

$$GL_\varepsilon(\mathbf{u}, \mathbf{A}) = \frac{1}{2} \int_\Omega (|\nabla \mathbf{u}|^2 + |\mathbf{A}|^2 |\mathbf{u}|^2 - 2\langle \nabla \mathbf{u}, i\mathbf{A}\mathbf{u}\rangle) + |\operatorname{curl} \mathbf{A} - h_{\mathrm{ex}}|^2 + \frac{(a_\varepsilon - |\mathbf{u}|^2)^2}{2\varepsilon^2},$$

we deduce that GL_ε is H^1-weakly lower semicontinuous, since:

- The term $\int_\Omega |\nabla \mathbf{u}|^2 + |\operatorname{curl} \mathbf{A} - h_{\mathrm{ex}}|^2$ is convex and H^1-strongly continuous. Therefore, it is H^1-weakly lower semicontinuous.
- The term $\frac{1}{2} \int_\Omega \frac{(a_\varepsilon - |\mathbf{u}|^2)^2}{2\varepsilon^2}$ is L^4-strongly continuous and, by the Rellich–Kondrachov theorem, also H^1-weakly continuous.
- By the Cauchy–Schwarz inequality, the term $\int_\Omega |\mathbf{A}|^2 |\mathbf{u}|^2$ is L^4-strongly continuous and, once again, by the Rellich-Kondrachov theorem it is also H^1-weakly continuous.

- The term $\int_\Omega \langle \nabla \mathbf{u}, i\mathbf{A}\mathbf{u}\rangle$ is also H^1-weakly continuous. To see this, if $(\mathbf{u}_n, \mathbf{A}_n)$ weakly converges to $(\mathbf{u}, \mathbf{A})$ in $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$, then, by the Rellich-Kondrachov theorem, $(\mathbf{u}_n, \mathbf{A}_n)$ strongly converges to $(\mathbf{u}, \mathbf{A})$ in $L^4(\Omega, \mathbb{C}) \times L^4(\Omega, \mathbb{R}^2)$ and therefore, $\mathbf{A}_n\mathbf{u}_n$ strongly converges to $\mathbf{A}\mathbf{u}$ in $L^2(\Omega)$. This means that $\int_\Omega \langle \nabla \mathbf{u}_n, i\mathbf{A}_n\mathbf{u}_n\rangle$ converges to $\int_\Omega \langle \nabla \mathbf{u}, i\mathbf{A}\mathbf{u}\rangle$.

Since GL_ε is H^1-weakly lower semicontinuous in a nonempty bounded set U, it follows that there exists $(\mathbf{u}_\varepsilon, \mathbf{A}_\varepsilon)$ that minimizes GL_ε over $\overline{U}$. Moreover, we have

$$F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, A_\varepsilon) \leq \varepsilon^\beta. \tag{4.1.1}$$

We claim that $(\mathbf{u}_\varepsilon, \mathbf{A}_\varepsilon) \in U$, which in turn implies that $(\mathbf{u}_\varepsilon, \mathbf{A}_\varepsilon)$ is a critical point and thus, a solution of (1.2.1). From now on we drop the ε subscript.

Since $(\mathbf{u}, \mathbf{A})$ is a minimizing configuration in $\overline{U}$,

$$GL_\varepsilon(\mathbf{u}, \mathbf{A}) \leq GL_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}} A^0_\varepsilon). \tag{4.1.2}$$

By combining (0.0.1) with (4.1.2), we deduce that

$$F_{\varepsilon,\rho_\varepsilon}(u, A) \leq h_{\mathrm{ex}} \int_\Omega \mu(u, A)\xi_\varepsilon - R_0. \tag{4.1.3}$$

First, let us bound the vorticity term. Since $F_{\varepsilon,\rho_\varepsilon}(u, A) \leq \varepsilon^\beta$, we can apply Proposition 2.3.1, which provide us with a collection of balls $\mathcal{B} = \{B_i\}_i = \{B(a_i, r_i)\}$, with $\sum_i r_i \leq r = \varepsilon^\mu$ and where $\mu \in (\alpha, 1)$ is a fixed number.

By combining (4.1.1) and (2.3.4), we obtain

$$\sum_i |d_{B_i}| \leq C\frac{F_{\varepsilon,\rho_\varepsilon}(u, A)}{|\log\varepsilon|} \leq C\frac{\varepsilon^\beta}{|\log\varepsilon|} = o(1).$$

It follows that $\sum_i |d_{B_i}| = 0$, which implies $d_{B_i} = 0$ for all i. Hence, it follows from

(2.4.2) and the hypothesis $h_{\mathrm{ex}} \leq \varepsilon^{-\alpha}$, that

$$\begin{aligned} h_{\mathrm{ex}} \left| \int_\Omega \mu(u,A)\xi_\varepsilon \right| &\leq C\varepsilon^{-\alpha}\varepsilon^{\mu} F_{\varepsilon,\rho_\varepsilon}(u,A) \left\| \nabla \xi_\varepsilon \right\|_{L^\infty(\Omega)} \\ &\overset{(2.1.3)}{\leq} C\varepsilon^{-\alpha+\beta+\mu} \\ &\overset{\mu\geq\alpha}{=} o(\varepsilon^\beta). \end{aligned} \tag{4.1.4}$$

An analogous argument shows that we have weak vortexlessness, that is,

$$\|\mu(u,A)\|_{\left(C_0^{0,1}(\Omega)\right)^*} = o(1).$$

Let us now provide an upper bound for $|R_0|$. Combining $h_{\mathrm{ex}} \leq \varepsilon^{-\alpha}$ with (4.1.1), (2.1.3) and (3.1.4) yields

$$\begin{aligned} |R_0| &\leq C h_{\mathrm{ex}}^2 \varepsilon F_{\varepsilon,\rho_\varepsilon}(u,A)^{\frac{1}{2}} \left\| \nabla \xi_\varepsilon \right\|_{L^\infty(\Omega)} \\ &\leq C\varepsilon^{\frac{\beta}{2}+1-2\alpha}. \end{aligned}$$

Observe that since $\beta < 2-4\alpha$, we have

$$\frac{\beta}{2} + 1 - 2\alpha > \frac{\beta}{2} + \frac{\beta}{2} = \beta,$$

which means that

$$|R_0| = o(\varepsilon^\beta). \tag{4.1.5}$$

Therefore, inserting (4.1.4) and (4.1.5) into (4.1.3), we deduce that

$$F_{\varepsilon,\rho_\varepsilon}(u,A) \leq h_{\mathrm{ex}} \left| \int_\Omega \mu(u,A)\xi_\varepsilon \right| + |R_0| \leq o(\varepsilon^\beta). \tag{4.1.6}$$

The claim is thus proved, that is, $(u,A) \in U$ for small enough ε. Moreover, since U is open, the configuration $(\mathbf{u},\mathbf{A})$ must be a local minimizer of GL_ε.

Finally, by combining (4.1.1), (4.1.4) and (4.1.5), we conclude that Item (1) holds, since (recall (4.1.2))

$$GL_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}}A_\varepsilon^0) \geq GL_\varepsilon(\mathbf{u}, \mathbf{A}) = GL_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}}A_\varepsilon^0) + O(\varepsilon^\beta).$$

Let us now prove that $(\mathbf{u}, \mathbf{A})$ is a strongly vortexless configuration. Since $(\mathbf{u}, \mathbf{A})$ is a local minimizer, it solves the Ginzburg–Landau equations (1.2.1). Since we have (4.1.6), it follows from (1.4.3) that $\|A\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon}$ and therefore, from (1.4.4), that

$$\|\nabla |u|\|_{L^\infty(\Omega)} \leq \|\nabla u\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon}.$$

Thus, by Proposition 3.1.1, $\|1 - |u|\|_{L^\infty(\Omega)} = o(1)$.

Step 2 *(Closeness to the Meissner configuration. Proof of Item (2) and Item (3)).* We start by estimating $\|\nabla u\|_{L^2(\Omega)}$. Note that

$$\int_\Omega |\nabla u|^2 \leq 2 \left(\int_\Omega |\nabla_A u|^2 + |A|^2 |u|^2 \right).$$

On the other hand, the Coulomb gauge estimate (1.4.1) yields

$$\|A\|_{H^1(\Omega)} \leq C \|\mathrm{curl}\, A\|_{L^2(\Omega)} \leq C F_{\varepsilon,\rho_\varepsilon}(u, A)^{\frac{1}{2}} \overset{(4.1.6)}{=} o(\varepsilon^{\frac{\beta}{2}}). \tag{4.1.7}$$

This together with the uniform convergence from the strong vortexlessness $\|1 - |u|\|_{L^\infty(\Omega)} = o(1)$, leads us to

$$\int_\Omega |\nabla u|^2 \leq C \left(F_{\varepsilon,\rho_\varepsilon}(u, A) + \|A\|^2_{L^2(\Omega)} \right) = o(\varepsilon^\beta). \tag{4.1.8}$$

Let us now provide an estimate for $\|u\|_{L^2(\Omega)}$. Defining $\overline{u} = \frac{1}{|\Omega|} \int_\Omega u$, by the Poincaré–Wirtinger inequality, we have

$$\int_\Omega |u - \overline{u}|^2 \leq C \int_\Omega |\nabla u|^2 = o(\varepsilon^\beta)$$

We then deduce that

$$\int_\Omega (1-|\overline{u}|)^2 \leq 2\left(\int_\Omega |1-u|^2+|u-\overline{u}|^2\right) \leq CF_{\varepsilon,\rho_\varepsilon}(u,A) = o(\varepsilon^\beta).$$

Since $\overline{u}$ is constant in Ω, we deduce that $u = e^{i\theta_\varepsilon} + o(\varepsilon^{\frac{\beta}{2}})$. Combining this with (4.1.7) and (4.1.8) yields that (0.0.10) holds, since

$$\inf_{\theta\in[0,2\pi]} \left\|u-e^{i\theta}\right\|_{H^1(\Omega)} + \|A\|_{H^1(\Omega)} = \inf_{\theta\in[0,2\pi]} \left\|u-e^{i\theta}\right\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} + \|A\|_{H^1(\Omega)} = o(\varepsilon^{\frac{\beta}{2}}). \tag{4.1.9}$$

Finally, we prove (0.0.11) and (0.0.12). The estimate on $\|\mathbf{A}-h_{\mathrm{ex}}A_\varepsilon^0\|_{H^1(\Omega)}$ follows immediately, since $\mathbf{A}-h_{\mathrm{ex}}A_\varepsilon^0 = A$. On the other hand, for $r\in[1,2)$, let $s>2$ such that $\frac{1}{r}=\frac{1}{s}+\frac{1}{2}$. Then, using Hölder's inequality and a Sobolev embedding, we deduce that (recall $\mathbf{u}=\rho_\varepsilon u$)

$$\begin{aligned}
\left\|\mathbf{u}-\rho_\varepsilon e^{i\theta}\right\|_{W^{1,r}(\Omega)} &\leq \left\|\rho_\varepsilon(u-e^{i\theta})\right\|_{L^r(\Omega)} + \|\rho_\varepsilon\nabla u\|_{L^r(\Omega)} + \left\|(u-e^{i\theta})\nabla\rho_\varepsilon\right\|_{L^r(\Omega)} \\
&\overset{\rho_\varepsilon\leq 1}{\leq} C\left(\left\|u-e^{i\theta}\right\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}\right) + \left\|u-e^{i\theta}\right\|_{L^s(\Omega)}\|\nabla\rho_\varepsilon\|_{L^2(\Omega)} \\
&\leq C\left\|u-e^{i\theta}\right\|_{H^1(\Omega)}\left(1+\|\nabla\rho_\varepsilon\|_{L^2(\Omega)}\right) \\
&\overset{(4.1.9)}{\leq} C\left\|u-e^{i\theta}\right\|_{H^1(\Omega)}(1+\varepsilon^{-\gamma}),
\end{aligned}$$

where in the last inequality we used the hypothesis $\|\nabla\rho_\varepsilon\|_{L^2(\Omega)}\leq\varepsilon^{-\gamma}$, for $\gamma<1-2\alpha$. Since, until this point, the choice of $\beta\in(0,2-4\alpha)$ was arbitrary, we may change it if necessary, so that $\frac{\beta}{2}\in(\gamma,1-2\alpha)$. Hence

$$\inf_{\theta\in[0,2\pi]}\left\|\mathbf{u}-\rho_\varepsilon e^{i\theta}\right\|_{W^{1,r}(\Omega)} \leq C\varepsilon^{-\gamma}\inf_{\theta\in[0,2\pi]}\left\|u-e^{i\theta}\right\|_{H^1(\Omega)} \overset{(4.1.9)}{\leq} o(\varepsilon^{\frac{\beta}{2}})\varepsilon^{-\gamma} = o(1).$$

This concludes the proof.

□

4.2 Uniqueness modulo gauge equivalence

We now prove the uniqueness (up to a gauge transformation) of a vortexless minimizing configuration.

