PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE

Complex abelian varieties

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1 Introduction

The present work is devoted to the study of complex abelian varieties and the action of a group on it. In the main part of the text, we describe the theoretic frame regarding abelian varieties, for which we need to talk about complex manifolds, complex tori and topics surrounding those. Most of the results here will not be proved; these contents can be found in classical books. In section 2 we define complex manifolds and the main concepts we will need later, especially vector bundles, sections and differential forms. The next section gives a one-dimensional flavor to the theory of complex manifolds, stating a few classical results in compact Riemann surfaces, without proof. This section is not strictly necessary to understand what follows. In section 4 we describe how the theory of complex tori works. A complex torus is the compact quotient of $\mathbb{C}^n$ by some discrete additive subgroup. One of the most important things here is understanding how divisors can be seen as line bundles, for what we need to speak about theta functions, among others. Next we talk about abelian varieties, which are complex tori with a bilinear form satisfying certain properties, and describe how this can be written in terms of matrices. We also mention the important fact that an abelian variety can always be seen as a submanifold of some projective space. To understand how a group acts on an abelian variety, in section 6 we study part of the algebra we need regarding group representations, for in section 7 apply this theory to our case, stating Lange and Recillas' isotypical decomposition. The algebra needed can be found in classical books like [3], while its applications to abelian varieties are studied in specific articles such as [13], [7] or [2], for instance. Finally, we try to work out specific group actions on abelian varieties. In [12], Rodríguez completely describes abelian varieties admitting an involution. In the same spirit, the present works makes an attempt to describe the action of an order-three group in an abelian variety. For this we need to study symplectic matrices. Further interesting work would be to work out the way different finite groups act on a complex abelian variety.

2 Complex manifolds

Roughly speaking, a complex manifold is a space that locally looks like $\mathbb{C}^n$, so that we can define holomorphic functions on it. Here we describe these ideas more precisely.

Definition 2.1. *A function f from an open set $D \subset \mathbb{C}^n$ to $\mathbb{C}$ is said to be holomorphic if every point $a \in D$ has a neighborhood in which f can be expressed as a power series*

$$f(z) = \sum_{k_1,\dots,k_n=0}^{\infty} c_{k_1,\dots,k_n}(z_1 - a_1)^{k_1} \cdots (z_n - a_n)^{k_n}.$$

A function between open sets $D \subset \mathbb{C}^n$, $E \subset \mathbb{C}^m$ is said to be holomorphic if every coordinate function is so.

Holomorphic functions in several variables quite behave like single-variable holomorphic functions. We state a couple of useful lemmas, whose proofs can be found, for instance, in [14]. The first one is known as Osgood's lemma.

Lemma 2.2. *Let $f : D \to \mathbb{C}$ be a continuous function. If f is holomorphic in each variable separately, then it is holomorphic on D.*

As a consecuence of Osgood's lemma, we have that a C^1 function $f : D \to \mathbb{C}$ is holomorphic iff $\frac{\partial f}{\partial \bar{z}_1} = \cdots = \frac{\partial f}{\partial \bar{z}_n} = 0$. The following is the identity principle for several variables.

Lemma 2.3. *If f, g are holomorphic functions on a connected open set $D \subset \mathbb{C}^n$, and $U \subset D$ is nonempty open subset satisfying $f|_U = g|_U$, then $f = g$.*

Definition 2.4. *Let X be a topological space. An atlas is a covering of X given by a family of open subsets $U_i \subset X$, together with a set of homeomorphisms $\phi_i : U_i \to D_i$, where D_i is an open set of $\mathbb{C}^n$, which satisfy that*

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is a biholomorphism for every par (i, j)*. We call a Hausdorff, second countable space* X *with such an atlas a complex manifold.*

The functions ϕ_i are called charts. We may always assume that the family $\{(U_i, \phi_i)\}$ is maximal with respect to the condition above. We also assume that X is connected, unless otherwise stated. In this case, the number n does not depend on the chart and will be called the dimension of X. A function $f : X \to Y$ between complex manifolds is said to be holomorphic if every composition $\varphi_j \circ f \circ \phi_i^{-1}$ is holomorphic, where ϕ_i and φ_j are charts for X, Y, respectively. Of course, the condition need only be checked for one chart at each point. Note that $\mathbb{C}$ itself has a natural structure of complex manifold. Holomorphic functions between X and $\mathbb{C}$ will be referred to simply as holomorphic functions on X.

Example 2.5. *Consider* $\mathbb{C}^{n+1} \backslash \{0\}$ *with the equivalence relation given by:* $x \sim y$ *iff* $x = ty$ *for some* $t \in \mathbb{C}^*$*. The quotient is called the projective space* $\mathbb{P}^n$ *and is made into a complex manifold by setting* $U_i = \{[x_0 : \ldots : x_n] \mid x_i \neq 0\}$ *and*

$$\begin{aligned} \phi_i : U_i &\to \mathbb{C}^n \\ [x_0 : \ldots : x_n] &\mapsto \left(\frac{x_0}{x_1}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_n}{x_i} \right). \end{aligned}$$

$\mathbb{P}^n$ *is the set of lines in* $\mathbb{C}^{n+1}$*. For* $n = 1$*, we obtain the Riemann sphere, that can be seen as* $\mathbb{C}$ *plus one point called infinity,* $\infty = [1 : 0]$*. We denote it by* $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = \mathbb{P}^1$*.*

Definition 2.6. *A meromorphic function on* X *is a holomorphic function* $f : X \to \hat{\mathbb{C}}$ *which is not identically* ∞*. The space of such functions is denoted* $\mathcal{M}(X)$*.*

Let f, g be meromorphic functions on X, with g not identically zero. If $f(z) = [f_1(z) : f_2(z)]$ and $g(z) = [g_1(z) : g_2(z)]$, we define the quotient

f/g simply as $[f_1g_2 : f_2g_1]$. It is not difficult to check that this indeed is well defined and gives a meromorphic function, so this makes $\mathcal{M}(X)$ into a field. A holomorphic function h on X can also be regarded as a meromorphic function by identifying it with $[h : 1]$.

Next we will come to an alternative definition of complex manifold using the language of sheaves.

Definition 2.7. *Let X be a topological space. A geometric structure $\mathcal{O}$ is an assignment $\mathcal{O}(U)$ for every open subset $U \subset X$, where $\mathcal{O}(U)$ is a ring of continuous functions defined on U, which satisfies the following conditions:*

(1) The constants are in $\mathcal{O}(U)$ for every open subset U.

(2) If $V \subset U$ are open sets, then $f|_V \in \mathcal{O}(V)$ for every $f \in \mathcal{O}(U)$.

(3) If U_i is a collection of open subsets and $f_i \in \mathcal{O}(U_i)$ satisfy $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ $\forall i, j$, then there exists a unique $f \in \mathcal{O}(U)$ such that $f_i = f|_{U_i}$ $\forall i$, where $U = \bigcup U_i$.

We call such a pair $(X, \mathcal{O})$ a geometric space.

Note that every open subset $U \subset \mathbb{C}^n$ has a natural structure of geometric space by letting $\mathcal{O}(V)$ be the set of holomorphic functions on V. If X is a geometric space and $U \subset X$ is open, U inherits a structure of geometric space that we denote $(U, \mathcal{O}|_U)$, given by $\mathcal{O}|_U(V) = \mathcal{O}(V)$ for $V \subset U$ an open set.

Definition 2.8. *Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ be geometric spaces. A morphism between them is a continuous function $f : X \to Y$ such that for every $U \subset Y$ open, $g \circ f \in \mathcal{O}_X(f^{-1}(U))$. An isomorphism is a morphism with a two sided inverse that is also a morphism.*

Now we give an alternative definition for a complex manifold.

Definition 2.9. *Let X be a second countable, Hausdorff topological space with a geometric structure $\mathcal{O}$. We say that $(X, \mathcal{O})$ is a complex manifold if it has a covering by open sets U_i such that every $(U_i, \mathcal{O}(U_i))$ is isomorphic to some open subset of $\mathbb{C}^n$.*

It is not too difficult to see that both definitions coincide. If we have a geometric space X satisfying the conditions above and $f_i : (U_i, \mathcal{O}(U_i)) \to (D_i, \mathcal{O}_{D_i})$ are isomorphisms with open subsets of $\mathbb{C}^n$, then the mappings $f_i : U_i \to D_i$ give coordinate charts for X, and the compatibility conditions are easily checked to be satisfied. Conversely, if we have a complex manifold with respect to the first definition, we can give it a complex structure putting $\mathcal{O}(U)$ as the holomorphic functions in U for every open subset $U \subset X$, and the coordinate charts give isomorphisms of geometric spaces. We leave the details to the reader.

Remark 2.10. *Likewise, we can define a differentiable manifold X asking for $\mathcal{O}(U)$ to be the smooth functions on the open subset U. Smooth functions are defined analogously.*

Remark 2.11. *Any open subset of a complex (smooth) manifold inherits a natural structure of complex (smooth) manifold itself.*

2.1 Vector bundles

Definition 2.12. *Let X be a topological space and $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. A $\mathbb{K}$-vector bundle is a topological space L together with a continuous mapping $\pi : L \to X$ such that there exists a covering of X given by open subsets U_i and homeomorphisms*

$$\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{K}^r$$

mapping $\pi^{-1}(p)$ into $\{p\} \times \mathbb{K}^r$ for every $p \in U_i$ and satisfying the condition that

$$\varphi_i \circ \varphi_j^{-1} : U_i \cap U_j \times \mathbb{K}^r \to U_i \cap U_j \times \mathbb{K}^r$$

is of the form (id, g_{ij}), for a continuous mapping $g_{ij} : U_i \cap U_j \to GL(r, \mathbb{K})$.

We call the functions φ_i *trivializations* and the g_{ij} *transition functions.* If $\mathbb{K} = \mathbb{C}$ and X is a complex manifold, we say that L is a holomorphic vector bundle if the g_{ij} are holomorphic. In this case, we can give L a structure of complex manifold using the φ_i as charts, and then the mapping π happens to be holomorphic. We define smooth vector bundles analogously.

It is easy to see that the transition functions given by a vector bundle necessarily satisfy the following conditions:

$$g_{ij} \cdot g_{jk} \cdot g_{ki} = id \quad \text{on} \quad U_i \cap U_j \cap U_k, \tag{2.1}$$

$$g_{ii} = id \quad \text{on} \quad U_i. \tag{2.2}$$

We say that two vector bundles $\pi : L \to X$, $p : E \to X$ are isomorphic if there exists a homeomorphism $h : L \to E$ such that $\pi = p \circ h$ and every composition $\phi_i \circ h \circ \varphi_i^{-1}$ is linear in the second variable, where φ_i, ϕ_i are trivializations for L and E, respectively. For holomorphic vector bundles, we also ask h to be holomorphic. Now we will prove that a vector bundle is determined by its transition functions. In fact, every collection of transition functions satisfying the conditions above will drive a unique vector bundle up to isomorphism.

Proposition 2.13. *Let X be a topological space, and $\{U_i\}$ an open covering. Suppose we have a collection of continuous mappings $g_{ij} : U_i \cap U_j \to GL(r, \mathbb{K})$ satisfying the compatibility conditions (2.1) and (2.2). Then there exists a unique (up to isomorphism) vector bundle E having g_{ij} as its transition functions. If X is a complex manifold and the g_{ij} are holomorphic (smooth),* *then E is a holomorphic (smooth) vector bundle.*

Proof. Consider the disjoint union $\bigsqcup U_i$ and the equivalence relation given by: $(p, v) \in U_i \times \mathbb{K}^r$ and $(q, w) \in U_j \times \mathbb{K}^r$ are equivalent iff $p = q$ and $v = g_{ij}(p)w$. The compatibility conditions ensure that this is in fact an equivalence relation. Let E be the quotient space $\bigsqcup U_i / \sim$. The projection $\pi : (p, v) \mapsto p$ is a well defined continuous map from E to X with $\pi^{-1}(U_i)$ being the quotient image of $U_i \times \mathbb{C}^r$ in E. Then E has the required properties, using the obvious trivializations. If $\pi_0 : L \to X$ is another vector bundle with trivializations φ_i and satisfying the same properties, define $h : L \to E$ by setting $h(z)$ to be the equivalence class of $\varphi_i(z)$ in E, for any i satisfying $\pi_0(z) \in U_i$. Then h is a well defined biholomorphism and makes E and L into isomorphic vector bundles. □

By abuse of language, two isomorphic vector bundles will be considered to be the same.

Definition 2.14. *Let $\pi : E \to X$ be a vector bundle. A section over an open set $U \subset X$ is a continuous map $s : U \to E$ such that $\pi \circ s$ is the identity in U. We call s a holomorphic (smooth) section if $s : U \to E$ is so.*

The set of sections over U is denoted $\Gamma(U, E)$. It will be clear from the context whether we are referring to holomorphic, smooth or just continuous sections. When we say just sections we will be talking about sections over the whole X. Consider a section s for a vector bundle $\pi : E \to X$. For each trivialization φ_i, let $f_i = \varphi_i \circ s$. By the definitions, f_i must be of the form (id, s_i) for a continuous (resp. holomorphic, smooth) mapping $s_i : U_i \to \mathbb{K}^r$. It is easy to verify that these functions satisfy the condition

$$g_{ij} \cdot s_j = s_i \tag{2.3}$$

on $U_i \cap U_j$. In fact, any such collection of mappings satisfying (2.3) gives rise to a section.

Proposition 2.15. *Let $\pi : E \to X$ be a vector bundle with covering $\{U_i\}$ and trivializations φ_i. Suppose we have a collection of mappings $s_i : U_i \to \mathbb{K}^r$ satisfying (2.3). Then there is a unique section $s : X \to E$ such that $\varphi_i \circ s = (id, s_i)$ for every i. If the functions s_i are holomorphic or smooth, then so will be the section s.*

Proof. For $p \in X$, define $s(p)$ to be $\varphi_i^{-1}(p, s_i(p))$. The map s is well defined because of the hypothesis. It is not difficult to check that s is the unique section with the desired properties. □

From this we see that, considering each section as the collection $\{s_i\}$ given above, we can give $\Gamma(U, E)$ a K-vector space structure, since equation (2.3) is linear.

2.2 Tangent bundle

We shall first define the tangent bundle for a smooth real manifold. Suppose X is a connected smooth real manifold of dimension m, and ϕ_α is a coordinate chart in the open set U_α. Call $\phi_\alpha = (x_1, \ldots, x_m)$, composing with the usual projections in $\mathbb{R}^m$. For every $p \in U_\alpha$ we consider the following operator acting on smooth functions defined about p, which we shall call $\frac{\partial}{\partial x_i}$, by abuse of notation:

$$\frac{\partial}{\partial x_i} : C^\infty(U_\alpha) \to \mathbb{R}$$
$$f \mapsto \frac{\partial}{\partial x_i}(f \circ \phi_\alpha^{-1})\Big|_{\varphi_\alpha(p)}$$

Note that we can replace $C^\infty(U_\alpha)$ by $C^\infty(V)$ for any open set $V \subset U_\alpha$ containing p. The tangent space at p will be

$$T_{\mathbb{R},p}X = span_{\mathbb{R}}\left\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}\right\}$$

There is a smooth vector bundle over X whose fibers are of the form $T_{\mathbb{R},p}X$, and we call it the *tangent bundle* of X. For this, we think of every $(p, v) \in U_\alpha \times \mathbb{R}^m$ as $p \in U_\alpha$, $v \in T_{\mathbb{R},p}X \cong \mathbb{R}^m$. The tangent bundle is obtained as the quotient of the disjoint union $\bigsqcup_\alpha U_\alpha \times \mathbb{R}^m$ by the following equivalence relation: if $(p, v) \in U_\alpha \times \mathbb{R}^m$, $(q, w) \in U_\beta \times \mathbb{R}^m$, they are equivalent if and only if $p = q \in U_\alpha \cap U_\beta$ and v, $w \in T_{\mathbb{R},p}X$ define the same operator acting on smooth functions over $U_\alpha \cap U_\beta$. Let us understand the meaning of the latter condition. Suppose that associated to U_α, U_β are the coordinate charts $\phi_\alpha = (x_1, \dots, x_m)$, $\phi_\beta = (y_1, \dots, y_m)$. Write v as a linear combination $\sum_j a_j \frac{\partial}{\partial x_j}$, and w as $\sum_i b_i \frac{\partial}{\partial y_i}$. For a smooth function f defined in a neighborhood of p, we see that

$$\begin{aligned} v(f) &= \sum_j a_j \frac{\partial}{\partial x_j}(f \circ \phi_\alpha^{-1}) \\ &= \sum_j a_j \frac{\partial}{\partial x_j}(f \circ \phi_\beta^{-1} \circ h), \end{aligned}$$

where $h = \phi_\beta \circ \phi_\alpha^{-1}$. Using the chain rule, we get

$$v(f) = \sum_{i,j} a_j \frac{\partial y_i}{\partial x_j} \frac{\partial}{\partial y_i}(f \circ \phi_\beta^{-1}).$$

This is, $(p, v) \sim (p, w)$ iff $b_i = \sum_j a_j \frac{\partial y_i}{\partial x_j}$ $\forall i$, where the $\frac{\partial y_i}{\partial x_j}$ are the partial derivatives of the funcion h. From what we have said, we can state the definition of the tangent bundle as follows:

Definition 2.16. *Let X be a connected smooth real manifold of dimension m, and suppose we have a system of coordinate charts given by $\phi_\alpha : U_\alpha \to V_\alpha$. Set $h_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$. The (real) tangent bundle $T_\mathbb{R}X$ is the smooth vector bundle achieved by setting*

$$g_{\alpha\beta} = J_\mathbb{R}(h_{\alpha\beta}) \circ h_\beta^{-1} : U_\alpha \cap U_\beta \to GL(m, \mathbb{R}),$$

as transition functions, where $J_{\mathbb{R}}(h_{\alpha\beta}) = \frac{\partial h_{\alpha\beta}}{\partial x_\beta}$ *is the jacobian matrix for* $h_{\alpha\beta}$*.*

It is easy to see that the functions $g_{\alpha\beta}$ above are smooth and satisfy the relations (2.1) and (2.2), so that the definition makes sense. The sections of the tangent bundle are called vector fields. Suppose $f : X \to Y$ is a smooth function between smooth manifolds of dimension m and t, respectively. It is not difficult to prove that we can define a map $df : T_{\mathbb{R}}X \to T_{\mathbb{R}}Y$ such that, if $(x_1, \dots, x_m)$ and $(y_1, \dots, y_t)$ are coordinates about p and $f(p)$, respectively, then $df\left(p, \frac{\partial}{\partial x_i}\right) = \left(p, \sum_j \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j}\right)$ and is linear on each tangent space $T_{\mathbb{R},p}X$. Let $V \subset X$ be a subset and $i : V \hookrightarrow X$ the inclusion map. We will say that V is a submanifold if it is a manifold itself having the subspace topology and the map di is of maximal rank in each point.

Consider now a connected complex manifold X of (complex) dimension n. Then X is a real smooth manifold of dimension $2n$, and it has a *real* tangent bundle, where every tangent space $T_{\mathbb{R},p}X$ can be written as

$$span_{\mathbb{R}}\left\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_n}\right\},$$

using local coordinates. We define the complexified tangent space at p as

$$T_{\mathbb{C},p}X = T_{\mathbb{R},p}X \otimes \mathbb{C} = span_{\mathbb{C}}\left\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_n}\right\}. \tag{2.4}$$

Let us call $\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$, $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$, so that the $T_{\mathbb{C},p}X$ is spanned in $\mathbb{C}$ by the elements of the form $\frac{\partial}{\partial z_j}$, $\frac{\partial}{\partial \bar{z}_j}$. We define the holomorphic and antiholomorphic tangent spaces at p as

$$T'_pX = span_{\mathbb{C}}\left\{\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}\right\},$$
$$T''_pX = span_{\mathbb{C}}\left\{\frac{\partial}{\partial \bar{z}_1}, \cdots, \frac{\partial}{\partial \bar{z}_n}\right\},$$

respectively, so that $T_{\mathbb{C},p}X$ decomposes as a direct sum $T'_pX \oplus T''_pX$. It is not too difficult to show that the spaces $T_{\mathbb{C},p}X$ are in fact the fibers of a holomorphic vector bundle, that we denote $T_{\mathbb{C}}X$. The bundle whose fibers are the holomorphic tangent spaces T'_pX will be called the *holomorphic tangent bundle* and will be denoted by $T'X$.

Definition 2.17. *A holomorphic vector field on X is a holomorphic section of $T'X$.*

2.3 Differential forms

Consider again the real tangent bundle $T_{\mathbb{R}}X$ for a smooth manifold X, and let k be a nonnegative integer. There exists a vector bundle whose fibers are $\bigwedge^k T^*_{\mathbb{R},p}X$, and we will call it $\bigwedge^k T^*_{\mathbb{R}}X$.

Definition 2.18. *A differential k-form on X is a section of the vector bundle $\bigwedge^k T^*_{\mathbb{R}}X$. The space of k-forms on X will be denoted as $A^k(X, \mathbb{R})$.*

If in an open set we have a coordinate chart given by $\phi = (x_1, \ldots, x_m)$, we write $\{dx_1, \ldots, dx_m\}$ for the basis of $T^*_{\mathbb{R},p}X$ which is dual to $\left\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}\right\}$ in each point. Then, a k-form can be expressed locally as

$$\omega = \sum_{i_1 < \cdots < i_k} \varphi_{i_1,\ldots,i_k}(x_1, \ldots, x_m) \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \tag{2.5}$$

where the $\varphi_{i_1,\ldots,i_k}$ are smooth functions that must satisfy the compatibility relations given by the chain rule. We will not work out these compatibility relations now. To simplify notation, let us write $I = \{i_1, \ldots, i_k\}$ and $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

Definition 2.19. *For a k-form ω defined locally by (2.5), its exterior derivative $d\omega$ is the $(k+1)$-form defined locally by*

$$d\omega = \sum_{j=1}^{n} \sum_{I} \frac{\partial \varphi_I}{\partial x_j} \cdot dx_j \wedge dx_I.$$

It is an exercise to see that this in fact defines a $(k+1)$-form. This gives us a map $d_k : A^k(X,\mathbb{R}) \to A^{k+1}(X,\mathbb{R})$ for every k. One can check that $d_{k+1} \circ d_k = 0$ (we simply write $d \circ d = 0$). From this we can define the de Rahm cohomology of X.

Definition 2.20. *Let $k \geq 1$. The k-th de Rahm cohomology group of X is*

$$H^k(X,\mathbb{R}) = \frac{ker(d_k)}{im(d_{k-1})}.$$

A k-form ω for which $d\omega = 0$ is called closed. We say that ω is exact if, in addition, $\omega = d\nu$ for some $(k-1)$-form ν.

If we have a smooth map $f : X \to Y$, the map $df : T_{\mathbb{R}}X \to T_{\mathbb{R}}Y$ induces a dual map between $\bigwedge^k T^*_{\mathbb{R}}Y$ and $\bigwedge^k T^*_{\mathbb{R}}X$. Call this map df^*_k. Then, if ω is a k-form on Y, we can define the pullback of ω as $f^*(\omega) = df^*_k \circ \omega \circ f$, which is a k-form on Y. To see how $f^*(\omega)$ looks locally, we find that if $(x_1,\ldots,x_m)$ and $(y_1,\ldots,y_t)$ are coordinates about p and $f(p)$, respectively, then

$$f^*dy_i = \sum_{j=1}^{m} \frac{\partial f_i}{\partial x_j} dx_j.$$

It can be shown that f^* descends to a homomorphism $f^* : H^k(Y,\mathbb{R}) \to H^k(X,\mathbb{R})$ for every k.

Now suppose X is a complex manifold of dimension n. Analogously, we can define (complex) k-forms as the sections of the complex vector bundle $\Lambda^k T^*_{\mathbb{C}}X$. From now on, when talking about forms we will always mean complex forms. The set of (complex) k-forms will be denoted simply by $A^k(X)$. Set $dz_j = dx_j + idy_j$, $d\bar{z}_j = dx_j - idy_j$ using local coordinates. A k-form can be written locally as

$$\omega = \sum_{I,J} \varphi_{I,J} \cdot dz_I \wedge d\bar{z}_J,$$

where the sum runs over the subsets $I, J \subset \{1,\ldots,n\}$ such that $|I|+|J| = k$.

Definition 2.21. *Let $k = p + q$. A k-form ω is said to be of type (p, q) if it can locally be written as*

$$\omega = \sum_{|I|=p} \sum_{|J|=q} \varphi_{I,J} \cdot dz_I \wedge d\bar{z}_J$$

We denote $A^{p,q}(X)$ the space of (p, q)-forms. Note that, in a similar way as before, we can define maps $\partial : A^{p,q} \to A^{p+1,q}$, $\bar{\partial} : A^{p,q} \to A^{p,q+1}$, such that $d = \partial + \bar{\partial}$. These maps also satisfy $\partial \circ \partial = 0$, $\bar{\partial} \circ \bar{\partial} = 0$. The de Rahm cohomology can be defined in a similar way. The k-th (complex) group of cohomology will be denoted just by $H^k(X)$. It is not too difficult to see that, in fact, it is isomorphic to $H^k(X, \mathbb{R}) \otimes \mathbb{C}$. As in the real case, the pull back of a smooth map descends to a homomorphism between de de Rahm cohomology groups.

Definition 2.22. *A k-form ω on X is said to be holomorphic if it is of type $(k, 0)$ and can be locally written as*

$$\omega = \sum_{|I|=k} f_I dz_I$$

for holomorphic functions f_I.