Proof of Theorem 4. To prove uniqueness up to a gauge transformation, we will prove that there is a unique minimizer in the Coulomb gauge. Suppose $(\mathbf{u}_j, \mathbf{A}_j) = (\rho_\varepsilon u_j, A_j + h_{\mathrm{ex}} A_\varepsilon^0)$ are distinct local minimizers, where $\mathbf{A}_j$ satisfies the Coulomb gauge condition (0.0.2) for $j = 1, 2$. Since A_ε^0 also satisfies (0.0.2), we deduce that A_j does it as well.

By (1.4.4) and Proposition 3.1.1, we have that $|u_j|$ converges uniformly to 1. In particular, we have $|u_j| \geq \frac{3}{4}$ for small enough ε[1]. Therefore, we can write $u_j = \eta_j e^{i\phi_j}$, where $\eta_j = |u_j|$. Note that (u_j, A_j) is gauge-equivalent to (η_j, A_j'), where $A_j' = A_j - \nabla\phi_j$. Let $A_j^\circ = A_j + h_{\mathrm{ex}} A_\varepsilon^0 - \nabla\phi_j$, which is gauge-equivalent to $(\mathbf{u}_j, \mathbf{A}_j)$ and therefore is a local minimizer.

Step 1 *(Proving that $\left\|A_j^\circ\right\|_{L^\infty(\Omega,\mathbb{R}^2)} = o(\varepsilon^{-1})$).* Observe that

$$\left\|A_j^\circ\right\|_{L^\infty(\Omega,\mathbb{R}^2)} \leq \left\|A_j + h_{\mathrm{ex}} A_\varepsilon^0\right\|_{L^\infty(\Omega,\mathbb{R}^2)} + \left\|\nabla\phi_j\right\|_{L^\infty(\Omega,\mathbb{R}^2)}.$$

From (1.4.3), we have that

$$\left\|A_j + h_{\mathrm{ex}} A_\varepsilon^0\right\|_{L^\infty(\Omega)} = o(\varepsilon^{-1}), \tag{4.2.1}$$

since we are assuming $E_\varepsilon(\rho_\varepsilon) \ll \frac{1}{\varepsilon^2}$ and $F_{\varepsilon,\rho_\varepsilon}(u_j, A_j) < \varepsilon^\beta$. We are then left to prove $\left\|\nabla\phi_j\right\|_{L^\infty(\Omega)} = o(\varepsilon^{-1})$.

By gauge-invariance, $(\rho_\varepsilon \eta_j, A_j^\circ)$ is also a local minimizer and thus, it satisfies (1.2.1).

---

[1] Actually, any c in the domain of convexity of $(1 - x^2)^2$ will do, that is $|u_j| \geq c > \frac{1}{\sqrt{3}}$. We choose $\frac{3}{4}$ as in [Ser99].

In particular, we have

$$-\nabla^{\perp}\operatorname{curl} A_j^{\circ} = \langle i\rho_\varepsilon\eta_j\, , \nabla_{A_j^{\circ}}(\rho_\varepsilon\eta_j)\rangle = -(\rho_\varepsilon\eta_j)^2 A_j^{\circ} \quad \text{in } \Omega,$$

which implies that

$$\operatorname{div}\left(\rho_\varepsilon^2\eta_j^2 A_j^{\circ}\right) = \operatorname{div}\left(\rho_\varepsilon^2\eta_j^2(A_j + h_{\mathrm{ex}}A_\varepsilon^0 - \nabla\phi_j)\right) = \operatorname{div}\left(\nabla^{\perp}\operatorname{curl} A_j^{\circ}\right) = 0 \quad \text{in } \Omega.$$

Moreover, since $\rho_\varepsilon^2 A_\varepsilon^0 = -\nabla^{\perp}\xi_\varepsilon$, we have that $\operatorname{div}(\rho_\varepsilon^2 A_\varepsilon^0) = 0$ in Ω. Recalling that A_j satisfies (0.0.2), a direct calculation then yields

$$2\rho_\varepsilon^2\eta\nabla\eta_j\cdot A_j^{\circ} + 2\eta_j^2\rho_\varepsilon\nabla\rho_\varepsilon\cdot A_j' - \eta_j^2\rho_\varepsilon^2\Delta\phi_j = 0 \quad \text{in } \Omega.$$

On the other hand, from the second boundary condition in (1.2.1), we have that

$$\nabla_{A_j^{\circ}}(\rho_\varepsilon\eta_j)\cdot\nu = 0 \quad \text{on } \partial\Omega.$$

Recalling the boundary condition in (0.0.3) and that both A and A_ε^0 satisfy (0.0.2), we deduce that

$$(\nabla\eta_j - i\nabla\phi_j)\cdot\nu = 0 \quad \text{on } \partial\Omega$$

and, in particular, $\nabla\phi_j\cdot\nu = 0$ on $\partial\Omega$.

Hence, ϕ_j solves the following elliptic PDE

$$\begin{cases} -\Delta\phi_j &= -2\left(\dfrac{\nabla\eta_j}{\eta_j}\cdot A_j^{\circ} + \dfrac{\nabla\rho_\varepsilon}{\rho_\varepsilon}\cdot A_j'\right) \quad \text{in } \Omega \\ \dfrac{\partial\phi_j}{\partial\nu} &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega. \end{cases}$$

Since $\eta_j \geq \frac{3}{4} > 0$ and $\rho_\varepsilon \geq \sqrt{b} > 0$, we have, for any $q > 1$, that

$$\|\Delta\phi_j\|_{L^q(\Omega)} \leq C\left(\left\|\nabla\eta_j\cdot A_j^{\circ}\right\|_{L^q(\Omega)} + \left\|\nabla\rho_\varepsilon\cdot A_j'\right\|_{L^q(\Omega)}\right)$$

We now estimate the terms in the RHS by interpolating between $L^2(\Omega)$ and $L^\infty(\Omega)$.

The L^∞-bounds come from our estimates for critical points of GL_ε in the Coulomb gauge, whereas the L^2-bounds follow from the smallness of $F_{\varepsilon,\rho_\varepsilon}(u_j, A_j)$, since we have

$$F_{\varepsilon,\rho_\varepsilon}(u_j, A_j) = \frac{1}{2}\int_\Omega \rho_\varepsilon^2 \left(|\nabla\eta_j|^2 + |\eta_j|^2|A_j - \nabla\phi_j|^2\right) + |\operatorname{curl} A_j|^2 + \rho_\varepsilon^4\frac{(1-\eta^2)^2}{2\varepsilon^2} < \varepsilon^\beta. \tag{4.2.2}$$

First, for any $q > 2$, we have

$$\begin{aligned}\|\nabla\eta_j\|_{L^q(\Omega)} &\le \|\nabla\eta_j\|_{L^\infty(\Omega)}^{1-\frac{2}{q}}\|\nabla\eta_j\|_{L^2(\Omega)}^{\frac{2}{q}}\\ &\overset{(1.4.4)}{\le} C(\varepsilon^{-1})^{1-\frac{2}{q}}\left(\frac{1}{b}F_{\varepsilon,\rho_\varepsilon}(u_j, A_j)\right)^{\frac{1}{q}}\\ &\overset{(4.2.2)}{\le} C\varepsilon^{\frac{2}{q}-1}\varepsilon^{\frac{\beta}{q}} = C\varepsilon^{\frac{2+\beta}{q}-1}.\end{aligned} \tag{4.2.3}$$

Second, for any $q > 2$, we have

$$\begin{aligned}\|\nabla\phi_j\|_{L^q(\Omega,\mathbb{R}^2)} &\le \|\nabla\phi_j\|_{L^\infty(\Omega,\mathbb{R}^2)}^{1-\frac{2}{q}}\|\nabla\phi_j\|_{L^2(\Omega,\mathbb{R}^2)}^{\frac{2}{q}}\\ &\overset{(1.4.4)}{\le} C(\varepsilon^{-1})^{1-\frac{2}{q}}\left(\|\nabla\phi_j - A_j\|_{L^2(\Omega,\mathbb{R}^2)} + \|A_j\|_{L^2(\Omega,\mathbb{R}^2)}\right)^{\frac{2}{q}}\\ &\overset{(1.4.1)}{\le} C\varepsilon^{\frac{2}{q}-1}\left(\frac{1}{b}F_{\varepsilon,\rho_\varepsilon}(u_j, A_j)^{\frac{1}{2}} + \|\operatorname{curl} A_j\|_{L^2(\Omega)}\right)^{\frac{2}{q}}\\ &\overset{(4.2.2)}{\le} C\varepsilon^{\frac{2}{q}-1}\varepsilon^{\frac{\beta}{q}} = C\varepsilon^{\frac{2+\beta}{q}-1}.\end{aligned} \tag{4.2.4}$$

Hence, from (4.2.1), (4.2.3), and (4.2.4), we conclude that, for any $q \in (2, 2+\beta)$, we have

$$\begin{aligned}\left\|\nabla\eta_j \cdot A_j^\circ\right\|_{L^q(\Omega)} &\le \left\|\nabla\eta_j \cdot (A_j + h_{\text{ex}}A_\varepsilon^0)\right\|_{L^q(\Omega)} + \|\nabla\eta_j \cdot \nabla\phi_j\|_{L^q(\Omega)}\\ &\le \|\nabla\eta_j\|_{L^q(\Omega,\mathbb{R}^2)}\left\|A_j + h_{\text{ex}}A_\varepsilon^0\right\|_{L^\infty(\Omega,\mathbb{R}^2)} + \|\nabla\eta_j\|_{L^q(\Omega,\mathbb{R}^2)}\|\nabla\phi_j\|_{L^\infty(\Omega,\mathbb{R}^2)}\\ &\overset{(1.4.4)\&(4.2.1)\&(4.2.3)}{\le} o(\varepsilon^{-1})\end{aligned}$$

and, recalling that $\|\nabla \rho_\varepsilon\|_{L^\infty(\Omega,\mathbb{R}^2)} \leq \frac{C}{\varepsilon}$, we have

$$\begin{aligned}\left\|\nabla \rho_\varepsilon \cdot A_j'\right\|_{L^q(\Omega,\mathbb{R}^2)} &\leq \|\nabla \rho_\varepsilon\|_{L^\infty(\Omega,\mathbb{R}^2)} \|A_j\|_{L^q(\Omega,\mathbb{R}^2)} + \|\nabla \rho_\varepsilon\|_{L^\infty(\Omega,\mathbb{R}^2)} \|\nabla \phi_j\|_{L^q(\Omega,\mathbb{R}^2)} \\ &\overset{(4.2.4)}{\leq} \frac{C}{\varepsilon} \|A_j\|_{H^1(\Omega,\mathbb{R}^2)} + o(\varepsilon^{-1}) \\ &\overset{(1.4.1)}{\leq} \frac{C}{\varepsilon} \|\operatorname{curl} A_j\|_{L^2(\Omega)} + o(\varepsilon^{-1}) \\ &\overset{(4.2.2)}{\leq} o(\varepsilon^{-1}),\end{aligned}$$

where after the first inequality we used Proposition 1.3.2 and Sobolev embedding. It follows that

$$\|\Delta \phi_j\|_{L^q(\Omega)} = o(\varepsilon^{-1})$$

and, since $q > 2$, by elliptic regularity and a Sobolev embedding, we have

$$\|\nabla \phi_j\|_{L^\infty(\Omega,\mathbb{R}^2)} = o(\varepsilon^{-1}).$$

This finally yields that

$$\left\|A_j^\circ\right\|_{L^\infty(\Omega,\mathbb{R}^2)} \leq \left\|A_j + h_{\text{ex}} A_\varepsilon^0\right\|_{L^\infty(\Omega,\mathbb{R}^2)} + \|\nabla \phi_j\|_{L^\infty(\Omega,\mathbb{R}^2)} = o(\varepsilon^{-1}).$$

Step 2 *(Convexity argument)* By gauge-invariance, we have

$$GL_\varepsilon(\mathbf{u}_j, \mathbf{A}_j) = GL_\varepsilon(\rho_\varepsilon \eta_j, A_j^\circ) = E_\varepsilon(\rho_\varepsilon \eta_j, A_j^\circ) + \frac{1}{2}\int_\Omega |\operatorname{curl} A_j^\circ - h_{\text{ex}}|^2.$$

Using (1.3.4), we have

$$GL_\varepsilon(\rho_\varepsilon \eta_j, A_j^\circ) = E_\varepsilon(\rho_\varepsilon) + \frac{1}{2}\int_\Omega \rho_\varepsilon^2 \left(|\nabla \eta_j|^2 + \eta_j^2 |A_j^\circ|^2\right) + \rho_\varepsilon^4 \frac{(1-\eta_j^2)^2}{2\varepsilon^2} + |\operatorname{curl} A_j^\circ - h_{\text{ex}}|^2.$$

Let us define

$$Y := \frac{GL_\varepsilon(\rho_\varepsilon\eta_1, A_1^\circ) + GL_\varepsilon(\rho_\varepsilon\eta_2, A_2^\circ)}{2} - GL_\varepsilon\left(\rho_\varepsilon\frac{\eta_1+\eta_2}{2}, \frac{A_1^\circ + A_2^\circ}{2}\right).$$

We claim that $Y > 0$. To prove this, let us write $Y = \frac{1}{2}(Y_1 + Y_2 + Y_3)$, where

$$\begin{aligned}
Y_1 &= \left(\int_\Omega \rho_\varepsilon^2\left(\frac{|\nabla\eta_1|^2 + |\nabla\eta_2|^2}{2}\right) - \int_\Omega \rho_\varepsilon^2\left|\nabla\left(\frac{\eta_1+\eta_2}{2}\right)\right|^2\right)\\
&\quad + \left(\int_\Omega \frac{|\operatorname{curl} A_1^\circ - h_{\text{ex}}|^2 + |\operatorname{curl} A_2^\circ - h_{\text{ex}}|^2}{2} - \int_\Omega \left|\operatorname{curl}\left(\frac{A_1^\circ + A_2^\circ}{2}\right) - h_{\text{ex}}\right|^2\right),\\
Y_2 &= \int_\Omega \frac{\rho_\varepsilon^4}{2\varepsilon^2}\left(\frac{(1-\eta_1^2)^2 + (1-\eta_2^2)^2}{2}\right) - \int_\Omega \frac{\rho_\varepsilon^4}{2\varepsilon^2}\left(1 - \left(\frac{\eta_1+\eta_2}{2}\right)^2\right)^2, \text{ and}\\
Y_3 &= \int_\Omega \rho_\varepsilon^2\left(\frac{|A_1^\circ|^2|\eta_1|^2 + |A_2^\circ|^2|\eta_2|^2}{2}\right) - \int_\Omega \rho_\varepsilon^2\left(\left|\frac{\eta_1+\eta_2}{2}\right|^2\left|\frac{A_1^\circ + A_2^\circ}{2}\right|^2\right).
\end{aligned}$$

By convexity, we have that $Y_1 \geq 0$.