2.4 Integration

In this part we briefly talk about what it means to integrate a differential form on a manifold, and state a few lemmas that are used later to prove some fundamental results. The details can be found in classic books about manifolds.

It is a theorem that if X is a real manifold, there exists a partition of unity $\{f_\alpha\}$ such that the support of each f_α is contained in a coordinate neighborhood U_α for which $\phi_\alpha : U_\alpha \to D_\alpha$ is a chart. If X is orientable, we can also assume that the charts ϕ_α define the same orientation, this is, the jacobian matrices of the functions $\phi_\beta \circ \phi_\alpha^{-1}$ have positive determinant. Now suppose ω is a smooth n-form with compact support, where n is the dimension of X. We can define

$$\int_X \omega = \sum_\alpha \int_{D_\alpha} \phi_\alpha d\lambda,$$

where λ is the Lebesgue measure in D_α. This integral is well defined because the ϕ_α must satisfy the compatibility relations given by the chain rule.

Definition 2.23. *Let Z be an oriented r-dimensional submanifold of X with the inclusion map $i : Z \hookrightarrow X$. For a compact supported r-form ω defined on an open set containing $i(Z)$, we define its integral over Z as*

$$\int_Z \omega = \int_X i^* Z.$$

The following is known as Stokes' theorem.

Theorem 2.24. *If $Z \subset X$ is an oriented r-dimensional submanifold with smooth boundary and ω is a closed compact supported $(r-1)$-form, then*

$$\int_Z d\omega = \int_{\partial Z} \omega$$

Using this one can show that integration can be defined in a slightly more general context: the expression $\int_Z \omega$ makes sense when $\omega \in H^r(X, \mathbb{R})$ and Z is an element of the r-th homology group $H_r(X, \mathbb{Z})$. We now state Poincaré's Lemma. For a proof see [4].

Lemma 2.25. *Every closed differential form in an open set of $\mathbb{R}^m$ which is a star domain is exact.*

Definition 2.26. *A differential k-form ω in a complex manifold X is said* *to be integer if for every closed k-dimensional submanifold $Z \subset X$ we have*

$$\int_Z \omega \in \mathbb{Z}.$$

3 Riemann surfaces

In this section we give a brief discussion about Riemann surfaces and state a few results without proving them. We are not going through the main topics, but just try to give an idea of how the theory works on the one-dimensional case. That can be useful for understanding some other concepts later.

Definition 3.1. *A Riemann surface is a one-dimensional connected complex manifold.*

Note that every Riemann surface can also be regarded as a two-dimensional real manifold. Because the coordinate changes satisfy the Cauchy-Riemann equations, it must necessarily be orientable. We shall assume the fact that every compact and orientable two-dimensional real manifold is homeomorphic to a sphere with g handles, where g is a topological invariant and is called the genus of the surface. The surface can also be achieved by identifying pairwise the edges of a $4g$-gon in the following way: we go over the edges counterclockwise naming them correlatively $a_1, b_1, a_1^{-1}, b_1^{-1}, \dots, a_g, b_g, a_g^{-1}, b_g^{-1}$. Then we identify each side a_j with a_j^{-1} with reversed orientation and do the same with b_j and b_j^{-1}. The quotient topological space obtained is a sphere with g handles, and the images of the curves $a_1, \dots, a_g, b_1, \dots, b_g$ generate the homology group $H_1(X, \mathbb{Z})$. For further references, see [8].

Proposition 3.2. *Let $f : X \to Y$ be a holomorphic map between Riemann surfaces, and suppose X is compact. Then f is either surjective or constant.*

Proof. If f is not constant, then it must be an open mapping. Thus $f(X)$ is an open set, but it is also closed since X is compact. As Y is connected, this completes the proof. □

As a consecuence, we see that for a compact Riemann surface, the only holomorphic functions are the constants. The following proposition is not very difficult to prove.

Proposition 3.3. *Let $f : X \to Y$ be a non-constant holomorphic map between Riemann surfaces and let $p \in X$. There exist coordinate charts about p and $f(p)$ such that, with respect to these charts, f is written as z^k for some integer $k \geq 1$, this is, $\varphi \circ f \circ \phi^{-1}(z) = z^k$ for some charts ϕ and φ, respectively.*

This says that, about p, the map f is k to 1. Of course, k does not depend on the chart chosen. Suppose U is a neighborhood of p and $\varphi \circ f \circ \phi^{-1}(z) = z^k$ on $\phi(U)$, as above. Note that every point of U other than p has a smaller neighborhood in which f is one to one. This says that the points for which $k \geq 2$ are isolated. We shall call them ramification points.

Corollary 3.4. *Let $f : X \to Y$ be a holomorphic map between Riemann surfaces, where X is compact. Then the set of ramification points is finite.*

Definition 3.5. *For a non-constant holomorphic map $f : X \to Y$ and a point $p \in X$, we define the branch number of f at p as $b_f(p) = k - 1$, where k is the number described above.*

Note that for almost every point in X the branch number of f is zero. Now, if f is a nonzero meromorphic function on X (taking $Y = \hat{\mathbb{C}}$), we define the order of f at a point $p \in X$ as

$$ord_f(p) = \begin{cases} b_f(p) + 1 & \text{if } f(p) = 0 \\ -(b_f(p) + 1) & \text{if } f(p) = \infty \\ 0 & \text{otherwise} \end{cases}$$

The following proposition will not be proved.

Proposition 3.6. *Let $f : X \to Y$ be a non-constant holomorphic mapping between compact Riemann surfaces. There exists an integer m such that for every $q \in Y$*

$$\sum_{p \in f^{-1}(q)} b_f(p) + 1 = m.$$

m is called the degree of f.

If m is the degree of f, then every point in Y has m preimages, counted by multiplicity. Except for a finite set, the number of preimages of a point in Y is actually m. The next result, known as Riemann-Hurwitz theorem, is a key step in studying maps between Riemann surfaces.

Theorem 3.7. *Let $f : X \to Y$ be a non-constant holomorphic mapping of degree m between compact Riemann surfaces. Let g be the genus of X, γ the genus of Y and call $B = \sum_{p \in X} b_f(p)$. Then*

$$2g - 2 = m(2\gamma - 2) + B. \tag{3.1}$$

Note that a holomorphic 1-form is just a section of the bundle whose fibers are $T_p'^{*}X$. Locally, it can be written as $f(z)dz$ with f holomorphic, and must satisfy compatibility relations given by (2.3). This means that, if z, w represent local coordinates and we let $\frac{dw}{dz}$ be the derivative of the change of coordinates $z \mapsto w$, then $f(z)dz$ is equal to $g(w)dw$ in the corresponding intersection if and only if

$$f(z) = g(w)\frac{dw}{dz}. \tag{3.2}$$

Using this we can define meromorphic differentials in an analogous way.

Definition 3.8. *Let X be a Riemann surface with an atlas given by $\{(U_i, \phi_i)\}$. A meromorphic* 1*-form on X is a collection of functions f_i, each defined on $\phi_i(U_i)$, satisfying the condition (3.2). The space of meromorphic* 1*-forms* *in X is denoted by $K^1(X)$, while the space of holomorphic 1-forms will be written as $\Omega^1(X)$.*

For a meromorphic differential ω, we can define its order at a point $p \in X$ in an analogous way as we did for meromorphic functions, using a local expression for ω. It is easy to see that this is well defined. We denote it simply by $ord_\omega(p)$. We also see that the quotient of two meromorphic 1-forms defines a meromorphic function on X. An important theorem is the following.

Theorem 3.9. *For a compact Riemann surface of genus g, the space $\Omega^1(X)$ has dimension g.*

3.1 Divisors on Riemann surfaces

Definition 3.10. *For a Riemann surface X, $Div(X)$ is the free abelian group with basis X. An element $D \in Div(X)$ is called a divisor.*

For a divisor $D = \sum_{p \in X} \alpha(p)p$, we define its degree as the number $deg(D) = \sum_{p \in X} \alpha(p) \in \mathbb{Z}$. From now on, suppose that X is compact. If f is a meromorphic function on X, we can define its divisor as $(f) = \sum_{p \in X} ord_f(p)p$. As a consequence of proposition (3.6), the divisor of a meromorphic function is always of degree zero. Likewise, the divisor of a nonzero meromorphic 1-form ω is $(\omega) = \sum_{p \in X} ord_\omega(p)p$.

Definition 3.11. *A divisor D is said to be principal if there is a meromorphic function f on X such that $(f) = D$. Two divisors are said to be linearly equivalent if they differ by a principal divisor.*

Remark 3.12. *The set of principal divisors is clearly a subgroup of $Div(X)$. Also, it is easy to see that any two nonzero meromorphic 1-forms define linearly equivalent divisors.*

A partial order relation on $Div(X)$ can be defined pointwise. Next we are going to state Riemann-Roch theorem, without proof. For this we need a few definitions. For a better understanding and a proof one can see classic books on Riemann surfaces.

Definition 3.13. *Let D be a divisor on a compact Riemann surface X. We define*

$$\mathcal{L}(D) = \{f \in \mathcal{M}(X)^* \mid (f) + D \geq 0\} \cup \{0\}$$
$$\Omega(D) = \{\omega \in K^1(X) \mid (\omega) \geq 0\} \cup \{0\}.$$

These are clearly vector spaces, and can be proved to be finite dimensional. We denote their dimensions by $l(D)$, $i(D)$, respectively.

Theorem 3.14. *If X is a compact surface of genus g and D a divisor on X, the formula*

$$r(D) = deg(D) - g + 1 + i(D)$$

holds.

3.2 Group actions on Riemann surfaces

For a Riemann surface X, call $Aut(X)$ the group of automorphisms of X, this is, bijective holomorphic maps between X and itself. Consider G a finite subgroup of $Aut(X)$. The quotient space X/G is made into a topological space in an obvious manner, making the projection $\pi : X \to X/G$ a continuous open map. Now we want to make it into a Riemann surface. The following is a technical lemma and requires complex analysis.

Lemma 3.15. *Let $G \leq Aut(X)$ be a finite subgroup. For $p \in X$ call $G_p = \{h \in G \mid h(p) = p\}$, the stabilizer of p. Then G_p is cyclic. Further, if $|G_p| = k$ and h generates G_p, there exists a neighborhood U of p and a coordinate chart ϕ on U such that $\phi \circ h \circ \phi^{-1}(z) = e^{2\pi i/k} z$.*

Proposition 3.16. *For a finite subgroup $G \leq Aut(X)$, the set of points with nontrivial stabilizer is discrete.*

Proof. Suppose p_n is a sequence of points with nontrivial stabilizer converging to p. As G is finite, we can assume that there exists a nontrivial $g \in G$ such that $g \in G_{p_n}$ for every n. By continuity, $g(p) = p$ too. But then g and id_X agree on a set with an accumulation point, which contradicts the identity principle. □

In particular, if we assume that X is compact, the set of points with nontrivial stabilizer is finite. Now we give X/G a complex structure as follows. Let $q \in X/G$. If $q = \pi(p)$, choose a neighborhood U about p and a chart ϕ as in lemma (3.15). We define a coordinate chart on $\pi(U)$ by setting $\varphi(w) = \phi(x)^k$ for any $x \in U$ such that $\pi(x) = w$. Recall that $k \neq 1$ only for a discrete subset.

Proposition 3.17. *The above construction makes X/G into a Riemann surface. The projection $\pi : X \to X/G$ is a holomorphic map.*

3.3 Jacobian variety

Let X be a compact surface of genus g. The space of holomorphic 1-forms has then dimension g. We can define an injection of $H_1(X, \mathbb{Z})$ into $\Omega^1(X)^*$ by sending a chain α to the application

$$\omega \mapsto \int_\alpha \omega.$$

This makes $H_1(X, \mathbb{Z})$ into a subgroup of $\Omega^1(X)^*$ isomorphic to $\mathbb{Z}^{2g}$. The quotient space $\Omega^1(X)^*/H_1(X, \mathbb{Z})$ is called the *jacobian* of X and denoted by JX. Recall that $\Omega^1(X)^*$ is a g dimensional complex vector space, so it is a complex manifold in a natural way. It is not difficult to see that JX can be made into a complex manifold by composing these charts with the projection onto the quotient. We now define a holomorphic map $\varphi : X \to JX$ as follows. Choose any $p_0 \in X$ and put

$$\begin{aligned} \varphi(p) : H^1(X) &\to \mathbb{C} \\ \omega &\mapsto \int_{p_0}^{p} \omega. \end{aligned}$$

Proposition 3.18. *φ is a well-defined holomorphic map and makes X into a complex submanifold of JX.*

We can extend φ linearly to $Div(X)$, so it is a group homomorphism. Call $Div^0(X)$ the subgroup of $Div(X)$ consisting of the divisors of degree zero, and let $\tilde{\varphi}$ be the restriction of φ to $Div^0(X)$. Of course, $\tilde{\varphi}$ does not depend on the choice of p_0. The following two theorems are due to Abel and Jacobi, respectively.