Let us now provide an estimate for Y_2. A direct calculation yields (see [Ser99, Section 2] for the details)

$$\frac{(1-\eta_1^2)^2 + (1-\eta_2^2)^2}{2} - \left(1 - \left(\frac{\eta_1+\eta_2}{2}\right)^2\right)^2 = \frac{1}{16}(\eta_1 - \eta_2)^2(7(\eta_1+\eta_2)^2 - 4\eta_1\eta_2 - 8).$$

Therefore, we have

$$Y_2 = \frac{1}{32\varepsilon^2}\int_\Omega \rho_\varepsilon^4(\eta_1 - \eta_2)^2(7(\eta_1+\eta_2)^2 - 4\eta_1\eta_2 - 8),$$

which combined with $\frac{3}{4} \leq \eta_j \leq 1$ and $\rho_\varepsilon^4 \geq b^2$, yields

$$Y_2 \geq \frac{b^2}{32\varepsilon^2}\left(7\left(\frac{3}{4}+\frac{3}{4}\right)^2 - 12\right)\|\eta_1 - \eta_2\|_{L^2(\Omega)}^2 = \frac{C_1}{\varepsilon^2}\|\eta_1 - \eta_2\|_{L^2(\Omega)}^2, \tag{4.2.5}$$

where $C_1 > 0$ is a constant that depends on b only.

Let us now estimate Y_3. A direct calculation shows that (see [Ser99, Section 2] for the details)

$$\begin{aligned}\frac{|A_1^\circ|^2|\eta_1|^2 + |A_2^\circ|^2|\eta_2|^2}{2} - \left|\frac{\eta_1+\eta_2}{2}\right|^2 \left|\frac{A_1^\circ + A_2^\circ}{2}\right|^2 \\ = \frac{1}{8}(\eta_1-\eta_2)^2|A_1^\circ + A_2^\circ|^2 + \frac{1}{2}\eta_1^2|A_1^\circ - A_2^\circ|^2 \\ - \frac{1}{8}(\eta_1-\eta_2)(A_1^\circ - A_2^\circ)\cdot\left(A_1^\circ(2\eta_1+4\eta_2) + A_2^\circ(6\eta_1+8\eta_2)\right).\end{aligned}$$

Therefore

$$\begin{aligned}Y_3 = \frac{1}{8}\int_\Omega \rho_\varepsilon^2\big((\eta_1-\eta_2)^2|A_1^\circ + A_2^\circ|^2 + 4\eta_1^2|A_1^\circ - A_2^\circ|^2\big) \\ - \frac{1}{8}\int_\Omega \rho_\varepsilon^2\big((\eta_1-\eta_2)(A_1^\circ - A_2^\circ)\cdot(A_1^\circ(2\eta_1+4\eta_2) + A_2^\circ(6\eta_1+8\eta_2)\big)\end{aligned}$$

which combined with $\rho_\varepsilon\eta_j \leq 1$, yields

$$\begin{aligned}Y_3 \geq \frac{1}{8}\int_\Omega \rho_\varepsilon^2\big((\eta_1-\eta_2)^2|A_1^\circ + A_2^\circ|^2 + 4\eta_1^2|A_1^\circ - A_2^\circ|^2\big) \\ - \frac{1}{8}\int_\Omega \rho_\varepsilon|\eta_1-\eta_2||A_1^\circ - A_2^\circ|(6|A_1^\circ| + 14|A_2^\circ|). \quad (4.2.6)\end{aligned}$$

Note that $Y_3 \geq 0$ if $\eta_1 \equiv \eta_2$ or $A_1^\circ \equiv A_2^\circ$, which in turn yields that $Y > 0$. Indeed, if $\eta_1 \equiv \eta_2$, $A_1^\circ \not\equiv A_2^\circ$, we have $Y_3 > \frac{1}{2}\int_\Omega \eta_1^2|A_1^\circ - A_2^\circ|^2 > 0$. Hence, $Y \geq \frac{1}{2}Y_3 > 0$. On the other hand, if $A_1^\circ \equiv A_2^\circ$, then $\eta_1 \not\equiv \eta_2$, and therefore $Y \geq \frac{1}{2}Y_2 > 0$. For this reason, we assume from now on that $\eta_1 \not\equiv \eta_2$ and $A_1^\circ \not\equiv A_2^\circ$.

From the L^∞-bound obtained in **Step 1**, we deduce that

$$\begin{aligned}\int_\Omega \rho_\varepsilon |\eta_1-\eta_2||A_1^\circ - A_2^\circ|(6|A_1^\circ| + 14|A_2^\circ|)&\\ &\le \|\eta_1-\eta_2\|_{L^2(\Omega)}\|A_1^\circ - A_2^\circ\|_{L^2(\Omega,\mathbb{R}^2)}\big(14(\|A_1^\circ\|_{L^\infty(\Omega,\mathbb{R}^2)} + \|A_2^\circ\|_{L^\infty(\Omega,\mathbb{R}^2)})\big)\\ &\le o(\varepsilon^{-1})\|\eta_1-\eta_2\|_{L^2(\Omega)}\|A_1^\circ - A_2^\circ\|_{L^2(\Omega,\mathbb{R}^2)}. \qquad (4.2.7)\end{aligned}$$

On the other hand, using once again that $\eta_1 \ge \frac{3}{4}$ and $\rho_\varepsilon^2 \ge b$, from Young's inequality, we deduce that

$$\frac{1}{2}\int_\Omega \rho_\varepsilon^2\eta_1^2|A_1^\circ - A_2^\circ|^2 + \frac{C_1}{\varepsilon^2}\|\eta_1-\eta_2\|^2_{L^2(\Omega)} \ge \frac{C_2}{\varepsilon}\|\eta_1-\eta_2\|_{L^2(\Omega)}\|A_1^\circ - A_2^\circ\|_{L^2(\Omega,\mathbb{R}^2)}, \qquad (4.2.8)$$

where $C_2 > 0$ is a constant that depends on b only. Finally, by combining (4.2.5), (4.2.6), (4.2.7), and (4.2.8), we are led to

$$Y_2 + Y_3 \ge \|\eta_1-\eta_2\|_{L^2(\Omega)}\|A_1^\circ - A_2^\circ\|_{L^2(\Omega,\mathbb{R}^2)}\left(\frac{C}{\varepsilon} - o(\varepsilon^{-1})\right).$$

Hence, for sufficiently small ε, we have $Y > 0$ on all cases.

Step 3 *(Contradiction)* Assume without loss of generality that

$$GL_\varepsilon(\rho_\varepsilon\eta_1, A_1^\circ) \le GL_\varepsilon(\rho_\varepsilon\eta_2, A_2^\circ).$$

Since $Y > 0$, we have

$$GL_\varepsilon(\rho_\varepsilon\eta_2, A_2^\circ) \ge \frac{GL_\varepsilon(\rho_\varepsilon\eta_1, A_1^\circ) + GL_\varepsilon(\rho_\varepsilon\eta_2, A_2^\circ)}{2} > GL_\varepsilon\left(\rho_\varepsilon\frac{\eta_1+\eta_2}{2}, \frac{A_1^\circ + A_2^\circ}{2}\right).$$

A standard argument then shows that, for any $t \in (0,1)$, we have

$$GL_\varepsilon(\rho_\varepsilon\eta_2, A_2^\circ) > GL_\varepsilon\Big(\rho_\varepsilon(t\eta_1 + (1-t)\eta_2), tA_1^\circ + (1-t)A_2^\circ\Big),$$

which contradicts the local minimality of $(\rho_\varepsilon\eta_2, A_2^\circ)$. Therefore, $(\mathbf{u}_1, \mathbf{A}_1) = (\mathbf{u}_2, \mathbf{A}_2)$, which concludes the proof.

□

Chapter 5

The Ginzburg–Landau problem in 3D

We dedicate this section to the 3D problem and the required tools. We also indicate the differences with the 2D problem and how to adapt their tools in the 3D framework. Most of the tools from Chapter 1 directly generalize to the 3D problem, including the Lassoued–Mironescu decoupling result (Proposition 1.3.2 and Lemma 1.3.1), the regularity properties from ρ_ε (Proposition 1.3.2), and the gauge invariance of physically relevant quantities.

The space of minimization changes to $H^1(\Omega,\mathbb{C})\times[A_{\text{ex}}+H_{\text{curl}}(\mathbb{R}^3,\mathbb{R}^3)]$. The Ginzburg–Landau equations are similar in nature, but now we have to keep in mind that the magnetic component $|H-H_{\text{ex}}|^2$ extends outside Ω. The *3D Ginzburg–Landau equations* solved by minimizers of $GL_\varepsilon^{3\text{D}}$ are

$$\begin{cases} -(\nabla_A)^2 u &= \dfrac{u(a_\varepsilon-|u|^2)}{\varepsilon^2} & \text{in } \Omega \\ \operatorname{curl}(H-H_{\text{ex}}) &= \chi_\Omega\langle iu,\nabla_A u\rangle & \text{in } \mathbb{R}^3 \\ [H-H_{\text{ex}}]\times\nu &= 0 & \text{on } \partial\Omega \\ \nabla_A u\cdot\nu &= 0 & \text{on } \partial\Omega, \end{cases}$$

Here, $[\cdot]$ denotes the jump across $\partial\Omega$. Thus, the first boundary condition essentially means that $H - H_{\text{ex}}$ crosses the boundary continuously in the normal direction. The rest of this chapter is essentially a generalization of Chapter 2, which starts with a useful tool from L^p-analysis of vector fields to derive the Meissner configuration and ends with a proof of Theorem 5.

5.1 Helmholtz-Hodge decomposition of vector fields

The fact that we can gauge transform any configuration into one in the Coulomb gauge follows from a deeper result in L^p-analysis of vector fields. It is a well-known fact that any vector field defined in $\mathbb{R}^3$ can be decomposed as $A = \operatorname{curl} B + \nabla\phi$, that is, A can be decomposed into solenoidal and potential components. This result also holds for bounded domains, but there is a dependence on the topological structure of the domain and its boundary. The main result of this section the following proposition.