Theorem 3.19. *The kernel of $\tilde{\varphi}$ is exactly the set of principal divisors.*

Theorem 3.20. *Let X_g be the set of divisors $D \geq 0$ with $deg(D) = g$. The restriction*

$$\varphi|_{X_g} : X_g \to JX$$

is surjective.

As a consequence, we have

Corollary 3.21. *As a group, JX is isomorphic to $Div^0(X)/PDiv(X)$, where $PDiv(X)$ is the set of principal divisors.*

Now suppose $a_1, b_1, \dots, a_g, b_g$ is a basis for $H_1(X, \mathbb{Z})$ given by the construction mentioned above with a $4g$-gon. One can define a homomorphism $E : H_1(X, \mathbb{Z}) \times H_1(X, \mathbb{Z}) \to \mathbb{Z}$ such that $E(a_i, a_j) = E(b_i, b_j) = 0$ and $E(a_i, b_j) = -E(a_j, b_i) = \delta_{ij}$. Intuitively, $E(c_1, c_2)$ counts the number of times c_2 passes across c_1, with the convention that passing from the right side is counted with a 1 and from the left with a -1. E can be extended $\mathbb{R}$-linearly to $H^1(X)^*$. This is called the *intersection product*.

4 Complex Tori

Let $V = \mathbb{C}^n$. A subgroup $\Lambda \subset V$ is said to be a lattice if it is generated over $\mathbb{Z}$ by $2n$ vectors which are $\mathbb{R}$-linearly independent.

Definition 4.1. *A complex torus is a quotient V/Λ, where Λ is a lattice in V.*

It is not difficult to see that a torus $X = V/\Lambda$ naturally inherits a structure of complex manifold. If $\pi : V \to X$ is the projection map and $U \subset V$ is an open set such that $\pi|_U$ is injective, then $\pi|_U^{-1}$ is a coordinate chart. The changes of coordinates correspond to translations by elements of Λ. The number n is called the dimension of the torus. A torus of dimension 1 is called an elliptic curve.

Lemma 4.2. *Let $\Lambda \subset V$ be a discrete subgroup. Then Λ is a lattice iff V/Λ is compact.*

Example 4.3. *Let X be a compact Riemann surface. The group $H_1(X, \mathbb{Z})$ can be seen as a lattice in $\Omega^1(X)^* \cong \mathbb{C}^g$, so the jacobian JX is a complex torus.*

Let X be a torus. Choose a $\mathbb{C}$-basis $\{e_1, \ldots, e_n\}$ of V and a $\mathbb{Z}$-basis $\{\lambda_1, \ldots, \lambda_{2n}\}$ of Λ. Write $\lambda_i = \sum_j \lambda_{ji} e_j$. Then the torus X is completely determined by the matrix

$$\Pi = \begin{bmatrix} \lambda_{11} & \cdots & \lambda_{1,2n} \\ \vdots & & \vdots \\ \lambda_{n1} & \cdots & \lambda_{n,2n} \end{bmatrix}.$$

We call Π the period matrix. Now we describe the holomorphic maps between complex tori.

Proposition 4.4. *Let $X = V/\Lambda$, $X' = V'/\Lambda'$ be two complex tori and $h : X \to X'$ a holomorphic map. Then there exists a group homomorphism $f : X \to X'$ such that $h(x) = f(x) + h(0)$ for all $x \in X$. There also exists a linear map $F : V \to V'$ with $F(\Lambda) \subset \Lambda'$ inducing f. The maps f and F are uniquely determined by h.*

Proof. Call $f = h - h(0)$. V and V' are the universal coverings of X and X', respectively, so f lifts to a unique holomorphic map $F : V \to V'$ satisfying $F(0) = 0$. For every $\lambda \in \Lambda$ we have that $F(z + \lambda) - F(z)$ is a holomorphic function defined in V with image contained in Λ', hence constant. This tells that the partial derivatives of F are Λ-periodic, hence constant. In other words, F is linear and f is a homomorphism. $\square$

For a homomorphism f, the map F above is called the *analitic representation* of f and is denoted by $\rho_a(f)$. Its restriction to the lattice Λ is denoted by $\rho_r(f)$ and is called the *rational representation* of f. If $\Pi \in M(n \times 2n, \mathbb{C})$ and $\Pi' \in M(n' \times 2n', \mathbb{C})$ are period matrices for X and X', with respect to some bases V, Λ and V', Λ', then, in these bases, $\rho_a(f)$ is represented by an $n' \times n$ complex matrix A, while $\rho_r(f)$ is represented by a $2n' \times 2n$ integer matrix R, satisfying

$$A\Pi = \Pi' R.$$

Definition 4.5. *Let X be a complex torus. We write $End(X)$ for the algebra of homomorphisms from X into itself. We also call $End_{\mathbb{Q}}(X) = End(X) \otimes \mathbb{Q}$.*

Consider the torus $X = V/\Lambda$. If $W \leq V$ is a subspace such that $W \cap \Lambda$ is a lattice in W, we say that $W/W \cap \Lambda$ is a subtorus of X. Note that a subtorus is a complex submanifold according to our definition.

Proposition 4.6. *Let $f : X \to X'$ be a homomorphism. Call $(kerf)_0$ the connected component of $kerf$ containing zero. Then*

(1) imf is a subtorus of X'.

(2) $(kerf)_0$ is a subtorus of X.

Definition 4.7. *A surjective homomorphism between complex tori is called an isogeny if it has finite kernel. Two complex tori X, Y are said to be isogenous if there is an isogeny between them. In this case we write $X \sim Y$.*

4.1 Differential forms

We call $\Lambda^k V^*_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \Lambda^k V^*$, this is, the space of maps $\omega : V^k \to \mathbb{C}$ that are $\mathbb{R}$-multilinear and alternating. These maps are also called *forms*. It is not very difficult to see that $\Lambda^k V^*_{\mathbb{C}}$ is a complex vector space of dimension $\binom{2n}{k}$. Similar to the case of forms in a manifold, we say that a form $\omega \in \Lambda^k V^*_{\mathbb{C}}$ is of type (p, q) if it is a complex linear combination of elements of the form $dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}$. The space of such forms is denoted by $\Lambda^{p,q} V^*_{\mathbb{C}}$. A form ω is real if $\omega = \bar{\omega}$. One can check that, for $k = 2$, a form $\omega \in \Lambda^2 V^*_{\mathbb{C}}$ is of type $(1, 1)$ if and only if $\omega(ix, iy) = \omega(x, y)$ for every $x, y \in V$. Recall that a Hermitian form on V is a bilinear form H which is linear on the first variable and antilinear on the second one. The following proposition establishes a correspondence between $(1, 1)$ real forms and Hermitian forms on V.

Proposition 4.8. *There is a bijection between the space of real $(1, 1)$-forms and the space of Hermitian forms on V, given by $\omega \mapsto H$, with $H(x, y) = \omega(ix, y) + i\omega(x, y)$. Its inverse is simply $H \mapsto \Im H$.*

Proof. This is a straightforward calculation. $\square$

Definition 4.9. *We say that a real $(1, 1)$-form ω is non-degenerated if its corresponding hermitian form is positive definite.*

Let $X = V/\Lambda$ be a torus and $\pi : V \to X$ the projection map. For $\lambda \in \Lambda$, we set

$$\begin{aligned} t_\lambda : V &\to V \\ x &\mapsto x + \lambda. \end{aligned}$$

Taking the pullbacks t_λ^* defines an action of Λ on $A^k(V)$. Recall that V, as a complex manifold, can be covered by a single coordinate neighborhood, using the identity map as a chart. Thus, a k-form $\nu \in A^k(V)$ can be completely described by global functions $\nu_{I,J} : V \to \mathbb{C}$, for subsets of indices with $|I| + |J| = k$, such that

$$\nu = \sum \nu_{I,J} dz_I \wedge d\bar{z}_J.$$

For $\omega \in A^k(X)$, write $\nu = \pi^*\omega$. Since $\pi = \pi \circ t_\lambda$, ν must be invariant under the action of Λ, this is, the functions $\nu_{I,J}$ above must be Λ-periodic.

Proposition 4.10. *The pullback of π gives a bijection between k-forms on X and k-forms on V that are invariant under the action of Λ.*

Now, for any $a \in X$ we can define a homeomorphism

$$\begin{aligned} t_a : X &\to X \\ x &\mapsto x + a. \end{aligned}$$

Lemma 4.11. *For every $a \in X$, the induced homomorphism $t_a^* : H^k(X) \to H^k(X)$ is the identity.*

A form $\nu \in \Lambda^k V_{\mathbb{C}}^*$ can be regarded as a k-form in V by considering the functions $\nu_{I,J}$ as constants. As we have seen, ν can then be seen as a k-form in X, since constant functions are in particular Λ-periodic. Further, this $\nu \in A^k(X)$ is clearly closed, so we have a canonical map $p : \Lambda^k V_{\mathbb{C}}^* \to H^k(X)$. The next theorem is a key step in studying complex tori.

Theorem 4.12. *The map p is an isomorphism between $\Lambda^k V_{\mathbb{C}}^*$ and $H^k(X)$.*

In $\mathbb{C}^{n+1}\backslash\{0\}$, consider the $(1,1)$-form defined by

$$\frac{i}{2\pi|z|^4}\left(|z|^2\sum_{j=0}^{n} dz_j \wedge d\bar{z}_j - \left(\sum_{j=0}^{n}\bar{z}_j dz_j\right)\wedge\left(\sum_{r=0}^{n} z_r d\bar{z}_r\right)\right).$$

It can be shown that it descends to a $(1,1)$-form ω in $\mathbb{P}^n$, which is real, closed, integer and at each point gives a positive definite hermitian form. For a closed submanifold $i : Z \hookrightarrow \mathbb{P}^n$, the pullback $i^*\omega$ will have the same properties.

Theorem 4.13. *Let $Z \subset \mathbb{P}^n$ be a closed complex submanifold. Then Z must possess a closed $(1,1)$-form, that is integer and positive definite at each point.*

For the case of a complex torus $X = V/\Lambda$, this theorem says that, if there is a holomorphic injection $i : X \hookrightarrow \mathbb{P}^n$ which makes X into a submanifold, then there is necessarily some positive definite hermitian form H on V whose imaginary part is integer on the lattice, this is, $\Im H(\Lambda \times \Lambda) \subset \mathbb{Z}$.

4.2 Theta functions

Consider a lattice Λ in V, as always. For a non-constant holomorphic function defined over V, we of course cannot expect it to be Λ-periodic. The following notion is perhaps the most close to periodicity we can expect, and is related to the construction of meromorphic functions on $X = V/\Lambda$.

Definition 4.14. *A theta function on V with respect to Λ is a holomorphic function $\theta : V \to \mathbb{C}$ such that for every $\lambda \in \Lambda$ there exists a linear operator $a_\lambda \in V^*$ and a number $b_\lambda \in \mathbb{C}$ satisfying*

$$\theta(z+\lambda) = e^{2\pi i(a_\lambda z + b_\lambda)}\theta(z), \forall z \in V.$$

The collection $\{(a_\lambda, b_\lambda)\}$ is called the *type* of θ. If θ_1 and θ_2 are theta functions of the same type, then the quotient θ_1/θ_2 is a meromorphic function that is Λ-invariant, so it defines a meromorphic function on X.

Example 4.15. *For a polynomial $Q(z)$ of degree less or equal than 2, the function $e^{Q(z)}$ is a theta function.*

We can use this example to define an equivalence relation on the set of theta functions.

Definition 4.16. *A theta function achieved by the previous example is said to be trivial. Two theta functions θ_1, θ_2 are said to be equivalent if the quotient θ_1/θ_2 is a trivial theta function.*

Let $\mathbb{H}_n$ be the space of symmetric matrices with positive definite imaginary part, this is,

$$\mathbb{H}_n = \{\tau \in M_n(\mathbb{C}) \mid \tau^t = \tau, \Im\tau > 0\}.$$

For $\tau \in \mathbb{H}_n$ and $a, b \in \mathbb{R}^n$, we define

$$\theta_{a,b}(z,\tau) = \sum_{m\in\mathbb{Z}^n} e^{\pi i((m+a)^t\tau(m+a)+2(m+a)^t(z+b))}.$$

It can be shown that the sum converges and that for every $p, q \in \mathbb{Z}^n$

$$\theta_{a,b}(z+\tau p+q) = e^{2\pi i\left(-\frac{1}{2}p^t\tau p - p^t z + a^t q - p^t b\right)}\theta_{a,b}(z,\tau)$$

or, in other words, $\theta_{a,b}(\cdot,\tau)$ is a theta function for the lattice whose period matrix is $\Pi = [id\ \tau]$. This is called the Riemann theta function for the parameters a, b and τ.