Proposition 5.1.1 (Helmholtz-Hodge decomposition). *Let Ω be a smooth simply connected domain. For every $A \in L^2(\Omega, \mathbb{R}^3)$, there exists a unique pair $(B_A, \nabla\phi_A)$, $B_A \in H^1_{\sigma,T}(\Omega, \mathbb{R}^3)$ and $\phi_A \in H^1(\Omega)$ such that*

$$A = \operatorname{curl} B_A + \nabla\phi_A \tag{5.1.1}$$

This decomposition is continuous in $L^2(\Omega, \mathbb{R}^3)$, that is, there exists $C = C(\Omega) > 0$ such that for any $A \in L^2(\Omega, \mathbb{R}^3)$

$$\|\nabla\phi_A\|_{L^2(\Omega,\mathbb{R}^3)} + \|B_A\|_{H^1(\Omega,\mathbb{R}^3)} \leq C\,\|A\|_{L^2(\Omega,\mathbb{R}^3)}\,. \tag{5.1.2}$$

Remark 5.1.1. *This decomposition holds for any bounded or exterior domain with smooth boundary. In particular, it also holds for $\mathbb{R}^3$.*

Proof. The proof follows from a nontrivial inequality [AS13, Theorem 3.1] (the additional terms in the referred Theorem come from holes in the domain which are not present in this case): There exists $C = C(\Omega)$ such that for $B \in H^1(\Omega, \mathbb{R}^3)$ with $B \times \nu = 0$ on $\partial\Omega$

$$\|B\|_{H^1(\Omega,\mathbb{R}^3)} \leq C(\|\operatorname{div} B\|_{L^2(\Omega)} + \|\operatorname{curl} B\|_{L^2(\Omega,\mathbb{R}^3)}). \tag{5.1.3}$$

Now fix $A \in C^\infty(\Omega, \mathbb{R}^3)$ and consider the following PDE

$$\begin{cases} \operatorname{curl}^2 B_A &= \operatorname{curl} A & \text{in } \Omega \\ \operatorname{div} B_A &= 0 & \text{in } \Omega \\ B_A \times \nu &= 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1.4}$$

We can define a weak formulation using the bilinear form $a\colon H^1_{\sigma,T}(\Omega, \mathbb{R}^3) \times H^1_{\sigma,T}(\Omega, \mathbb{R}^3)$

$$a(V, W) := \int_\Omega \operatorname{curl} V \cdot \operatorname{curl} W.$$

Equation (5.1.4) is equivalent to finding $B_A \in H^1_{\sigma,T}(\Omega, \mathbb{R}^3)$ such that for any $V \in H^1_{\sigma,T}(\Omega, \mathbb{R}^3)$

$$a(B_A, V) = \int_\Omega A \cdot \operatorname{curl} V.$$

The bilinear form a is continuous with the inherited H^1-norm and coercivity follows from inequality (5.1.3). Thus, we deduce using Lax–Milgram that there exists B_A such that

$$a(B_A, V) = \int_\Omega \operatorname{curl} A \cdot V = \int_\Omega A \cdot \operatorname{curl} V$$

for any $V \in H^1_{\sigma,T}(\Omega, \mathbb{R}^3)$. In particular, we can take the L^2-closure on A to deduce that this result holds for any $A \in L^2(\Omega, \mathbb{R}^3)$. Moreover, we have

$$\|B_A\|_{H^1(\Omega,\mathbb{R}^3)} \leq \frac{1}{C} \|A\|_{L^2(\Omega,\mathbb{R}^3)},$$

where C is the coercivity constant from (5.1.3).

Finally, note that $\operatorname{curl}(A - \operatorname{curl} B_A) = 0$. Since Ω is simply connected, there exists ϕ_A such that $A - \operatorname{curl} B_A = \nabla\phi_A$. Moreover, we deduce from the triangle inequality that

$$\|\nabla\phi\|_{L^2(\Omega,\mathbb{R}^3)} \leq \|A\|_{L^2(\Omega,\mathbb{R}^3)} + \|\operatorname{curl} B\|_{L^2(\Omega,\mathbb{R}^3)} \leq \left(1+\frac{1}{C}\right)\|A\|_{L^2(\Omega,\mathbb{R}^3)}.$$

Uniqueness of the pair $(B_A, \nabla\phi_A)$ follows from the L^2-orthogonality of $\nabla H^1(\Omega)$ and $L^2_{\sigma,N}(\Omega,\mathbb{R}^3)$, since we have for any $B \in H^1_{\sigma,T}(\Omega,\mathbb{R}^3)$ and $\phi \in H^1(\Omega)$ that

$$\int_\Omega \operatorname{curl} B\cdot\nabla\phi = \int_\Omega B\cdot\operatorname{curl}\nabla\phi + \int_{\partial\Omega}(B\times\nu)\cdot\nabla\phi = 0.$$

□

5.2 Meissner configuration in 3D

The derivation of the Meissner configuration in 3D follows a similar idea to the derivation of its 2D counterpart: We minimize $GL^{3\mathrm{D}}_\varepsilon$ while fixing $|u| = \rho_\varepsilon$. We can choose an adequate gauge transformation to ensure coercivity with respect to A and thus the existence of a minimizer. However, this means that we cannot choose favorable boundary conditions on $\partial\Omega$ for the minimizer, which are crucial for the splitting result. We can solve this problem by preemptively choosing a particular phase $(\rho_\varepsilon e^{i\phi_A}, A) \sim (\rho_\varepsilon, A-\nabla\phi_A)$. Before we state our result, let us give some context for our choices of gauge and admissible space.

To deduce the structure of this gauge, we start by deriving A^0_ε as in 2D: Using the energy decoupling (1.3.1), we can write the energy as

$$GL_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}}A) = GL_\varepsilon(\rho_\varepsilon\cdot 1, h_{\mathrm{ex}}A) = E_\varepsilon(\rho_\varepsilon) + \frac{h_{\mathrm{ex}}^2}{2}\int_\Omega \rho_\varepsilon^2|A|^2 + \int_{\mathbb{R}^3}|\operatorname{curl} A - H_{0,\mathrm{ex}}|^2.$$

Minimizing this functional is equivalent to minimize

$$J(A) := \frac{1}{2}\int_\Omega \rho_\varepsilon^2|A|^2 + \frac{1}{2}\int_{\mathbb{R}^3}|\operatorname{curl} A - H_{0,\mathrm{ex}}|^2.$$

If we preemptively enforce a gauge $(\rho_\varepsilon e^{i\phi_A}, A) \sim (\rho_\varepsilon, A - \nabla\phi_A)$, $J(A)$ becomes

$$J(A) := \frac{1}{2}\int_\Omega \rho_\varepsilon^2 |A - \nabla\phi|^2 + \frac{1}{2}\int_{\mathbb{R}^3} |\operatorname{curl} A - H_{0,\mathrm{ex}}|^2.$$

Furthermore, if ϕ_A depends linearly on A and take the associated Euler–Lagrange equation solved by a minimizer A_ε^0, we would have for any A in the space of admissible functions

$$\int_\Omega \rho_\varepsilon^2 (A_\varepsilon^0 - \nabla\phi_{A_\varepsilon^0}) \cdot (A - \nabla\phi_A) + \int_{\mathbb{R}^3} (\operatorname{curl} A_\varepsilon^0 - H_{0,\mathrm{ex}}) \cdot \operatorname{curl} A = 0.$$

The appropriate choice for ϕ_A is to take it as the unique solution with zero average in $H^1(\Omega)$ of

$$\begin{cases} \operatorname{div} \rho_\varepsilon^2 (A - \nabla\phi_A) & = & 0 \quad \text{in } \Omega \\ \rho_\varepsilon^2 (A - \nabla\phi_A) \cdot \nu & = & 0 \quad \text{on } \partial\Omega, \end{cases} \tag{5.2.1}$$

The existence of ϕ_A follows by a standard application of the Poincaré–Wirtinger inequality and the Lax–Milgram theorem. This choice of ϕ_A ensures that, by Lemma 1.1.1, we can write $\rho_\varepsilon^2 (A_\varepsilon^0 - \nabla\phi_{A_\varepsilon^0}) = \operatorname{curl} B_\varepsilon^0$ for $B_\varepsilon^0 \in H^1_{\sigma,T}(\Omega, \mathbb{R}^3)$. Thus, by replacing it in (5.2.1) and using L^2-orthogonality, we have

$$\int_\Omega \operatorname{curl} B_\varepsilon^0 \cdot A + \int_{\mathbb{R}^3} (\operatorname{curl} A_\varepsilon^0 - H_{0,\mathrm{ex}}) \cdot \operatorname{curl} A = 0.$$

On the other hand, the adequate space for minimization is the *homogeneous Sobolev space* $\dot{H}^1(\mathbb{R}^3, \mathbb{R}^3)$, which is the closure of $C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$ with respect to the norm $\|D(\cdot)\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)}$. Essentially, this enforces a Dirichlet condition at infinity to allow integration by parts. Moreover, this norm is equivalent to

$$(\|\mathrm{div}(\cdot)\|^2_{L^2(\mathbb{R}^3)} + \|\mathrm{curl}(\cdot)\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^3)})^{\frac{1}{2}}.$$

To additionally ensure coercivity, we will restrict ourselves to the subspace $\dot{H}^1_\sigma(\mathbb{R}^3, \mathbb{R}^3)$ of vector fields in $\dot{H}^1(\mathbb{R}^3, \mathbb{R}^3)$ with zero divergence. The norm in $\dot{H}^1_\sigma(\mathbb{R}^3, \mathbb{R}^3)$ inherited from $\dot{H}^1(\mathbb{R}^3, \mathbb{R}^3)$ is just $\|\mathrm{curl}(\cdot)\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)}$.

Now, we formalize the previous explanation in the following proposition.

Proposition 5.2.1. *Let*

$$J(A) := \frac{1}{2}\int_\Omega \rho_\varepsilon^2 |A - \nabla\phi_{A,\rho_\varepsilon^2}|^2 + \frac{1}{2}\int_{\mathbb{R}^3} |\operatorname{curl} A - H_{0,\mathrm{ex}}|^2,$$

where $\phi_{A,\rho_\varepsilon^2}$ is the unique solution modulo $\mathbb{R}$ of

$$\begin{cases} \operatorname{div} \rho_\varepsilon^2 (A - \nabla\phi_{A,\rho_\varepsilon^2}) &= 0 \quad \text{in } \Omega \\ \rho_\varepsilon^2 (A - \nabla\phi_{A,\rho_\varepsilon^2})\cdot\nu &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5.2.2}$$

There exists a unique minimizer $A_\varepsilon^0 \in [A_{0,\mathrm{ex}} + \dot{H}^1_\sigma(\mathbb{R}^3,\mathbb{R}^3)]$ for J. A_ε^0 satisfies

(1) *$J(A_\varepsilon^0) \le C\,\|H_{0,\mathrm{ex}}\|_{L^2(\Omega,\mathbb{R}^3)}$ for some $C = C(\Omega) > 0$.*

(2) *The Euler–Lagrange equation solved by the minimizer A_ε^0 is given by*

$$\operatorname{curl}(\operatorname{curl} A_\varepsilon^0 - H_{0,\mathrm{ex}}) + \chi_\Omega \operatorname{curl} B_\varepsilon^0 = 0 \quad \text{in } \mathbb{R}^3,$$

where χ_Ω denotes the indicator function of the set Ω.

(3) *The vector field B_ε^0 given by the relation $\operatorname{curl} B_\varepsilon^0 = \rho_\varepsilon^2(A_\varepsilon^0 - \nabla\phi^0_{\varepsilon,\rho_\varepsilon^2})$ solves the following variational equation for all $V \in L^2_{\sigma,N}(\Omega,\mathbb{R}^3)$.*

$$\begin{cases} \displaystyle\int_\Omega \left(\operatorname{curl}\frac{\operatorname{curl} B_\varepsilon^0}{\rho_\varepsilon^2} + B_\varepsilon^0\right)\cdot V &= \displaystyle\int_\Omega H_{0,\mathrm{ex}}\cdot V \quad \text{in } \Omega \\ B_\varepsilon^0 \times \nu &= 0 \qquad\qquad \text{on } \partial\Omega \end{cases} \tag{5.2.3}$$

Before proving this proposition, we introduce the notation $H_\varepsilon^0 := \operatorname{curl} A_\varepsilon^0$.

Proof. **Step 1** *(Existence and uniqueness. Proof of item (1))* Since $\|A\|_{\dot{H}^1_\sigma(\mathbb{R}^3,\mathbb{R}^3)} = \|\operatorname{curl} A\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)}$, the coercivity for J in $[A_{0,\mathrm{ex}} + \dot{H}^1_\sigma(\mathbb{R}^3,\mathbb{R}^3)]$ follows directly. On the other hand, by the linearity of equation (5.2.2), we deduce that the map $A \to \nabla\phi_{A,\rho_\varepsilon^2}$ is linear. Furthermore, by using $\phi_{A,\rho_\varepsilon^2}$ as a test function in (5.2.2)

$$\int_\Omega \rho_\varepsilon^2 A \cdot \nabla \phi_{A,\rho_\varepsilon^2} = \int_\Omega \rho_\varepsilon^2 |\nabla \phi_{A,\rho_\varepsilon^2}|^2$$

By the Cauchy-Schwarz inequality and the uniform bounds for ρ_ε, we conclude that the map $A \to \nabla \phi_{A,\rho_\varepsilon^2}$ is a bounded linear operator from $L^2(\Omega, \mathbb{R}^3)$ to $L^2(\Omega, \mathbb{R}^3)$. We conclude that J is both (strongly) continuous and strictly convex, which implies it is also weakly lower semicontinuous. It follows from the direct method of the calculus of variations that J admits a unique minimizer in $[A_{0,\mathrm{ex}} + \dot{H}^1_\sigma(\mathbb{R}^3, \mathbb{R}^3)]$ denoted by A_ε^0, that is, $A_\varepsilon^0 - A_{0,\mathrm{ex}} \in \dot{H}^1_\sigma(\mathbb{R}^3, \mathbb{R}^3)$. Finally, we see that item (1) is satisfied by noticing that $A_{0,\mathrm{ex}}$ is admissible for J. Therefore

$$\begin{aligned} J(A_\varepsilon^0) \le J(A_{0,\mathrm{ex}}) &= \frac{1}{2} \int_\Omega \rho_\varepsilon^2 |A_{0,\mathrm{ex}} - \nabla \phi_{A_{0,\mathrm{ex}},\rho_\varepsilon^2}|^2 \\ &\le C \left\| A_{0,\mathrm{ex}} \right\|_{L^2(\Omega,\mathbb{R}^3)} \\ &\le C \left\| H_{0,\mathrm{ex}} \right\|_{L^2(\Omega,\mathbb{R}^3)}, \end{aligned}$$

where we recall that we chose $A_{0,\mathrm{ex}} \in L^2_{\sigma,N}(\Omega, \mathbb{R}^3)$ to use (5.1.2) for the last inequality.

Step 2 *(Euler-Lagrange equation for A_ε^0. Proof of items (2) and (3))* Since A_ε^0 minimizes J, it satisfies the following weak equation for all $A \in \dot{H}^1_\sigma(\mathbb{R}^3, \mathbb{R}^3)$

$$\int_\Omega \rho_\varepsilon^2 (A_\varepsilon^0 - \nabla \phi^0_{\varepsilon,\rho_\varepsilon^2}) \cdot (A - \nabla \phi_{A,\rho_\varepsilon^2}) + \int_{\mathbb{R}^3} (H_\varepsilon^0 - H_{0,\mathrm{ex}}) \cdot \operatorname{curl} A = 0. \tag{5.2.4}$$

Since $\rho_\varepsilon^2 (A_\varepsilon^0 - \nabla \phi^0_{\varepsilon,\rho_\varepsilon^2}) \in L^2_{\sigma,N}(\Omega, \mathbb{R}^3)$, it follows from Lemma 1.1.1 that we can write $\rho_\varepsilon^2 (A_\varepsilon^0 - \nabla \phi^0_{\varepsilon,\rho_\varepsilon^2})$ as $\operatorname{curl} B_\varepsilon^0$, where $B_\varepsilon^0 \in H^1_{\sigma,T}(\Omega, \mathbb{R}^3)$. To recap, B_ε^0 satisfies

$$\begin{cases} \operatorname{div} B_\varepsilon^0 &= 0 & \text{in } \Omega \\ \operatorname{curl} B_\varepsilon^0 &= \rho_\varepsilon^2 (A_\varepsilon^0 - \nabla \phi^0_{\varepsilon,\rho_\varepsilon^2}) & \text{in } \Omega \\ B_\varepsilon^0 \times \nu &= 0 & \text{on } \partial\Omega \\ \operatorname{curl} B_\varepsilon^0 \cdot \nu &= 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2.5}$$

Additionally, we have for any $\phi \in H^1(\Omega)$

$$\int_\Omega \rho_\varepsilon^2(A_\varepsilon^0 - \nabla\phi^0_{\varepsilon,\rho_\varepsilon^2})\cdot\nabla\phi \overset{(5.2.5)}{=} \int_\Omega \operatorname{curl} B_\varepsilon^0\cdot\nabla\phi = \int_\Omega B_\varepsilon^0\cdot\operatorname{curl}\nabla\phi - \int_{\partial\Omega}\nabla\phi\cdot(B_\varepsilon^0\times\nu) \overset{(5.2.5)}{=} 0 \tag{5.2.6}$$

Using (5.2.6) with $\phi = \phi_{A,\rho_\varepsilon^2}$, equation (5.2.4) becomes

$$\int_\Omega \operatorname{curl} B_\varepsilon^0 \cdot A + \int_{\mathbb{R}^3} (H_\varepsilon^0 - H_{0,\mathrm{ex}}) \cdot \operatorname{curl} A = 0. \tag{5.2.7}$$

By the Helmholtz-Hodge decomposition (see Remark 5.1.1), we conclude that equation (5.2.7) holds for any $A \in \dot{H}^1(\mathbb{R}^3, \mathbb{R}^3)$. An integration by parts on (5.2.7) yields the Euler-Lagrange equation for J, which is the equation stated in item (2).

$$\chi_\Omega \operatorname{curl} B_\varepsilon^0 + \operatorname{curl}(H_\varepsilon^0 - H_{0,\mathrm{ex}}) = 0 \tag{5.2.8}$$

The remaining item follows from using vector fields supported in Ω as test vector fields. If $\Phi \in C_0^\infty(\Omega, \mathbb{R}^3)$, it follows from equation (5.2.7) and integration by parts that

$$\int_\Omega (B_\varepsilon^0 + H_\varepsilon^0 - H_{0,\mathrm{ex}}) \cdot \operatorname{curl}\Phi = 0.$$

In particular, for any $\Phi \in C_0^\infty(\Omega, \mathbb{R}^3)$ such that $\operatorname{div}\Phi = 0$

$$\int_\Omega (B_\varepsilon^0 + H_\varepsilon^0 - H_{0,\mathrm{ex}}) \cdot \Phi = 0.$$

Therefore, by Lemma 1.1.1, we have that for any $A \in L^p_{\sigma,N}(\Omega, \mathbb{R}^3)$

$$\int_\Omega (B_\varepsilon^0 + H_\varepsilon^0 - H_{0,\mathrm{ex}}) \cdot A \overset{(5.2.5)}{=} \int_\Omega \left(B_\varepsilon^0 + \operatorname{curl}\frac{\operatorname{curl} B_\varepsilon^0}{\rho_\varepsilon^2} - H_{0,\mathrm{ex}}\right) \cdot A = 0.$$

This concludes the proof of item (3).

□

5.2.1 Regularity of the Meissner configuration

Regarding regularity, we are looking for bounds that do not depend on ε. We can deduce from item (1) in the preceding proposition that

$$\left\|\operatorname{curl} B_\varepsilon^0\right\|_{L^2(\Omega,\mathbb{R}^3)} + \left\|A_\varepsilon^0 - A_{0,\text{ex}}\right\|_{H^1(\mathbb{R}^3,\mathbb{R}^3)} \leq C \left\|H_{0,\text{ex}}\right\|_{L^2(\Omega,\mathbb{R}^3)},$$

where $C = C(\Omega) > 0$. As for B_ε^0, we can deduce $C_T^{0,1}$ regularity analogous to (2.1.3).

Proposition 5.2.2. *Let* A_ε^0 *be the minimizer of* J *given by Proposition 5.2.1 and* $B_\varepsilon^0, \phi_{\varepsilon,\rho_\varepsilon^2}$ *given by the relation* $\rho_\varepsilon^2(A_\varepsilon^0 - \nabla\phi_{\varepsilon,\rho_\varepsilon^2}) = \operatorname{curl} B_\varepsilon^0$. *For any* $p > 3$ *there exists* $C > 0$ *not depending on* ε *such that*

$$\left\|\operatorname{curl} B_\varepsilon^0\right\|_{L^\infty(\Omega,\mathbb{R}^3)} \leq C \left\|H_{0,\text{ex}}\right\|_{L^p(\Omega,\mathbb{R}^3)}. \tag{5.2.9}$$

In particular, $B_\varepsilon^0 \in W^{1,\infty}(\Omega,\mathbb{R}^3)$ *and* $\|\operatorname{curl} B_\varepsilon^0\|_{L^\infty(\Omega,\mathbb{R}^3)}$ *are uniformly bounded.*

Proof. In this proof, C will denote a generic constant independent of ε which might change from line to line. By interior elliptic regularity on equation (5.2.8), we have for any $W \subset \mathbb{R}^3$

$$\left\|A_\varepsilon^0 - A_{0,\text{ex}}\right\|_{H^2(W,\mathbb{R}^3)} \leq C \left\|\operatorname{curl} B_\varepsilon^0\right\|_{L^2(\Omega,\mathbb{R}^3)}. \tag{5.2.10}$$

On the other hand,

$$\begin{aligned}\left\|\operatorname{curl} B_\varepsilon^0\right\|_{L^p(\Omega,\mathbb{R}^3)} &\overset{(5.2.5)}{=} \left\|\rho_\varepsilon^2(A_\varepsilon^0 - \nabla\phi_{\varepsilon,\rho_\varepsilon^2}^0)\right\|_{L^p(\Omega,\mathbb{R}^3)} \\ &\leq C(\left\|A_\varepsilon^0\right\|_{L^p(\Omega,\mathbb{R}^3)} + \left\|\nabla\phi_{\varepsilon,\rho_\varepsilon^2}^0\right\|_{L^p(\Omega,\mathbb{R}^3)}).\end{aligned}$$

We would like to bound the term $\left\|\nabla\phi_{\varepsilon,\rho_\varepsilon^2}^0\right\|_{L^p(\Omega,\mathbb{R}^3)}$ in terms of A_ε^0. Recall equation (5.2.2) which relates A_ε^0 and $\phi_{\varepsilon,\rho_\varepsilon^2}^0$. Using the dual characterization of the norm, we have for any $p \in [1,\infty]$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$

$$\left\|\nabla \phi^0_{\varepsilon,\rho^2_\varepsilon}\right\|_{L^p(\Omega,\mathbb{R}^3)} = \sup_{V \in L^q(\Omega,\mathbb{R}^3)} \frac{1}{\|V\|_{L^q(\Omega,\mathbb{R}^3)}} \int_\Omega V \cdot \nabla \phi^0_{\varepsilon,\rho^2_\varepsilon}$$

Using the Helmholtz-Hodge decomposition (5.1.1) to write

$$V = V_0 + \nabla\varphi,$$

where $V_0 \in L^q_{\sigma,N}(\Omega,\mathbb{R}^3)$ and using the L^2-orthogonality relation between the components, we deduce that

$$\begin{aligned}\left\|\nabla \phi^0_{\varepsilon,\rho^2_\varepsilon}\right\|_{L^p(\Omega,\mathbb{R}^3)} &\leq C \sup_{V=V_0+\nabla\varphi \in L^q(\Omega,\mathbb{R}^3)} \frac{1}{\|V_0\|_{L^q(\Omega,\mathbb{R}^3)} + \|\nabla\varphi\|_{L^q(\Omega,\mathbb{R}^3)}} \int_\Omega \nabla\varphi \cdot \nabla \phi^0_{\varepsilon,\rho^2_\varepsilon} \\ &\leq C \sup_{\nabla\varphi \in L^q(\Omega,\mathbb{R}^3)} \frac{1}{\|\nabla\varphi\|_{L^q(\Omega,\mathbb{R}^3)}}\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\left\|\nabla \phi^0_{\varepsilon,\rho^2_\varepsilon}\right\|_{L^p(\Omega,\mathbb{R}^3)} &\leq C \sup_{\nabla\varphi \in L^q(\Omega,\mathbb{R}^3)} \frac{1}{\|\nabla\varphi\|_{L^q(\Omega,\mathbb{R}^3)}} \int_\Omega \nabla\varphi \cdot \rho^2_\varepsilon \nabla \phi^0_{\varepsilon,\rho^2_\varepsilon} \\ &\overset{(5.2.2)}{=} C \sup_{\nabla\varphi \in L^q(\Omega,\mathbb{R}^3)} \frac{1}{\|\nabla\varphi\|_{L^q(\Omega,\mathbb{R}^3)}} \int_\Omega \nabla\varphi \cdot \rho^2_\varepsilon A^0_\varepsilon \\ &\leq C \left\|A^0_\varepsilon\right\|_{L^p(\Omega,\mathbb{R}^3)}.\end{aligned}$$

This yields the following bound for $\operatorname{curl} B^0_\varepsilon$

$$\left\|\operatorname{curl} B^0_\varepsilon\right\|_{L^p(\Omega,\mathbb{R}^3)} \leq C \left\|A^0_\varepsilon\right\|_{L^p(\Omega,\mathbb{R}^3)}, \tag{5.2.11}$$

where $C > 0$ is independent of ε. Finally, combining these estimates with a Sobolev embedding, we conclude that for $p > 3$

$$\begin{aligned}
\left\|\operatorname{curl} B_\varepsilon^0\right\|_{L^\infty(\Omega,\mathbb{R}^3)} &\overset{(5.2.11)}{\leq} C\left\|A_\varepsilon^0\right\|_{L^\infty(\Omega,\mathbb{R}^3)} \\
&\leq C(\left\|A_\varepsilon^0 - A_{0,\mathrm{ex}}\right\|_{L^\infty(\Omega,\mathbb{R}^3)} + \|A_{0,\mathrm{ex}}\|_{L^\infty(\Omega,\mathbb{R}^3)}) \\
&\overset{p>3}{\leq} C(\left\|A_\varepsilon^0 - A_{0,\mathrm{ex}}\right\|_{H^2(\mathbb{R}^3,\mathbb{R}^3)} + \|A_{0,\mathrm{ex}}\|_{W^{1,p}(\Omega,\mathbb{R}^3)}) \\
&\overset{(5.2.10)}{\leq} C(\left\|\operatorname{curl} B_\varepsilon^0\right\|_{L^2(\Omega,\mathbb{R}^3)} + \|H_{0,\mathrm{ex}}\|_{L^p(\Omega,\mathbb{R}^3)}) \\
&\leq C(\|H_{0,\mathrm{ex}}\|_{L^2(\Omega,\mathbb{R}^3)} + \|H_{0,\mathrm{ex}}\|_{L^p(\Omega,\mathbb{R}^3)}) \\
&\leq C\|H_{0,\mathrm{ex}}\|_{L^p(\Omega,\mathbb{R}^3)}
\end{aligned}$$

This concludes the proof. □

5.2.2 Energy splitting

Now we present a proof for the energy splitting result in 3D (0.0.5).