Let θ be a theta function of type $\{(a_\lambda, b_\lambda)\}$ and λ_1, λ_2 two elements of the lattice. Applying repeatedly the condition of being a theta function, for λ_2

and then for λ_1, we have

$$\theta(z+\lambda_1+\lambda_2) = e^{2\pi i[(a_{\lambda_1}+a_{\lambda_2})(z)+a_{\lambda_2}(\lambda_1)+b_{\lambda_2}+b_{\lambda_1}]}\theta(z).$$

On the other hand, using the property for $\lambda_1+\lambda_2 \in \Lambda$, we see that the above expression must be equal to

$$e^{2\pi i(a_{\lambda_1+\lambda_2}(z)+b_{\lambda_1+\lambda_2})}\theta(z).$$

In other words, the $\{(a_\lambda, b_\lambda)\}$ must satisfy the following relations

$$a_{\lambda_1+\lambda_2} = a_{\lambda_1} + a_{\lambda_2} \tag{4.1}$$
$$b_{\lambda_1+\lambda_2} \equiv b_{\lambda_1} + b_{\lambda_2} + a_{\lambda_2}(\lambda_1) \pmod{\mathbb{Z}}. \tag{4.2}$$

Define a mapping

$$\begin{aligned} a : \Lambda \times V &\to \mathbb{C} \\ (\lambda, z) &\mapsto a_\lambda(z). \end{aligned}$$

This mapping is $(\mathbb{Z},\mathbb{C})$-linear. Since Λ generates V over $\mathbb{R}$, we can extend a to an $\mathbb{R}$-bilinear map $a : V \times V \to \mathbb{C}$ which is $\mathbb{C}$-linear on the second variable. We can now define a form $\omega_\theta \in \Lambda^2_{\mathbb{C}} V^*$ by $\omega_\theta(x,y) = a(x,y) - a(y,x)$.

Proposition 4.17. *Let θ be a theta function. Then ω_θ is a real $(1,1)$-form that is integer on the lattice.*

Proof. Let $\lambda_1, \lambda_2 \in \Lambda$. The condition (4.1) implies that $\omega_\theta(\lambda_1,\lambda_2) = a_{\lambda_1}(\lambda_2) - a_{\lambda_2}(\lambda_1) \equiv 0 \pmod{\mathbb{Z}}$. Since Λ generates V over $\mathbb{R}$, then ω_θ must be real. Now a calculation shows that $\omega_\theta(ix,iy) - \omega_\theta(x,y) = i(\omega_\theta(ix,y) + \omega_\theta(x,iy))$. As ω_θ is real, both hand sides of the equation must be zero. This completes the proof. □

An immediate consequence of the previous proposition is that to every theta function θ we can associate the hermitian form H_θ achieved by the correspondence in (4.8). We do not prove the following proposition.

Proposition 4.18. *The hermitian form H_θ is positive, this is, $H_\theta(x,x) \geq 0 \; \forall x \in V$.*

Note that we are not saying that H_θ is positive *definite*; it still can be singular. What we are going to do now is to bring every theta function into a standard form.

Definition 4.19. *Let H be a hermitian form and ω the $(1,1)$-form associated to it. Call $S^1 \subset \mathbb{C}$ the unit circle. A semicharacter for H is an application $\alpha : \Lambda \to S^1$ satisfying $\alpha(\lambda_1 + \lambda_2) = \alpha(\lambda_1)\alpha(\lambda_2)(-1)^{\omega(\lambda_1,\lambda_2)}$ for every $\lambda_1, \lambda_2 \in \Lambda$.*

Lemma 4.20. *Let θ be a theta function. Then θ is equivalent to a unique theta function $\tilde{\theta}$ of type $\{(\tilde{a}_\lambda, \tilde{b}_\lambda)\}$, with*

$$\tilde{a}_\lambda = \frac{1}{2i} H_\theta(\cdot, \lambda)$$
$$\Im \tilde{b}_\lambda = -\frac{1}{4} H_\theta(\lambda, \lambda)$$

and satisfying $\tilde{\theta}(z+\lambda) = \alpha(\lambda) e^{\pi H_\theta(z,\lambda) + \frac{\pi}{2} H_\theta(\lambda,\lambda)} \tilde{\theta}(z)$ for some semicharacter α for H_θ.

Remark 4.21. *If $\theta_1 \sim \theta_2$, then $\omega_{\theta_1} = \omega_{\theta_2}$. In particular, the $H_{\tilde{\theta}}$ coincides with H_θ.*

The function $\tilde{\theta}$ is called a normalized theta function.

4.3 Divisors

Let X be a complex manifold. An *admissible family* is a collection $\{(U_\alpha, h_\alpha)\}$, where the U_α form an open covering of X and each h_α is a meromorphic function on U_α, satisfying the following condition: for every α, β, h_α/h_β is a holomorphic function on $U_\alpha \cap U_\beta$. We define an equivalence relation by declaring two admissible families to be equivalent iff their union is again admissible.

Definition 4.22. *A Cartier divisor on X is an equivalence class of admissible families. The set of Cartier divisors on X is called $Div(X)$.*

Remark 4.23. *It can be shown that, in a Riemann surface X, the notion we had for divisors coincide with the one of Cartier divisors on X.*

The divisor defined by an admissible family $\{(U_\alpha, h_\alpha)\}$ will simply be written as $[(U_\alpha, h_\alpha)]$. Now we define a binary operation on divisors as follows. Let $D_1 = [(U_\alpha, h_\alpha)]$, $D_2 = [(V_\beta, g_\beta)]$ be two divisors. We define $D_1 + D_2$ to be the equivalence class represented by the family of pairs of the form $(U_\alpha \cap V_\beta, h_\alpha g_\beta)$.

Lemma 4.24. *The operation $+$ is well defined and makes $Div(X)$ into an abelian group, whose identity element is $[(X, 1)]$ and the inverse of an element $D = [(U_\alpha, h_\alpha)]$ can be written as $[(U_\alpha, 1/h_\alpha)]$.*

Definition 4.25. *A divisor $D = [(U_\alpha, h_\alpha)]$ is said to be effective if all the functions h_α are holomorphic. We say that $D_1 \geq D_2$ if $D_1 - D_2$ is effective.*

The previous definition gives a partial order for $Div(X)$.

Definition 4.26. *Let f be a nonzero meromorphic function on X. The divisor of f is $[(X, f)]$, and we will write it as (f). A divisor of a meromorphic function on X is called a principal divisor, and the set of principal divisors is denoted by $PDiv(X)$.*

Proposition 4.27. *$PDiv(X)$ is a subgroup of $Div(X)$.*

Definition 4.28. *The Picard group of X is the quotient*

$$Pic(X) = Div(X)/PDiv(X).$$

Theorem 4.29. *Every divisor is the difference of two effective divisors.*

Now let $X = V/\Lambda$ be a torus. Choose a covering $\{U_\alpha\}$ such that $\pi|_{V_\alpha} : V_\alpha \to U_\alpha$ is bijective, this is, $\pi|_{V_\alpha}^{-1}$ is a coordinate chart. Now let θ be a theta function on X. By the definition of theta functions, the family $\{(U_\alpha, \theta \circ \pi|_{V_\alpha}^{-1})\}$ is admissible. The corresponding equivalence class is called the divisor of the function θ and will be denoted by (θ).

Theorem 4.30. *Every effective divisor on a complex torus is the divisor of a theta function.*

Corollary 4.31. *Every meromorphic function in a complex torus is the quotient of two theta functions of the same type.*

Corollary 4.32. *Let $u : X \to \mathbb{P}^n$ be a holomorphic function, where X is a torus. Then there exist theta functions $\theta_0, \ldots, \theta_n$ of the same type such that $u(x) = [\theta_0(x) : \ldots : \theta_n(x)]$.*

Recall that for every theta function we can associate a hermitian form H_θ, and for two theta functions θ_1, θ_2 we have $H_{\theta_1\theta_2} = H_{\theta_1} + H_{\theta_2}$. We can define a homomorphism φ from $Div(X)$ to the set of hermitian forms in V. Let $D = D_1 - D_2$ for effective divisors D_i. If D_i is the divisor of θ_i, set $\varphi(D) = H_{\theta_1} - H_{\theta_2}$. φ is well defined since, if $D = (\theta_1) - (\theta_2) = (\theta_3) - (\theta_4)$, then $(\theta_1\theta_4) = (\theta_2\theta_3)$, so $\theta_1\theta_4$ and $\theta_2\theta_3$ are theta functions of the same type, which implies that $H_{\theta_1} + H_{\theta_4} = H_{\theta_2} + H_{\theta_3}$.

Proposition 4.33. *φ descends to a homomorphism from $Pic(X)$ to the set of hermitian forms on V.*

Proof. If f is a meromorphic function, then $f = \theta_1/\theta_2$ for theta functions of the same type. Then $\varphi((f)) = H_{\theta_1} - H_{\theta_2} = 0$. $\square$

Definition 4.34. *The image of the mapping φ is called the Néron-Severi group of X, denoted by $NS(X)$.*

Remark 4.35. *It can be shown that $NS(X)$ is exactly the set of hermitian forms whose imaginary part is integer on the lattice, but we will not prove it here.*

4.4 Line bundles

Definition 4.36. *Let X be a complex manifold. A line bundle is a holomorphic vector bundle of (complex) dimension $r = 1$.*

Example 4.37. *The product $X \times \mathbb{C}$ with the projection on the first variable is a line bundle. It is called the trivial line bundle.*

One can show that a line bundle $p : L \to X$ is isomorphic to the trivial one iff it possess a nowhere vanishing holomorphic section. Now consider the projective space $\mathbb{P}^n$, covered by the open sets U_i as given in example (2.5).

Definition 4.38. *The line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$ is the one defined by the transition functions $g_{ij} = x_j/x_i$.*

Each preimage $p^{-1}(U_i)$ for the bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$ can bee seen as the set of pairs $(z, v) \in \mathbb{P}^n \times \mathbb{C}^{n+1}$ such that v belongs to the line defined by z, and the trivialization is

$$\begin{aligned} \varphi_i : p^{-1}(U_i) &\to U_i \times \mathbb{C} \\ (z, v) &\mapsto (z, \alpha), \end{aligned}$$

where α is the i-th coordinate of v.

Now we are going to define a product in the set of line bundles over X. Recall that a line bundle is determined by an open covering $\{U_i\}$ and transition functions $\{g_{ij}\}$. The g_{ij} are just nonzero scalars, since we are dealing with one-dimensional vector spaces. Note that the trivial bundle is achieved by just setting all the $g_{ij} = 1$.

Definition 4.39. *If two line bundles L, L' are given by $\{U_i, g_{ij}\}$, $\{V_r, h_{rk}\}$, respectively, we define the product $L \otimes L'$ as the line bundle given by the collection $\{U_i \cap V_r, g_{ij}h_{rk}\}$.*

Proposition 4.40. *The operation above is well defined on the set of line bundles modulo isomorphism and makes it into an abelian group. The identity element is the trivial bundle. The inverse of a line bundle is afforded by replacing the transition functions by their multiplicative inverses.*

Remark 4.41. *Let $\{(U_\alpha, h_\alpha)\}$ represent a Cartier divisor D on X. The functions h_α/h_β satisfy the conditions (2.1), (2.2), so we can define a holomorphic line bundle using the $(U_\alpha, h_\alpha/h_\beta)$. This is called $\mathcal{O}_X(D)$. Moreover, if f is a meromorphic function and (f) its divisor, then $\mathcal{O}_X(D+(f))$ is isomorphic to $\mathcal{O}_X(D)$. Thus, we can define $\mathcal{O}_X$ as a mapping from $Pic(X)$ to the set of line bundles modulo isomorphism. It is clearly a homomorphism of groups.*

For a line bundle with transition functions g_{ij}, we call a *meromorphic section* a collection of meromorphic functions $s_i : U_i \to \hat{\mathbb{C}}$ satisfying the relations (2.3).

Proposition 4.42. *The map $\mathcal{O}_X$ is an isomorphism between $Pic(X)$ and the group of holomorphic line bundles that admit a non zero meromorphic section.*

Proof. If $D = [(U_i, h_i)]$ and $D' = [(U_i, h'_i)]$ have the same image, without loss of generality we may assume $h_i/h_j = h'_i/h'_j$ for every i, j. Then the quotient $f = h'_j/h_j$ is a well defined meromorphic function on X satisfying $D' = D + (f)$, so $\mathcal{O}_X$ is injective. Further, the collection $\{h_i\}$ is clearly a meromorphic section for $\mathcal{O}_X(D)$. Conversely, if a line bundle admits a meromorphic section $\{s_i\}$, the quotients s_i/s_j must be holomorphic because they are exactly the transition functions, so the line bundle is the image under $\mathcal{O}_X$ of the divisor $[(U_i, s_i)]$. □

Further, it can be shown that every line bundle on X admits a non-zero meromorphic section, this is, $\mathcal{O}_X$ is an isomorphism between $Pic(X)$ and the group of all holomorphic line bundles. We are not going to prove this, but we assume it to be true.

For a divisor D we define its linear space

$$\mathcal{L}(D) = \{f \in \mathcal{M}(X)^* \mid (f) + D \geq 0\} \cup \{0\}.$$

Of course, it is a vector space.

Proposition 4.43. $\mathcal{L}(D)$ *is isomorphic to the space of holomorphic sections* $\Gamma(X, \mathcal{O}(D))$.

Proof. Write $D = [(U_i, h_i)]$. If $\{s_i\}$ is a holomorphic section for $\mathcal{O}_X(D)$, then $f = s_i/h_i$ is a well defined meromorphic function on X, clearly belonging to $\mathcal{L}(D)$ as the s_i are holomorphic. Conversely, for $f \in \mathcal{L}(D)$ the functions fh_i are holomorphic by hypothesis and satisfy the relations (2.3). This defines a linear bijection between both spaces. $\square$

Now we are going to find a way to construct line bundles over X. We try to extend the action of Λ on V to some action in $V \times \mathbb{C}$ such that $V \times \mathbb{C}$ results in a line bundle over X. For this to happen, we need that the action has the form

$$\lambda \cdot (z, t) = (z + \lambda, e_\lambda(z)t)$$

for holomorphic functions e_λ satisfying $e_0 = 1$ and

$$e_{\lambda_1\lambda_2}(z) = e_{\lambda_1}(z + \lambda_2)e_{\lambda_1}(z),$$

which is just the condition for being an action. But this is exactly the condition that must be satisfied by the quotient $\theta(z+\lambda)/\theta(z)$ for a function θ to be a theta function. Now let (H, α) be the type of a normalized theta function $\tilde{\theta}$. We define $L(H, \alpha)$ to be the line bundle afforded by setting

$$e_\lambda(z) = \frac{\tilde{\theta}(z + \lambda)}{\tilde{\theta}(z)}$$

in the construction above.

Lemma 4.44. *$L(H_1, \alpha_1) \otimes L(H_2, \alpha_2)$ is isomorphic to $L(H_1 + H_2, \alpha_1\alpha_2)$.*

Consider $\Pi : V \times \mathbb{C} \to L(H, \alpha)$ and $\pi : V \to X$ the canonic surjections and let $s \in \Gamma(X, L(H, \alpha))$, a holomorphic section. For every $z \in V$, $s(\pi(z))$ is the image under Π of a unique element in $V \times \mathbb{C}$ of the form (z, t), with $t \in \mathbb{C}$. We define $\theta_s(z) = t$. Then θ_s is a holomorphic function on V.

Lemma 4.45. *The application $s \mapsto \theta_s$ is an isomorphism of vector spaces between $\Gamma(X, L(H, \alpha))$ and the set of normalized theta functions of type (H, α).*

Proof. First, the condition $\Pi(z+\lambda, e_\lambda(z)t) = \Pi(z, t)$ implies that $\theta_s(z+\lambda) = e_\lambda(z)\theta_s(z)$, this is, θ_s is a normalized theta function of type (H, α). The application is linear simply because of the definition of the vector space structure in $\Gamma(X, L(H, \alpha))$. The injectivity is a direct consequence of the relation $s(\pi(z)) = \Pi(z, \theta_s(z))$. Now, let θ be a normalized theta function of type (H, α). Since $\theta(z + \lambda) = e_\lambda(z)\theta(z)$, then $\Pi(z, \theta(z)) = \Pi(z + \lambda, \theta(z + \lambda))$ for every $\lambda \in \Lambda$, so we can define a section s by $s(\pi(z)) = \Pi(z, \theta(z))$. One easily verifies that $\theta = \theta_s$, so the application is surjective. This completes the proof. $\square$

Now we will state without proof an important result characterizing the Picard group of a torus X. For a proof, see [4].

Theorem 4.46. *Every line bundle on a torus X is of the form $L(H, \alpha)$ for some $H \in NS(X)$. In fact, $Pic(X)$ can be decomposed as a direct sum $\oplus_{H \in NS(X)} Pic^H(X)$, where $Pic^H(X)$ is the set of line bundles of the form $L(H, \alpha)$ for some semicharacter α.*

5 Complex abelian varieties

Definition 5.1. *A polarized abelian variety is a torus* $X = V/\Lambda$ *together with a real non-degenerated* $(1,1)$*-form* ω *such that* $\omega(\Lambda \times \Lambda) \subset \mathbb{Z}$*.* ω *is called the polarization of* X*.*

Lemma 5.2. *Let* ω *be a non-degenerated* $(1,1)$*-form integer on the lattice. There exist positive integers* $d_1, \ldots, d_n$ *with* d_i *dividing* d_{i+1} *and a basis* $\{l_1, \ldots, l_{2n}\}$ *for the lattice (over* $\mathbb{R}$*) such that, in this basis, the matrix of* ω *writes*

$$\begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix},$$

where D *is the diagonal matrix with diagonal* $d_1, \ldots, d_n$*. The integers* d_i *are uniquely determined by* ω*.*

Definition 5.3. *Such a basis* $\{l_1, \ldots, l_{2n}\}$ *is called a symplectic basis of* Λ *for* ω*. The tuple* $(d_1, \ldots, d_n)$ *is called the type of the polarization. A principally polarized abelian variety is an abelian variety with a polarization of type* $(1, \ldots, 1)$*. We abbreviate it by ppav.*

Example 5.4. *If* S *is a Riemann surface of genus* g*, then its jacobian* JS *is a principally polarized abelian variety of dimension* g*, with the polarization given by the intersection product.*

Lemma 5.5. *In a torus* $X = V/\Lambda$*, there exists a non-degenerated* $(1,1)$*-form integer on the lattice if and only if there exists a complex basis* B *of* V*, positive integers* $d_i | d_{i+1}$ *and a matrix* $\tau \in \mathbb{H}_n$ *such that the period matrix for the torus with respect to* B *is* $\Pi = [D\ \tau]$*, where* D *is the diagonal matrix with entries* $d_1, \ldots, d_n$ *in the diagonal. The matrix of such* $(1,1)$*-form in the basis* B *is* $(\Im \tau)^{-1}$*.*

The basis B can be achieved by taking a symplectic basis $\{l_1, \ldots, l_{2n}\}$

and letting $v_j = l_j/d_j$, $j = 1, \ldots, n$. The tuple $(d_1, \ldots, d_n)$ is the type of the corresponding polarization.

As we saw in (4.13), if a torus X is a submanifold of $\mathbb{P}^n$, then it necessarilly carries a polarization, namely, the pullback of the Fubini-Study form. The following theorem, due to Lefschetz, states the converse.

Theorem 5.6. *Let $L = L(H, \alpha)$ be a line bundle on a torus, with $H \in NS(X)$ a nondegenerate Hermitian form. For $r \geq 3$, let $\theta_0, \ldots, \theta_n$ generate the space of sections of L^r, as in lemma (4.45). Then, the map*

$$\begin{aligned} \psi : X &\to \mathbb{P}^n \\ z &\mapsto [\theta_0(z) : \ldots : \theta_n(z)] \end{aligned}$$

is well defined and makes X into a complex submanifold of $\mathbb{P}^n$.

5.1 Homomorphisms of abelian varieties

Definition 5.7. *Let (X, ω_X), (Y, ω_Y) be two polarized abelian varieties, with $X = V/\Lambda$. A homomorphism of complex tori $f : X \to Y$ is said to preserve the polarization if*

$$\omega_X(x, y) = \omega_Y(\rho_a(f)(x), \rho_a(f)(y)) \quad \forall x, y \in V.$$

Such a homomorphism is called a homomorphism of abelian varieties. It is an isomorphism of abelian varieties if it possesses a two-sided inverse.

Remark 5.8. *If (Y, ω_Y) is a polarized abelian variety and $f : X \to Y$ a homomorphism of complex tori such that $\rho_a(f)$ is injective, then there is a unique polarization ω_X on X that makes f a homomorphism of abelian varieties. This is called the pullback of ω_Y by f. In particular, any subtorus of an abelian variety naturally inherits a structure of abelian variety itself.*

Fix the complex space V. Using lemma (5.5), we can find a correspondence between $\mathbb{H}_n$ and the set of threes of the form $(\Lambda, \omega, \{l_1, \ldots, l_{2n}\})$, where

Λ is a lattice for V, ω is a polarization of type $(1, \ldots, 1)$ for the torus V/Λ and $\{l_1, \ldots, l_{2n}\}$ is a symplectic basis. Now we would like to know which of those principally polarized abelian varieties are isomorphic to each other.

Definition 5.9. *We define the symplectic group as*

$$Sp(2n, \mathbb{Z}) = \{R \in M_{2n}(\mathbb{Z}) \mid R^t J R = J\},$$

where

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

One easily verifies that this is indeed a group with the product of matrices, and if R is a symplectic matrix, so is R^t. We can write any $2n \times 2n$ matrix as

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{5.1}$$

for A, B, C, D $n \times n$ matrices. It not difficult to see that an integer $2n \times 2n$ matrix R given by (5.1) belongs to $Sp(2n, \mathbb{Z})$ iff the following three conditions hold:

(1) AB^t is symmetric.

(2) CD^t is symmetric.

(3) $AD^t - BC^t = I$.

In this case, the inverse of R is

$$R^{-1} = \begin{bmatrix} D^t & -B^t \\ -C^t & A^t \end{bmatrix}.$$

Definition 5.10. *If*

$$R_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \in Sp(2n_i, \mathbb{Z}),$$

for $i = 1, 2$*, we define*

$$R_1 * R_2 = \begin{bmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{bmatrix} \in Sp(2(n_1 + n_2), \mathbb{Z}).$$

Note that if $R_i, S_i \in Sp(2n_i, \mathbb{Z})$, then $(R_1 * R_2)(S_1 * S_2) = (R_1S_1) * (R_2S_2)$.

Let $X_i = V/\Lambda_i$ be ppav's, $i = 1, 2$. From lemma (5.5), we can find bases B_1, B_2 of V such that the period matrices write $\Pi_i = [I\,\tau_i]$, respectively. Note that each ppav is completely determined by the matrix $\tau_i \in \mathbb{H}_n$. If $f : X_1 \to X_2$ is an isomorphism, there are matrices A, R representing $\rho_a(f), \rho_r(f)$ in the respective bases, satisfying $A[I\,\tau_1] = [I\,\tau_2]R$. The pullback of the (principal) polarization in X_2 is represented in X_1 by the matrix R^tJR, so f preserves it if and only if $R \in Sp(2n, \mathbb{Z})$. Also, if R is given by (5.1), one easily checks that $\tau_1 = (A + \tau_2 C)^{-1}(B + \tau_2 D)$.

Proposition 5.11. $Sp(2n, \mathbb{Z})$ *acts on* $\mathbb{H}_n$ *by*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \tau = (A + \tau C)^{-1}(B + \tau D) \tag{5.2}$$

Theorem 5.12. *Two elements in* $\mathbb{H}_n$ *define isomorphic ppav's if and only if they are in the same orbit in the action defined by* $Sp(2n, \mathbb{Z})$*. This is, the quotient* $\mathbb{H}_n/Sp(2n, \mathbb{Z})$ *parametrizes isomorphism classes of ppav's.*

The quotient $\mathbb{H}_n/Sp(2g, \mathbb{Z})$ is said to be the *moduli space* of the principally polarized abelian varieties.

Definition 5.13. *Let (X, ω) be a polarized abelian variety. A subtorus $Y \subset X$ with the induced polarization $\omega|_Y$ is called an abelian subvariety of (X, ω). An abelian variety is called simple if its only subtori are $\{0\}$ and itself.*

Note that if X, Y are two abelian varieties, then $X \times Y$ can be seen as an abelian variety in a natural way. Likewise, we can define the power X^k for some posive integer k. The following theorem is due to Poincaré.

Theorem 5.14. *Let X be an abelian variety. There exist simple abelian subvarieties $X_1, \ldots, X_r$, not isogenous to each other, and positive integers $k_1, \ldots, k_r$, such that X is isogenous to $X_1^{k_1} \times \ldots \times X_r^{k_r}$. The X_j and k_j are unique up to isogeny and permutation of the factors.*

6 Representations of finite groups

Now we want to talk about group actions on abelian varieties, so we first need some tools from representation theory.

6.1 Decomposition of semi-simple rings

In this part, we are mainly concerned about the group algebra KG, where K is a field and G a finite group. Some of the results we are going to give are achieved because these algebras belong to a larger class of rings, namely, rings with minimum condition.

Definition 6.1. *Let R be a ring with unit element. R is said to satisfy the minimum condition if every nonempty collection of left ideals has a minimal element.*

Throughout this section, R will always be a ring with 1 satisfying the minimum condition.