Proof of Proposition 0.0.2. We start by using the Lassoued-Mironescu decoupling (0.0.4) on the integral in Ω.

$$GL_\varepsilon(\mathbf{u},\mathbf{A}) = E_\varepsilon(\rho_\varepsilon) + \frac{1}{2}\int_\Omega \left(\rho_\varepsilon^2|\nabla_{\mathbf{A}} u e^{ih_{\mathrm{ex}}\phi_{\varepsilon,\rho_\varepsilon^2}}|^2 + \rho_\varepsilon^4\frac{(1-|u|^2)^2}{2\varepsilon^2}\right) + \frac{1}{2}\int_{\mathbb{R}^3} |\operatorname{curl} A + h_{\mathrm{ex}}\operatorname{curl}(A_\varepsilon^0 - A_{0,\mathrm{ex}})|^2. \quad (5.2.12)$$

First, by expanding the square in the second term of (5.2.12) we have

$$\begin{aligned}
\int_\Omega \rho_\varepsilon^2|\nabla_{\mathbf{A}} u e^{ih_{\mathrm{ex}}\phi_{\varepsilon,\rho_\varepsilon^2}}|^2 &= \int_\Omega \rho_\varepsilon^2(|\nabla_A u - ih_{\mathrm{ex}}(\nabla\phi_{\varepsilon,\rho_\varepsilon^2} - A_\varepsilon^0)u|^2) \\
&\overset{(5.2.5)}{=} \int_\Omega \rho_\varepsilon^2|\nabla_A u|^2 + h_{\mathrm{ex}}^2\rho_\varepsilon^2|\nabla\phi_{\varepsilon,\rho_\varepsilon^2}^0 - A_\varepsilon^0|^2|u|^2 + 2h_{\mathrm{ex}}\langle iu, \nabla_A u\rangle \operatorname{curl} B_\varepsilon^0.
\end{aligned}$$

On the other hand, by expanding the square in the rightmost term of (5.2.12)

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} |\operatorname{curl} A|^2 + h_{\mathrm{ex}}^2 |\operatorname{curl}(A_\varepsilon^0 - A_{0,\mathrm{ex}})|^2 + 2h_{\mathrm{ex}} \operatorname{curl} A \cdot \operatorname{curl}(A_\varepsilon^0 - A_{0,\mathrm{ex}}) \\
&\overset{(5.2.7)}{=} \int_{\mathbb{R}^3} |\operatorname{curl} A|^2 + h_{\mathrm{ex}}^2 |\operatorname{curl}(A_\varepsilon^0 - A_{0,\mathrm{ex}})|^2 - 2h_{\mathrm{ex}} \int_\Omega \operatorname{curl} B_\varepsilon^0 \cdot A.
\end{aligned}
$$

Since $B_\varepsilon^0 \times \nu = 0$ on $\partial\Omega$, we have

$$\int_\Omega \operatorname{curl} B_\varepsilon^0 \cdot A = \int_\Omega B_\varepsilon^0 \cdot \operatorname{curl} A.$$

By replacing in (5.2.12), we obtain

$$
\begin{aligned}
GL_\varepsilon(\mathbf{u}, \mathbf{A}) =& E_\varepsilon(\rho_\varepsilon) + F_{\varepsilon,\rho_\varepsilon}(u, A) - h_{\mathrm{ex}} \int_\Omega \mu(u, A) \cdot B_\varepsilon^0 \\
&+ \frac{h_{\mathrm{ex}}^2}{2} \int_\Omega \left(\rho_\varepsilon^2 |\nabla \phi_{\varepsilon,\rho_\varepsilon^2}^0 - A_\varepsilon^0|^2 |u|^2 + \int_{\mathbb{R}^3} |\operatorname{curl}(A_\varepsilon^0 - A_{0,\mathrm{ex}})|^2 \right).
\end{aligned}
$$

Finally, recall that

$$GL_\varepsilon(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi_{\varepsilon,\rho_\varepsilon^2}}, h_{\mathrm{ex}} A_\varepsilon^0) = E_\varepsilon(\rho_\varepsilon) + \frac{h_{\mathrm{ex}}^2}{2} \left(\int_\Omega \rho_\varepsilon^2 |\nabla \phi_{\varepsilon,\rho_\varepsilon^2}^0 - A_\varepsilon^0|^2 + \int_{\mathbb{R}^3} |\operatorname{curl}(A_\varepsilon^0 - A_{0,\mathrm{ex}})|^2 \right).$$

Thus, we conclude the proof by writing $|u|^2$ as $1 + (|u|^2 - 1)$. □

5.3 Vortex balls on a surface and ε-level estimates

Because vortices are one-dimensional objects, it is not possible to directly extend the vortex ball construction from Section 2.3 since they can no longer be covered by 3D balls. The tool for estimating the energy of vortices are ε-level estimates, which refer to a polyhedral approximation of vortices with ε-quantitative estimates. This construction is based on the Federer-Fleming deformation theorem. The main result of this section is the following.

Theorem 6. *[Work in preparation] For any $M, m, n > 0$ there exist C, ε_0 depending* *only on $M, m, n, \partial\Omega$ such that, for any $\varepsilon < \varepsilon_0$, there exists $\kappa = \kappa(\varepsilon)$ such that, if (u, A) satisfies $F_{\varepsilon,\rho_\varepsilon}(u, A) \leq M|\log\varepsilon|^m$, $[\rho_\varepsilon]_{C^{0,\alpha}} \leq M|\log\varepsilon|^m$ for some $\alpha \in (0, 1]$ and ρ_ε is constant in $\{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > \kappa\}$, then there exists a polyhedral 1-current ν_ε which satisfies the following.*

(1) *ν_ε/π is integer multiplicity.*

(2) *$\partial\nu = 0$ relative to Ω.*

(3) *There exists $S_{\nu_\varepsilon} \subset \overline{\Omega}$ such that $\operatorname{supp}\nu \subset S_{\nu_\varepsilon}$ and $|S_{\nu_\varepsilon}| \leq C|\log\varepsilon|^{-q}$, where $q(m, n) := \frac{3}{2}(m + n)$.*

(4) *(Weighted lower bound)*

$$\int_{S_\nu} \rho_\varepsilon^2|\nabla_A u|^2 + \rho_\varepsilon^4\frac{(1-|u|^2)^2}{2\varepsilon^2} + |\operatorname{curl} A|^2$$
$$\geq \pi\left(\int_{\nu_\varepsilon} \rho_\varepsilon^2\theta d\mathcal{H}^1\right)(|\log\varepsilon| - C\log|\log\varepsilon|) - \frac{C}{|\log\varepsilon|^n}, \quad (5.3.1)$$

where θ is the multiplicity of ν/π and $\mathcal{H}^1$ is the one dimensional Hausdorff measure.

(5) *(Vorticity estimate) For any $\gamma \in (0, 1]$, there exists $C_\gamma > 0$ depending only on $\gamma, \partial\Omega$ such that*

$$\|\mu(u, A) - \nu_\varepsilon\|_{(C_T^{0,\gamma})^*} \leq C_\gamma\frac{F_{\varepsilon,\rho_\varepsilon}(u, A) + 1}{|\log\varepsilon|^{q\gamma}} \quad (5.3.2)$$

This result was developed in [Rom19b] by C. Román, where the vorticity was approximated using an appropiately positioned grid of small cubes where the vortex is linearized in each cube and the restriction to the edges are approximated by Dirac masses using similar vorticity estimates as (2.4.2). The proof in the inhomogeneous case uses a similar strategy, but we need to impose additional hypothesis on ρ_ε.

One of the main tools used to prove these ε-level estimates is an extension of the vortex ball construction for surfaces in $\mathbb{R}^3$. The construction follows the same principles as the one in Section 2.3: Capture the vorticity regions $\{|u| \leq \frac{1}{2}\}$ in balls to obtain an initial lower bound for the free energy functional, then increase the radius of each ball in a way that preserves the bound and merges them when they overlap. We also have to consider the curvature of the surface in this construction.

Let $\tilde{\Sigma} \subseteq \Omega$ be a complete oriented surface with a distance function $\mathfrak{d}(\cdot, \cdot)$ and with bounded second fundamental form. We also introduce the weighted free energy without magnetic component in $\Sigma \subset \tilde{\Sigma}$.

$$E_{\varepsilon,\eta}(u, \Sigma) := \frac{1}{2} \int_{\Sigma} \eta^2 |\nabla u|^2 + \eta^4 \frac{(1 - |u|^2)^2}{2\varepsilon^2}.$$

We assume the following hypothesis on u, $\tilde{\Sigma}$ and Σ:

Hypothesis 5.3.1. *$\tilde{\Sigma}$ has a second fundamental form bounded by 1 and $\Sigma \subseteq \tilde{\Sigma}$ is open and bounded. Also, we let $u = u_\varepsilon \in H^1(\Sigma, \mathbb{C})$ such that for any $m, M > 0$ there exists $\varepsilon_0 = \varepsilon_0(m, M)$ such that*

$$E_{\varepsilon,\eta}(u; \Sigma) \leq M_\varepsilon, \quad |u(x)| \geq \frac{1}{2} \text{ if } \mathfrak{d}(x, \partial\Sigma) < 1,$$

where $M_\varepsilon := M |\log \varepsilon|^m$.

Our main energy estimate is the following.

Proposition 5.3.1. *If (u, Σ) are such that they satisfy Hypothesis 5.3.1, then we have*

$$E_{\varepsilon,\eta}(u, \Sigma) \geq \pi \underline{\eta}^2(\Sigma) |d| \left(\log \frac{1}{\varepsilon} - M_\varepsilon \right),$$

where d is the winding number of $u/|u|$ in $\partial\Sigma$.

The structure of the proof is similar to the 2D case: we state an initial lower bound for the energy in a ball, construct an initial configuration with an adequate lower bound and apply a growing and merging process. We will follow the structure of [Rom19b, Section 3.1], which generalizes the proof of [San01] to include the case of a possibly unbounded number of vortices. In what follows, the sets $B(x, t)$ and $S(x, t)$

denote respectively a geodesic (closed) ball and a geodesic sphere of radius t centered at $x \in \tilde{\Sigma}$. We will also use $S_e(x,t)$ and $B_e(x,t)$ to denote, respectively, the Euclidean sphere and the Euclidean ball of radius t centered at x in the tangent space $T_x\Sigma$.

Lemma 5.3.1. *Suppose u, Σ satisfy Hypothesis 5.3.1. There exist $\varepsilon_0, r_0, C > 0$ depending only on Σ such that, for any $\varepsilon < \varepsilon_0$, any $x \in \Sigma$, and any $t > 0$ such that $\frac{\varepsilon}{\underline{\eta}^2(B(x,t))} < t < r_0$, if $|u| \leq 1$ in $S(x,t)$, we have*

$$E_{\varepsilon,\eta}(u, S(x,t)) \geq \underline{\eta}^2(\tilde{\Sigma}) \left(\pi m^2 \left(\frac{|d|}{t} - C \right)^+ + \frac{1}{C\varepsilon}(1-m)^2 \right),$$

where $m := \inf_{S(x,t)} |u|$, and

$$d := \begin{cases} \deg\left(\frac{u}{|u|}, S(x,t)\right) & \text{if } m \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This result is an extension of [DVR24, Lemma A.1], which states that in $\Omega \subset \mathbb{R}^2$, for any $u \in H^1(\Omega, \mathbb{C})$ and any (Euclidean) ball $B_e(x,t) \subseteq \Omega$, we have

$$\frac{1}{2} \int_{S_e(x,t)} \eta^2 |\nabla |u||^2 + \frac{\eta^4}{2\varepsilon^2}(1-|u|^2)^2 \geq \frac{\underline{\eta}^2(B_e(x,t))}{C\varepsilon}(1-m)^2, \tag{5.3.3}$$

where $C > 0$ is a universal constant.

We extrapolate this result to surfaces in $\mathbb{R}^3$ using the exponential map defined locally in $\tilde{\Sigma}$. Fixing $u \in H^1(\Sigma, \mathbb{C})$, $x \in \Sigma$, $t > 0$ small enough, and using (5.3.3) with $\Omega = \exp_x^{-1}(\Sigma) \subseteq T_x\Sigma$, $\tilde{u} := u \circ \exp_x$, and $\tilde{\eta} := \eta \circ \exp_x$, yields

$$\frac{1}{2} \int_{S_e(x,t)} \tilde{\eta}^2 |\nabla |\tilde{u}||^2 + \frac{\tilde{\eta}^4}{2\varepsilon^2}(1-|\tilde{u}|^2)^2 \geq \frac{\underline{\tilde{\eta}}^2(B_e(x,t))}{C\varepsilon}(1-\tilde{m})^2, \tag{5.3.4}$$

where $\tilde{m} := \inf_{S(x,t)} |\tilde{u}|$. The bound on the second fundamental form of $\tilde{\Sigma}$ implies that there exists $r_0 > 0$ depending only on Σ such that, for $t < r_0$, $\exp_x$ is a quasiisometry in $B_e(x,t)$ and $\exp_x(S_e(x,t))$ is a geodesic sphere in Σ centered at x. By performing a change of variables on (5.3.4), we obtain

$$\frac{1}{2}\int_{S(x,t)} \eta^2|\nabla|u||^2 + \frac{\eta^4}{2\varepsilon^2}(1-|u|^2)^2 \geq \frac{\underline{\eta}^2(B(x,t))}{C\varepsilon}(1-m)^2 \geq \frac{\underline{\eta}^2(\tilde{\Sigma})}{C\varepsilon}(1-m)^2, \quad (5.3.5)$$

with potentially different values for the constants r_0 and C, after the change of coordinates. This yields the lower bound

$$\begin{aligned} E_{\varepsilon,\eta}(u,S(x,t)) &= \frac{1}{2}\int_{S(x,t)} \eta^2\left(|\nabla|u||^2 + |u|^2\left|\nabla\frac{u}{|u|}\right|^2\right) + \frac{\eta^4}{2\varepsilon^2}(1-|u|^2)^2 \\ &\overset{(5.3.5)}{\geq} \underline{\eta}^2(\tilde{\Sigma})\left(\frac{1}{C\varepsilon}(1-m)^2 + m^2\int_{S(x,t)}\left|\nabla\frac{u}{|u|}\right|^2\right). \end{aligned} \quad (5.3.6)$$

For the second term in the RHS of (5.3.6), we use the following result used in the proof of [San01, Lemma 3.12], which states that for small enough t,

$$\int_{S(x,t)}\left|\nabla\frac{u}{|u|}\right|^2 \geq \pi\left(\frac{|d|}{t} - C\right)^+. \quad (5.3.7)$$

By inserting (5.3.7) into (5.3.6), we complete the proof. □

With this modification at hand, the proof of Proposition 5.3.1 follows almost verbatim [Rom19b, Section 3.1], the only difference being that the lower bound $\underline{\eta}^2(\tilde{\Sigma})$ is carried through the subsequent steps.

5.4 The isoflux problem

Using the ε-level estimates with the energy splitting (0.0.9), we can heuristically try to deduce the leading order of the first critical field. If $GL_\varepsilon^{3\mathrm{D}}$ is minimized by $(\mathbf{u},\mathbf{A}) = (u\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi^0_{\varepsilon,\rho_\varepsilon^2}}, A + h_{\mathrm{ex}}A^0_\varepsilon)$, then $GL_\varepsilon^{3\mathrm{D}}(\mathbf{u},\mathbf{A}) \leq GL_\varepsilon^{3\mathrm{D}}(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi^0_{\varepsilon,\rho_\varepsilon^2}}, h_{\mathrm{ex}}A^0_\varepsilon)$.

Thus, we have

$$F_{\varepsilon,\rho_\varepsilon}(u,A) - h_{\mathrm{ex}} \int_\Omega \mu(u,A)\cdot B_\varepsilon^0 + R_0 \le 0.$$

Reasoning as in the 2D approach, vortices will appear when the Meissner configuration is not near a minimizer, that is,

$$F_{\varepsilon,\rho_\varepsilon}(u,A) - h_{\mathrm{ex}} \int_\Omega \mu(u,A)\cdot B_\varepsilon^0 \ll 0.$$

Using items (4) and (5) of Theorem 6 and ignoring lower order terms, we deduce this is possible when

$$\frac{\pi}{2}\left(\int_{\nu_\varepsilon} \rho_\varepsilon^2 \theta d\mathcal{H}^1\right)|\log\varepsilon| - h_{\mathrm{ex}}\langle \nu_\varepsilon\,, B_\varepsilon^0\rangle \le 0,$$

that is,

$$h_{\mathrm{ex}} \ge \frac{\left(\pi \int_{\nu_\varepsilon} \rho_\varepsilon^2 \theta d\mathcal{H}^1\right)}{2\langle \nu_\varepsilon\,, B_\varepsilon^0\rangle}|\log\varepsilon|. \tag{5.4.1}$$

This motivates the definition of the so-called *isoflux problem* to remove the dependence of ν_ε from the bound. This problem is an isoperimetric-like problem where, given a 1-form, we find a 1-current which maximizes the ratio between the duality product with the 1-form and the mass of the 1-current.

In line with the inhomogeneous framework, we work on the following *weighted variant* of the isoflux problem: Let $\mathcal{N}(\Omega)$ be the space of 1-currents supported in Ω with finite mass and boundary supported on $\partial\Omega$ with finite mass as well. Given a vector field $B\colon \Omega \to \mathbb{R}^3$, $B = (B_x, B_y, B_z)$, identified with the 1-form $B_x dx + B_y dy + B_z dz$ and a nonnegative nonzero weight function η, we seek to maximize over $\mathcal{N}(\Omega)$ the following functional

$$R_B^\eta(\Gamma) := \frac{\langle \Gamma\,, B\rangle}{|\eta\Gamma|}, \tag{5.4.2}$$

where the product $\eta\Gamma$ is defined by duality: For any 1-form ω supported in Ω

$$\langle \eta\Gamma\,, B\rangle := \langle \Gamma\,, \eta B\rangle.$$

To see that (5.4.1) is related to this problem, we can determine $|\eta\Gamma|$ in the case that Γ is an integer multiplicity rectifiable 1-current with associated varifold $v(\Gamma, \theta)$.

$$\begin{aligned} |\eta\Gamma| &= \sup_{\omega \in \mathcal{D}^1(\Omega), \|\omega\|_{L^\infty(\Omega,\mathbb{R}^3)}=1} \int_\Gamma \eta(x)\omega(x)\theta(x) d\mathcal{H}^1(x) \\ &= \sup_{\omega \in \mathcal{D}^1(\Gamma), \|\omega\|_{L^\infty(\Gamma,\mathbb{R}^3)}=1} \int_\Gamma \eta(x)\omega(x)\theta(x) d\mathcal{H}^1(x) \\ &= \int_\Gamma \eta(x)\theta(x) d\mathcal{H}^1(x). \end{aligned} \tag{5.4.3}$$

This last integral is the integral of η respect to the weight measure of the varifold $v(\Gamma, \theta)$. We will write this last integral as $\|\eta\|_{L^1(\Gamma)}$, and also keep in mind that there is an implicit dependence on the multiplicity of the varifold as well.

5.4.1 1-current associated to a 2-form

The following is a useful family of 1-currents to obtain lower bounds for the maximizing value of the isoflux functional (5.4.2). Any 2-form α supported on $\overline{\Omega}$ induces a 1-current $[\alpha]$ defined by the following action on a 1-form ω.

$$\langle [\alpha], \omega \rangle := \int_\Omega \alpha \wedge \omega$$

In order for $[\alpha]$ to be an admissible current, we will require that α is a closed 2-form, that is, $d\alpha = 0$. This is formalized in the following proposition.

Proposition 5.4.1. *If α is a smooth, continuous up to the boundary, closed 2-form supported on $\overline{\Omega}$, then $[\alpha] \in \mathcal{N}(\Omega)$ and*

$$\langle \partial[\alpha], f \rangle = \int_{\partial\Omega} f\alpha.$$

Proof. (1) *$[\alpha]$ has finite mass:* Note that by the duality characterization of L^p norms, the mass of a current generalizes the L^1 norm, as we will see in the following argument.

$$
\begin{aligned}
|[\alpha]| &= \sup_{\omega \in \mathcal{D}^1(\Omega), \|\omega\|_{L^\infty(\Omega,\mathbb{R}^3)}=1} \langle [\alpha], \omega \rangle \\
&= \sup_{\omega \in \mathcal{D}^1(\Omega), \|\omega\|_{L^\infty(\Omega,\mathbb{R}^3)}=1} \int_\Omega \alpha \wedge \omega \\
&= \|\alpha\|_{L^1(\Omega,\mathbb{R}^3)} \\
&< \infty.
\end{aligned}
$$

Note that α is integrable since it is continuous in $\overline{\Omega}$.

(2) *Boundary of* $[\alpha]$ *has finite mass and is supported on* $\partial\Omega$*:* Let f be a smooth 0-form, that is, f is a smooth function. In particular, the wedge product is just standard multiplication. Thus, we have

$$
\begin{aligned}
\langle \partial[\alpha], f \rangle &= \langle [\alpha], df \rangle \\
&= \int_\Omega \alpha \wedge df \\
&= \int_\Omega d(\alpha \wedge f) - d\alpha \wedge f \\
&= \int_\Omega d(f\alpha) - f d\alpha \\
&\overset{d\alpha=0}{=} \int_\Omega d(f\alpha)
\end{aligned}
$$

Finally, by Stokes' theorem

$$
\langle \partial[\alpha], f \rangle = \int_{\partial\Omega} f\alpha,
$$

which is supported on $\partial\Omega$. We conclude analogously that

$$
|\partial[\alpha]| = \|\alpha\|_{L^1(\partial\Omega,\mathbb{R}^3)} < \infty.
$$

We can conclude that $[\alpha]$ satisfies all the conditions of $\mathcal{N}(\Omega)$. $\square$

Now we can state a lower bound for the maximal energy of (5.4.2) in the special case where B is a solenoidal vector field.

Proposition 5.4.2. *Suppose* $\operatorname{div} B = 0$*. Then we have*

$$\max_{\Gamma \in \mathcal{N}(\Omega)} R_B^\eta(\Gamma) \geq \frac{\|B\|_{L^2(\Omega,\mathbb{R}^3)}}{\|\eta\|_{L^2(\Omega)}} \tag{5.4.4}$$

Proof. If $B = B_x dx + B_y dy + B_z dz$, then $\star B$ is the 2-form $B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$, where $\star$ is the Hodge star operator. Moreover, $\star B$ is closed since $d(\star B) = (\operatorname{div} B) dx \wedge dy \wedge dz = 0$. Additionally, we have $[\star B] \wedge B = (|B_x|^2 + |B_y|^2 + |B_z|^2) dx \wedge dy \wedge dz$ and, $|[\eta \star B]| = \|\eta B\|_{L^1(\Omega,\mathbb{R}^3)}$ Thus, by Proposition 5.4.1, $[\star B] \in \mathcal{N}(\Omega)$. By using $[\star B]$ as a competitor for the maximal energy of R_B, we deduce that

$$\begin{aligned}
\max_{\Gamma \in \mathcal{N}} R_B(\Gamma) &\geq \frac{\langle [\star B], B \rangle}{|\eta \star B|} \\
&= \frac{1}{\|\eta B\|_{L^1(\Omega,\mathbb{R}^3)}} \int_\Omega [\star B] \wedge B \\
&\geq \frac{\|B\|^2_{L^2(\Omega,\mathbb{R}^3)}}{\|\eta\|_{L^2(\Omega)} \|B\|_{L^2(\Omega,\mathbb{R}^3)}} \\
&= \frac{\|B\|_{L^2(\Omega,\mathbb{R}^3)}}{\|\eta\|_{L^2(\Omega)}}.
\end{aligned}$$

$\square$

5.5 Lower bound for the first critical field in 3D

We delegate this section to the proof of the lower bound for the first critical field in 3D, Theorem 5. With the tools we have developed, the proof will be similar to the proof of Theorem 1, the 2D lower bound. We need one additional result, and that is to prove that the maximizer of the isoflux problem for the Ginzburg–Landau setting does not go to zero under an additional hypothesis. We will use R^0_ε to denote $\max_{\Gamma\in\mathcal{N}(\Omega)} R^{\rho_\varepsilon^2}_{B^0_\varepsilon}(\Gamma)$.