Remark 6.2. *If K is a field and G is a finite group, the group algebra KG can be regarded as a finite dimensional vector space over K. The left ideals of KG are sub-vector spaces, hence KG satisfies the minimum condition.*

If M is a module over R, then we say that M is completely reducible if it decomposes into a direct sum of irreducible submodules. The next proposition is given without proof. For a proof, see [3].

Proposition 6.3. *If R is a ring with 1 satisfying the minimum condition, then the following are equivalent*

(1) Every R-module is completely reducible.

(2) R has no nilpotent ideals.

(3) If M is a left R-module, every submodule is a direct summand of M.

In the case the ring R satisfies the conditions for the previous proposition, we say that the ring R is semi-simple. To know whether a group algebra is a semi-simple ring, we have Maschke's theorem:

Theorem 6.4. *If G is a finite group and K is a field whose characteristic does not divide the order of G, then KG is semi-simple.*

Proof. If M is a left R-module and V a submodule, we can find a complementary vector space, so we can define a K-linear projection π onto V. Now call $\pi_0 = \frac{1}{n}\sum_{g\in G} g\pi g^{-1}$. Then π_0 is a projection onto V that is also a KG-module homomorphism. Its kernel is a KG-direct summand of V. $\square$

Let's look at R as a module over itself. The irreducible submodules are exactly the minimal left ideals, that is, left ideals containing no proper left R-ideals other than the trivial one. Now, if R is semi-simple, using induction and the minimum condition, it is not difficult to show that we can write R as a finite direct sum of minimal left ideals, $R = L_1 \oplus \cdots \oplus L_n$. Write

$$1 = e_1 + \cdots + e_n, \tag{6.1}$$

with $e_i \in L_i$. Then in particular $e_i = e_i e_1 + \cdots + e_i e_n$. Since $e_i \in L_i$ and the sum is direct, we find that

$$e_i e_j = \delta_{ij} e_i. \tag{6.2}$$

If we have a set of nonzero elements satisfying (6.2), we say that they are orthogonal idempotents. Also, equation (6.1) implies that $R = Re_1 + \cdots + Re_n$. This together with the fact that $Re_i \subset L_i$ says that $Re_i = L_i$. Note also that $xe_i = x$ for all $x \in Re_i$.

So, we now know that if R is semi-simple, then it can be decomposed into a finite direct sum of minimal left ideals, each of them generated by an idempotent, and these idempotents are orthogonal to each other. Note that we have not said at all that this decomposition is unique. Next we give a good reason for being interested in minimal left ideals.

Theorem 6.5. *If R is semi-simple, every irreducible R-module is isomorphic to some minimal left ideal of R.*

Proof. Write $R = Re_1 \oplus \cdots \oplus Re_n$. If M is any nonzero R-module, we can write

$$M = \sum_{m \in M} \sum_{i=1}^{n} Re_i m.$$

Some of the summands, say $Re_i m$, must be nonzero. Define

$$\begin{aligned} \varphi : Re_i &\to Re_i m \\ xe_i &\mapsto xe_i m. \end{aligned}$$

φ is an R-homomorphism of Re_i onto $Re_i m$, which is a nonzero submodule of M. Since M is irreducible, $Re_i m$ must be equal to M. Further, the kernel of φ is a left ideal of R contained in Re_i. Then φ is injective, hence an isomorphism. $\square$

Note that from the proof we can infer that if M is an irreducible R-

module, then for any decomposition $R = Re_1 \oplus \cdots \oplus Re_n$ into minimal left ideals, M will be isomorphic to one of the Re_i. In particular, if $R = \oplus Re'_i$ is another decomposition into minimal left ideals, then each of the Re'_i is isomorphic to some Re_i. Let's give a connection with group representation theory.

Definition 6.6. *A representation of a group G over a field K is a homomorphism $\varphi : G \to GL(V)$, where V is a K-vector space. We denote $g(v) = \varphi(g)(v)$ for $g \in G$, $v \in V$.*

We say that two representations $\varphi : G \to GL(V)$, $\phi : G \to GL(W)$ are equivalent if there exists an isomorphism of vector spaces $T : V \to W$ such that $T(gv) = g(Tv)$ for every $v \in V$, $g \in G$. The character of a representation is the function $\chi : G \to K$ assigning to each $g \in G$ the trace of $\varphi(g)$. Note that two equivalent representations have the same character.

Remark 6.7. *It is an easy exercise to show that a representation is uniquely determined by giving V a structure of KG-module, this is, giving a homomorphism of K-modules $\rho : KG \to End(V)$. Two representations are equivalent if the respective KG-modules are isomorphic.*

Using Maschke's theorem and the previous results, we have

Corollary 6.8. *Let G be a finite group of order not divisible by the characteristic of K. Then G has a finite number of non-equivalent irreducible representations over K.*

Now we are going to determine when two minimal left ideals are isomorphic.

Lemma 6.9. *Let R be semi-simple. The minimal left ideals L and L' are isomorphic if and only if $L' = La'$ for some $a' \in L'$. Otherwise $La' = 0$ $\forall a' \in L'$.*

Proof. Let $a' \in L'$. Define a R-homomorphism by

$$\begin{aligned} \varphi : L &\to L' \\ x &\mapsto xa' \end{aligned}$$

This must be either the zero homomorphism or an isomorphism, since its image and kernel will be left ideals contained in L' and L, respectively. If $L' = La'$, φ will be surjective, hence an isomorphism. Conversely, if $\varphi : L \to L'$ is an isomorphism, then for $x \in L$ we have $\varphi(x) = \varphi(xe) = x\varphi(e)$, where e is the idempotent for which $L = Re$. Choosing $a' = \varphi(e)$, we have $L' = La'$. □

Another way to state what we have just proved is that if L and L' are minimal left ideals, then they are isomorphic if and only if $LL' = L'$. Otherwise, $LL' = 0$. What we are going to do next is to find another decomposition of R, using two-sided ideals.

Definition 6.10. *A ring with* 1 *is called simple if it satisfies the minimum condition and has no nontrivial two-sided ideals.*

Theorem 6.11. *Let R be a semi-simple ring and L a minimal left ideal. Let B_L be the sum of all the minimal left ideals of R which are isomorphic to L. Then B_L is a simple ring and a two-sided ideal. Further, R is the direct sum of all the B_L obtained by letting L range over a full set of non-isomorphic minimal left ideals of R.*

Proof. B_L is easily seen to be a left ideal. By the previous lemma, if L' is another minimal left ideal, then $L \cong L'$ iff $B_L L' \neq 0$. But by definition of B_L, if $L \cong L'$ then $L' \subset BL$. Since every element of R can be written as a sum of elements contained in some minimal left ideals, we find that B_L is a two-sided ideal. Further, L is equivalent to L' if and only if $B_L B_{L'} \neq 0$. Since R is the sum of its minimal left ideals, then $R = B_{L_1} + \cdots + B_{L_n}$ where the L_i form a complete set of non-isomorphic minimal left ideals. Without loss of generality, we may assume $L_i \ncong L_j$ for $i \neq j$. Using the fact that $B_{L_i}B_{L_j} = 0$ if $i \neq j$ and that $R = B_{L_1} + \cdots + B_{L_n}$, it is not difficult to see that the sum is actually direct. If $I \subset B_L$ is a nonzero two-sided ideal, then it must contain a minimal left ideal L' so, by lemma (6.9), I must also contain the whole of $B_{L'}$. Note that $B_L = B_{L'}$ because the sum is direct. This proves that the B_L contain no two-sided ideals. Now write

$$1 = b_1 + \cdots + b_m,$$

for $b_i \in B_{L_i}$. Then b_i is a unit element for B_{L_i}. In fact, we have $a = ab_1 + \cdots + ab_m = b_1a + \cdots + b_ma$. Since the B_{L_j} are two-sided ideals, then each $ab_j, b_ja \in B_{L_j}$. If $a \in B_{L_i}$, then $a = ab_i = b_ia$, because the sum is direct. Finally, each of the B_L satisfies the minimum condition since R does. $\square$

We call the ideals B_L the simple components of R.

Proposition 6.12. *If R is semi-simple, any two-sided ideal of R is a sum of a certain number of its simple components.*

Proof. Let B be a two sided ideal, and L a minimal left ideal contained in B, and then $B \supset B_L$. Let B' be the sum of all the simple components of R contained in B. Since R is semi simple, we can write $B = B' \oplus B''$ for some left ideal B''. If $B'' \neq 0$, it must contain a minimal left ideal L'', and so $B \supset B_{L''}$, which is a contradiction because this would imply $B_{L''} \subset B' \cap B''$. Hence $B'' = 0$ and the proposition follows. $\square$

As a consequence of the previous proposition, we have that a semi-simple ring decomposes into two-sided ideals which are simple rings *in a unique manner*, namely, as the direct sum of its simple components. At this point we should take a look at what is happening in the case of group representations. A representation is given by a homomorphism

$$\rho : KG \to End(V).$$

Set $A = \rho(KG)$, which is a subalgebra of $End(V)$. Then A decomposes into simple algebras as $Ab_1 \oplus \cdots \oplus Ab_h$, with b_i central primitive idempotents. By the discussion above, each Ab_i is the direct sum of a certain number of irreducible representations equivalent to each other. We may write $Ab_i = Ae_{1i} \oplus \cdots \oplus Ae_{n_i i}$. Consider $V = \mathbb{C}^n$. One can calculate (see, for instance, [15]), that if $\phi : G \to GL(V)$ is a representation, then the central idempotents defining the simple components are given by

$$b_i = \frac{n_i}{|G|} \sum_{s \in G} \chi_i(s^{-1})\phi(s),$$

where the χ_i are the characters of a complete set of irreducible representations, each of degree n_i.

6.2 Rational central idempotents

What we are going to do know is to try to understand what happens when the field in which the representation is being considered is changed, and how to relate the new central idempotents with the previous ones. We will present some results without going trough the proofs. They can be found in [3].

Definition 6.13. *Let $T : G \to GL(n, K)$ be a representation of G, F a subfield of K. We say that T is realizable in F if there exists a representation $T' : G \to GL(n, F)$ such that T and T' are K-equivalent representations. Equivalently, let T be afforded by a KG-module M. T is realizable over F if there exists an FG-module N such that M and $N \otimes_F K$ as left KG-modules.*

In what follows, let K be the algebraic closure of F, where F is of characteristic zero.

Definition 6.14. *A subfield $E \subset K$ is said to be a splitting field for G if every irreducible K representation is realizable in E.*

Let χ be the character of an irreducible representation T of G over K. For $F \subset K$ we define $F(\chi) = F(\chi(s) \mid s \in G)$. Note that $|F(\chi) : F|$ is finite. Note also that any field in which T is realizable must contain $F(\chi)$.

Definition 6.15. *Let T be an irreducible K-representation of G with character χ and let $F \subset K$. The Schur index of T with respect to F is*

$$m_F(T) = min\{|L : F(\chi)|\},$$

where the minimum is taken over all extension fields L of $F(\chi)$ such that T is realizable in F.

Starting with an irreducible K-representation, one can ask about the "minimum" field in which it is realizable. For this we have the following result.

Theorem 6.16. *Let T be a K-irreducible representation with character χ.*

(1) There exists an algebraic extension field L of F in which T is realizable such that $|L : F(\chi)| = m_F(T)$.

(2) For every finite algebraic extension E of F in which T is realizable,

$$m_F(T) \mid |E : F(\chi)|.$$

(3) $m_F(T)$ is the minimal value of m such that the sum of m copies of T is realizable in $F(\chi)$.

(4) $m_F(T) \mid deg(T)$

Now let's restrict ourselves to the case of $\mathbb{Q}$. Let E be a splitting field for G. We may assume E is Galois over $\mathbb{Q}$. Let $\{U_1, \ldots, U_m\}$ be a complete set of E-irreducible representations of G. Let $\mathbb{Q} \subset F \subset E$ and T an irreducible

F-representation. It will be given by a homomorphism

$$\rho : FG \to End(V).$$

Looking T as an E-representation is the same as considering the induced homomorphism

$$\rho^E : EG \to End(V \otimes_F E)$$

Let's write the corresponding E-representation as T^E. Then T^E is equivalent to a sum of some of the U_i,

$$T^E \cong d(1)U_1 \oplus \cdots \oplus d(t)U_t. \tag{6.3}$$

Fix a basis for V and consider the U_i as matrix representations. Then the group $Gal(E/F)$ acts on each of these matrices by acting on its entries. Each $\sigma(U_i)$ will give a new irreducible representation, hence must be equivalent to one of the U_j.