Proposition 5.5.1. *Assume* $\operatorname{curl} H_{0,\mathrm{ex}} \neq 0$ *in* Ω. *Then we have*

$$\liminf_{\varepsilon\to 0^+} R^0_\varepsilon > 0. \tag{5.5.1}$$

Proof. Assume $\liminf_{\varepsilon\to0^+}\max_{\Gamma\in\mathcal{N}(\Omega)} R_\varepsilon(\Gamma)=0$. It follows by (5.4.4) that, by passing to a subsequence if necessary, $B^0_\varepsilon \to 0$ in $L^2(\Omega,\mathbb{R}^3)$. On the other hand, by using an arbitrary (but fixed in ε) vector field $\Phi\in C^\infty_0(\Omega,\mathbb{R}^3)$ with $\operatorname{div}\Phi=0$ as a test vector field in equation (5.2.3) we have

$$\begin{aligned}\left|\int_\Omega H_{0,\mathrm{ex}}\cdot\Phi\right| &= \left|\int_\Omega \frac{\operatorname{curl} B^0_\varepsilon\cdot\operatorname{curl}\Phi}{\rho_\varepsilon^2} + B^0_\varepsilon\cdot\Phi\right|\\ &\le C\left|\int_\Omega \operatorname{curl} B^0_\varepsilon\cdot\operatorname{curl}\Phi + B^0_\varepsilon\cdot\Phi\right|\\ &\le C\left|\int_\Omega B^0_\varepsilon\cdot\operatorname{curl}\operatorname{curl}\Phi + B^0_\varepsilon\cdot\Phi\right|\\ &\le C\left\|B^0_\varepsilon\right\|_{L^2(\Omega,\mathbb{R}^3)}\left\|\Phi\right\|_{H^2(\Omega,\mathbb{R}^3)}.\end{aligned}$$

As $\varepsilon\to0$, the right-hand side goes to zero while the left-hand side remains unchanged. This implies $\int_\Omega H_{0,\mathrm{ex}}\cdot\Phi=0$. Taking the L^2-closure, we deduce that $H_{0,\mathrm{ex}}\perp L^2_{\sigma,N}(\Omega,\mathbb{R}^3)$. We deduce from the Helmholtz-Hodge decomposition (5.1.1) that $H_{0,\mathrm{ex}}\in\nabla H^1(\Omega)$. Thus, $\operatorname{curl} H_{0,\mathrm{ex}}=0$ on Ω. □

Proof of Theorem 5. The sequence of steps is similar to the proof of 1, sans the strong vortexlessness. We will additionally assume that $\operatorname{curl} H_{0,\mathrm{ex}}\neq0$, which is required to use Proposition 5.5.1.

Step 1 *(Free energy bound)* Since we are assuming that $(\mathbf{u},\mathbf{A})$ minimizes GL_ε, we have $GL_\varepsilon(\mathbf{u}, \mathbf{A}) \leq GL_\varepsilon(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi^0_{\varepsilon,\rho_\varepsilon^2}}, h_{\mathrm{ex}}A^0_\varepsilon)$ This implies that

$$F_{\varepsilon,\rho_\varepsilon}(u, A) \overset{(0.0.9)}{\leq} h_{\mathrm{ex}} \int_\Omega \mu(u, A) \cdot B^0_\varepsilon + R_0. \tag{5.5.2}$$

On the other hand, the boundary condition $B^0_\varepsilon \times \nu = 0$ implies that we can integrate by parts to obtain

$$\begin{aligned} F_{\varepsilon,\rho_\varepsilon}(u, A) &\leq h_{\mathrm{ex}} \int_\Omega (\langle iu, \nabla_A u\rangle \cdot \operatorname{curl} B^0_\varepsilon + \operatorname{curl} A \cdot B^0_\varepsilon) + R_0 \\ &\leq C(h_{\mathrm{ex}} \|u\|_{L^2(\Omega,\mathbb{C})} \|\nabla_A u\|_{L^2(\Omega,\mathbb{C}^3)} \left\|\operatorname{curl} B^0_\varepsilon\right\|_{L^\infty(\Omega,\mathbb{R}^3)} \\ &+ \|\operatorname{curl} A\|_{L^2(\Omega,\mathbb{R}^3)} \left\|B^0_\varepsilon\right\|_{L^2(\Omega,\mathbb{R}^3)} + |R_0|) \\ &\overset{(5.2.9)}{\leq} C(h_{\mathrm{ex}} \|u\|_{L^2(\Omega,\mathbb{C})} \|\nabla_A u\|_{L^2(\Omega,\mathbb{C}^3)} + \|\operatorname{curl} A\|_{L^2(\Omega,\mathbb{R}^3)} + |R_0|). \end{aligned}$$

Let us bound the terms on the right hand side individually. For $\|u\|_{L^2(\Omega,\mathbb{C})}$, we have

$$\begin{aligned} \int_\Omega |u|^2 &= \int_\Omega 1 - (1 - |u|^2) \\ &\leq C(1 + \int_\Omega |1 - |u|^2|) \\ &\leq C(1 + \varepsilon F_{\varepsilon,\rho_\varepsilon}(u, A)^{1/2}). \end{aligned}$$

Thus, we have

$$\|u\|_{L^2(\Omega,\mathbb{C})} \leq C(1 + \varepsilon F_{\varepsilon,\rho_\varepsilon}(u, A)^{1/2})^{1/2}. \tag{5.5.3}$$

In the case of $\|\nabla_A u\|_{L^2(\Omega,\mathbb{C}^3)}$, we have

$$\|\nabla_A u\|_{L^2(\Omega,\mathbb{C}^3)} \leq C F_{\varepsilon,\rho_\varepsilon}(u, A)^{1/2}. \tag{5.5.4}$$

Similarly, for $\|\operatorname{curl} A\|_{L^2(\Omega,\mathbb{R}^3)}$,

$$\|\operatorname{curl} A\|_{L^2(\Omega,\mathbb{R}^3)} \leq C F_{\varepsilon,\rho_\varepsilon}(u, A)^{1/2}. \tag{5.5.5}$$

Finally, we have for $|R_0|$ that

$$|R_0| \leq \frac{h_{\text{ex}}^2}{2}\int_\Omega \frac{|\operatorname{curl} B_\varepsilon^0|^2}{\rho_\varepsilon^2}|(|u|^2-1)| \overset{(5.2.9)}{\leq} Ch_{\text{ex}}^2 \varepsilon F_{\varepsilon,\rho_\varepsilon}(u,A)^{1/2}. \tag{5.5.6}$$

Thus, we conclude from (5.5.3), (5.5.4), (5.5.5) and (5.5.6) that

$$F_{\varepsilon,\rho_\varepsilon}(u,A) = O(h_{\text{ex}}F_{\varepsilon,\rho_\varepsilon}(u,A)^{1/2}),$$

which implies that there exists $C>0$ not depending on ε such that

$$F_{\varepsilon,\rho_\varepsilon}(u,A) \leq Ch_{\text{ex}}^2.$$

Step 2 *(Proof of weak vortexlessness)* As we have stated that $h_{\text{ex}} = O(|\log\varepsilon|)$, the conclusion of the previous step yields

$$F_{\varepsilon,\rho_\varepsilon}(u,A) \leq C|\log\varepsilon|^2 \tag{5.5.7}$$

for some $C>0$ not depending on ε.

We proceed to use the tools from the ε-level estimates. Using (5.3.2), we can deduce that

$$\begin{aligned}\int_\Omega \mu(u,A)\cdot B &\leq \|\mu(u,A)-\nu_\varepsilon\|_{(C_T^{0,1}(\Omega,\mathbb{R}^3))^*}\left\|\operatorname{curl} B_\varepsilon^0\right\|_{L^\infty(\Omega,\mathbb{R}^3)} + \langle \nu_\varepsilon\,, B_\varepsilon^0\rangle \\ &\overset{(5.2.9)\&(5.5.7)}{\leq} \langle \nu_\varepsilon\,, B_\varepsilon^0\rangle + C(|\log\varepsilon|^{2-q}+|\log\varepsilon|^{-q})\end{aligned} \tag{5.5.8}$$

By inserting (5.3.1) and (5.5.8) in (5.5.2) and using (5.5.6) to control $|R_0|$, we deduce by choosing a large enough n that

$$\begin{aligned}&F_{\varepsilon,\rho_\varepsilon}(u,A) - h_{\text{ex}}\int_\Omega \mu(u,A)\cdot B_\varepsilon^0 \\ &\quad\geq \frac{\pi}{2}\left\|\rho_\varepsilon^2\right\|_{L^1(\nu_\varepsilon/\pi)}(|\log\varepsilon| - C\log|\log\varepsilon|) - h_{\text{ex}}\langle \nu_\varepsilon\,, B_\varepsilon^0\rangle + o(|\log\varepsilon|^{-2}).\end{aligned} \tag{5.5.9}$$

On the other hand, by the definition of R_ε^0 as a maximizer

$$\langle \nu_\varepsilon \, , B_\varepsilon^0 \rangle \leq |\rho_\varepsilon^2 \nu_\varepsilon| R_\varepsilon^0. \tag{5.5.10}$$

Since ν_ε/π is an integer valued current, we follow (5.4.3) to obtain

$$|\rho_\varepsilon^2 \nu_\varepsilon| = \pi |\rho_\varepsilon^2 (\nu_\varepsilon/\pi)| = \pi \left\| \rho_\varepsilon^2 \right\|_{L^1(\nu_\varepsilon/\pi)} . \tag{5.5.11}$$

We return to (5.5.9) with (5.5.10) and (5.5.11), where we find that

$$\begin{aligned} &\frac{\pi}{2} \left\| \rho_\varepsilon^2 \right\|_{L^1(\nu_\varepsilon/\pi)} (|\log \varepsilon| - C \log |\log \varepsilon|) - h_{\text{ex}} \langle \nu_\varepsilon \, , B_\varepsilon^0 \rangle \\ &\qquad \geq \frac{\pi}{2} \left\| \rho_\varepsilon^2 \right\|_{L^1(\nu_\varepsilon/\pi)} \big(|\log \varepsilon| - 2 h_{\text{ex}} R_\varepsilon^0 - C \log |\log \varepsilon| \big) + o(|\log \varepsilon|^{-2}). \end{aligned}$$

Using the hypothesis on h_{ex} we can remove the the leading term $|\log \varepsilon|$ from the bound, which leaves us with

$$F_{\varepsilon,\rho_\varepsilon}(u, A) - h_{\text{ex}} \int_\Omega \mu(u, A) \cdot B_\varepsilon^0 \geq \frac{\pi}{2} \left\| \rho_\varepsilon^2 \right\|_{L^1(\nu_\varepsilon/\pi)} (2K_0 R_\varepsilon^0 - C) \log |\log \varepsilon| + o(|\log \varepsilon|^{-2}).$$

Using the limiting behaviour of R_ε^0 (5.5.1), we can choose K_0 independently of ε such that $2K_0 R_\varepsilon^0 - C > 1$ to deduce

$$F_{\varepsilon,\rho_\varepsilon}(u, A) - h_{\text{ex}} \int_\Omega \mu(u, A) \cdot B_\varepsilon^0 \geq \frac{\pi}{2} \left\| \rho_\varepsilon^2 \right\|_{L^1(\nu_\varepsilon/\pi)} \log |\log \varepsilon| + o(|\log \varepsilon|^{-2}).$$

Combining this bound with the following upper bound given by the minimizer

$$F_{\varepsilon,\rho_\varepsilon}(u, A) - h_{\text{ex}} \int_\Omega \mu(u, A) \cdot B_\varepsilon^0 \overset{(5.5.2)}{\leq} |R_0| \overset{(5.5.6)}{=} O(\varepsilon |\log \varepsilon|^3), \tag{5.5.12}$$

yields

$$\left\| \rho_\varepsilon^2 \right\|_{L^1(\nu_\varepsilon/\pi)} \leq o(1).$$

Moreover, we also have a similar bound for $|\nu_\varepsilon|$

$$|\nu_\varepsilon| \leq \frac{\pi}{b} \|\rho_\varepsilon\|_{L^1(\nu_\varepsilon/\pi)} = o(1). \tag{5.5.13}$$

Thus, we can conclude that weak vortexlessness is satisfied following another use of the vorticity estimate (5.3.2), since for any $B \in C_T^{0,\gamma}(\Omega, \mathbb{R}^3)$

$$\begin{aligned}\int_\Omega \mu(u, A) \cdot B &\leq \|\mu(u, A) - \nu_\varepsilon\|_{(C_T^{0,\gamma})^*} \|B\|_{C_T^{0,\gamma}(\Omega,\mathbb{R}^3)} + |\nu_\varepsilon| \|B\|_{L^\infty(\Omega,\mathbb{R}^3)} \\ &\overset{(5.3.2)\&(5.5.13)}{\leq} o(1) \|B\|_{C_T^{0,\gamma}(\Omega,\mathbb{R}^3)} .\end{aligned} \tag{5.5.14}$$

Step 3 *(Meissner configuration approximation)* The Meissner approximation follows after (5.5.14), since equation (5.5.12) implies that

$$F_{\varepsilon,\rho_\varepsilon}(u, A) = o(1). \tag{5.5.15}$$

Finally, as we have proven in (5.5.15), (5.5.14) and (5.5.6) that all the terms from the energy splitting (0.0.5) are $o(1)$, we conclude that

$$GL_\varepsilon(\mathbf{u}, \mathbf{A}) = GL_\varepsilon(\rho_\varepsilon e^{ih_{\text{ex}}\phi^0_{\varepsilon,\rho_\varepsilon^2}}, h_{\text{ex}} A^0_\varepsilon) + o(1).$$

This finishes the proof.

□

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