Theorem 6.17. *With the notation above, the components U_i appearing in (6.3) are conjugates to each other, in the sense that for all i, j there exists $\sigma \in Gal(E/F)$ such that $\sigma(U_i) \cong U_j$. Further, the number of distinct conjugates appearing in* (6.3) *is exactly $|F(\chi_1) : F|$ and $d(1) = \cdots = d(t) = m_F(U_1)$.*

We finish giving a way to calculate the central idempotents for the rational group algebra $\mathbb{Q}G$. Let E be Galois over $\mathbb{Q}$ and a splitting field for G. Let $\chi_1, \ldots, \chi_r$ be the irreducible E-characters of G. If $L \subset E$ and θ is an irreducible $\mathbb{Q}$-character, then

$$\theta = m_F(\chi_i) \sum_{\sigma \in Gal(E/\mathbb{Q})} \sigma(\chi_i)$$

for some i. Now, if we want to write $\mathbb{Q}$ as a direct sum of its simple components, we have

$$\mathbb{Q}G = p_1\mathbb{Q}G \oplus \cdots \oplus p_h\mathbb{Q}G,$$

where the p_i are determined as follows: if p_i corresponds to the irreducible rational representation W_i, let χ_i be a character of one of the irreducible E-representations associated to W_i. Then

$$p_i = \frac{deg\chi_i}{|G|}\sum_{s\in G} Tr_{\mathbb{Q}(\chi_i)/\mathbb{Q}}(\chi_i(s^{-1}))s. \tag{6.4}$$

7 Group representations in abelian varieties

For a complex torus X, recall the definition of the algebra $End_{\mathbb{Q}}(X)$, given in (4.5). As a consequence of theorem (5.14), we have that if X is an abelian variety, then $End_{\mathbb{Q}}(X)$ is a semi-simple algebra. Indeed, keeping the notation as in the theorem, $End_{\mathbb{Q}}(X)$ is isomorphic to the product $M_{k_1}(F_1) \times \cdots \times M_{k_r}(F_r)$, where each $F_j = End_{\mathbb{Q}}(X_j)$ is a skew-field of finite dimension over $\mathbb{Q}$, as X_j is a simple torus.

Now we are going to describe an analogous construction developed by Lange and Recillas in [7] which involves the action of a group. Consider a finite group G acting on an abelian variety X or, in other words, suppose $\rho : \mathbb{Q}G \to End_{\mathbb{Q}}(X)$ is a homomorphism of $\mathbb{Q}$-algebras. For $h \in End_{\mathbb{Q}}(X)$, we define $X^h = jh(X)$, for any positive integer j such that $jh \in End(X)$. Let p_i be the central primitive idempotents for the algebra $\mathbb{Q}G$, defined by (6.4). Let

$$A_i = X^{\rho(p_i)}.$$

Then G acts on each A_i and does not act in any proper abelian subvariety of A_i. The map

$$\begin{aligned} A_1 \times \cdots \times A_m &\to X \\ (a_1, \ldots, a_m) &\mapsto a_1 + \ldots + a_m \end{aligned}$$

is an isogeny. For each i, p_i can be written as a sum $p_i = e_{i1} + \ldots + e_{in_i}$ of primitive idempotents, where $n_i = deg(\chi_i)/m_{\mathbb{Q}}(\chi_i)$ for χ_i any of the complex irreducible representations associated to p_i. This means that the KGe_{ij} are minimal left ideals (the way of writing so is not unique). Then the map

$$\begin{aligned} X^{\rho(e_{i1})} \times \cdots \times X^{\rho(e_{in_i})} &\to A_i \\ (b_1, \ldots, b_{n_i}) &\mapsto b_1 + \ldots + b_{n_i} \end{aligned}$$

is also an isogeny, and the action of G makes the $X^{\rho(e_{ij})}$ isogenous to each other. Call

$$B_i = X^{\rho(e_{ij})}$$

for any $j \in \{1, \ldots, n_i\}$. With this construction one can achieve the following result, proved in [7].

Theorem 7.1. *Let G be a finite group acting on an abelian variety X. Define B_i as above. Then G acts on each $B_i^{n_i}$ and the construction above defines an isogeny*

$$X \sim B_1^{n_1} \times \cdots \times B_m^{n_m}. \tag{7.1}$$

This is called the isotypical decomposition of X. Note that the A_i are unique, whereas the B_i are determined only up to isogeny.

7.1 Involutions on ppav's

In [12], Rodríguez characterizes prinicipally polarized abelian varieties admitting an involution. Recall that an isomorphism of ppav's is simply represented by a matrix $R \in Sp(2n, \mathbb{Z})$. If we are to study involutions, we need to know what involutions in $Sp(2n, \mathbb{Z})$ look like.

Remark 7.2. *A symplectic matrix R will correspond to an automorphism of a ppav X having period matrix $[I\,\tau]$ (in some basis) if $R \cdot \tau = \tau$, according to the action defined in (5.2). Note that here we are not saying that a symplectic involution R will necessarily be an involution for some ppav.*

Notation 7.3. *If M_j are square matrices for $j = 1, \ldots, m$, we will write $(M_1, \ldots, M_m)$ the block matrix whose diagonal blocks are the M_j or, in other words, $(M_1, \ldots, M_m)$ is the direct sum of the M_j.*

Call

$$J_1 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}.$$

Let x, y, z be nonnegative integers such that $2x + y + z = n$. Call $W(x, y, z)$ the block matrix $(J_1, \ldots, J_1, -I_y, I_z)$, written as in notation (7.3), having x blocks equal to J_1, and where I_r is the $r \times r$ identity matrix. In [9], Reiner proved the following.

Theorem 7.4. *The block matrices of the form $(W(x, y, z), W(x, y, z)^t)$ form a complete set of nonconjugate involutions in $Sp(2n, \mathbb{Z})$.*

Thus, if $X = V/\Lambda$ is a ppav and $h : X \to X$ an order two automorphism of ppav's, there is a symplectic basis for the lattice such that the matrix of h with respect to this basis is of the form $(W(x, y, z), W(x, y, z)^t)$. Let

$$B = \{\alpha_1, \ldots, \alpha_{2x}, \beta_1, \ldots \beta_y, \gamma_1, \ldots, \gamma_z, \alpha'_1, \ldots, \alpha'_{2x}, \beta'_1, \ldots \beta'_y, \gamma'_1, \ldots, \gamma'_z\}$$

be such basis. The group $\langle h \rangle$ has two irreducible rational representations, both of degree one, given by the identity and multiplication by -1. The rational central idempotents in this case are

$$p_1 = \frac{1}{2}(id_X + h)$$
$$p_2 = \frac{1}{2}(id_X - h).$$

Hence $\langle h \rangle$ induces an isogeny

$$X \sim A_1 \times A_2,$$

for $A_i = X^{p_i} = B_i$. Using this one can calculate the following.

Proposition 7.5. *With the notation above, let L_i be the sublattice corresponding to A_i, respectively. Then*

$$L_1 = \langle \{2\alpha_{2j-1} + \alpha_{2j}, \gamma_k, \alpha'_{2j-1}, \gamma'_k\}_{j=1,\ldots x,\, k=1,\ldots,z} \rangle_{\mathbb{Z}}$$
$$L_2 = \langle \{\alpha_{2j}, \beta_l, -\alpha'_{2j-1} + \alpha'_{2j}, \beta'_l\}_{j=1,\ldots x,\, l=1,\ldots,y} \rangle_{\mathbb{Z}}.$$

One can also find the dimensions of A_i, the induced polarizations on them and the kernel of the isogeny, in terms of the basis for the lattice. All of that is done in [12]. A principally polarized abelian variety admitting such an involution will be given by $\tau \in \mathbb{H}_n$ satisfying $R \cdot \tau = \tau$, as in (5.2). Knowing R, we can find the possible τ satisfying that, this is, we can find the period matrix of all the possible ppav admitting R as an involution. That is done in [12] too.

7.2 Automorphisms of order three on ppav's

Let $h : X \to X$ be an order-3 automorphism (preserving the polarization). The group $\langle h \rangle$ has two irreducible representations: the trivial one and the two-dimensional one sending h to the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}.$$

The latter is the direct sum of the two nontrivial complex irreducible representations, given by multiplication by ζ and ζ^2, where ζ is a cubic root of unity. The corresponding central idempotents are

$$\begin{aligned} p_1 &= \frac{1}{3}(id_X + h + h^2) \\ p_2 &= \frac{1}{3}(2id_X - h - h^2). \end{aligned}$$

As in the case of involutions, $\langle h \rangle$ induces a decomposition

$$X \sim A_1 \times A_2,$$

for $A_i = X^{p_i} = B_i$.

Recall that a ppav with an automorphism of order three is given by a matrix $\tau \in \mathbb{H}_n$ and an order-three symplectic matrix R, satisfying the equation $R \cdot \tau = \tau$. Now we make an attempt to determine all the symplectic order-three matrices, up to symplectic conjugation, as symplectic conjugation only means a change of the basis. We begin by considering the group Γ_m of $m \times m$ matrices with integer coefficients and determinant ± 1. Let p be a prime. A matrix $R \in \Gamma_m$ satisfying $R^p = I$ makes $\mathbb{Z}^m$ into a $\mathbb{Z}G$-module, where $G = \langle g \rangle$ is a cyclic group of order p. Two such matrices are conjugated in Γ_m if and only if the corresponding $\mathbb{Z}G$-modules are isomorphic. In [10], Reiner proves the following.

Theorem 7.6. *Let G be the cyclic group of prime order p generated by g and let ζ_p be a primitive p-th root of* 1. *Any finitely generated torsion-free $\mathbb{Z}G$-module is isomorphic, as a $\mathbb{Z}G$-module, to a direct sum of a number of irreducible finitely generated torsion-free $\mathbb{Z}G$-modules. Any such irreducible* *module V is isomorphic to one of the following three kinds.*

(1) $V = \mathbb{Z}y$, with $g(y) = y$.

(2) $V = \mathfrak{u}$, where $\mathfrak{u}$ is an ideal in $\mathbb{Z}[\zeta_p]$, with $g(v) = \zeta_p v$ for every $v \in V$.

(3) $V = \mathfrak{u} \oplus_{\mathbb{Z}} \mathbb{Z}y$, with $\mathfrak{u}$ an ideal of $\mathbb{Z}[\zeta_p]$, $g(v) = \zeta_p v$ for every $v \in V$ and $g(y) = y + \beta$, where β can be chosen to be any element in $\mathfrak{u}$ not belonging to $(\zeta_p - 1)\mathfrak{u}$.

In the case when $p = 3$, write $\zeta = \zeta_3$ and let $\mathfrak{u}$ be an ideal in $\mathbb{Z}[\zeta]$. We know that $\mathbb{Z}[\zeta]$ is a principal ideal domain, so let $\mathfrak{u} = (a)$. Then $\mathfrak{u}$ can be seen as a rank-two free $\mathbb{Z}$-module, generated over $\mathbb{Z}$ by a and ζa. Using this basis, we see

$$\begin{aligned} g(a) &= \zeta a \\ g(\zeta a) &= -a - \zeta a. \end{aligned}$$

As a conclusion, an irreducible $\mathbb{Z}G$-module of kind (2) can be represented by the matrix

$$S = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

acting on $\mathbb{Z}^2$. Now, for a nonzero ideal $\mathfrak{u} = \langle a \rangle$, neither a nor ζa can belong to $(\zeta - 1)\mathfrak{u}$, so we can represent an irreducible $\mathbb{Z}G$-module of type (3), for instance, by the matrix

$$T = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

For a square matrix M, we denote by $M^{(x)}$ the block matrix with x diagonal blocks equal to M. The discussion above together with theorem (7.6) implies the following.

Corollary 7.7. *Let $R \in \Gamma_m$ satisfy $R^3 = I$. Then there exist non-negative integers x, y, z such that $3x + 2y + z = m$ and R is conjugated in Γ_m to the block matrix $(T^{(x)}, S^{(y)}, I_z)$, where T and S are the matrices described above and I_z is the $z \times z$ identity matrix.*

Now we know all matrices in Γ_m satisfying $R^3 = I$, up to conjugation in Γ_m. We work towards describing symplectic matrices satisfying the same condition, up to symplectic conjugation. Let

$$P_r = \begin{bmatrix} 0 & -I_r \\ I_r & -I_r \end{bmatrix}, \quad Q_s = \begin{bmatrix} 0 & I_s \\ -I_s & -I_s \end{bmatrix}.$$

Note that S is just P_1. Call

$$F(r, s, t, u, v) = P_r * Q_s * (I_t, S^{(u)}, T^{(v)}, I_t, (S^{2t})^{(u)}, (T^{2t})^{(v)}),$$

where $*$ is the operation defined in (5.10) and we are using notation (7.3) for block matrices. It is easy to see that $F = F(r, s, t, u, v)$ is a symplectic $2n \times 2n$ matrix, where $n = r + s + t + 2u + 3v$, satisfying $F^3 = I$. We now go back to the case of a ppav with an automorphism h of order 3 and restrict to the case when h has precisely this shape. So, suppose

$$B = \{\alpha_j, \beta_k, \gamma_l, \delta_m, \epsilon_q, \alpha'_j, \beta'_k, \gamma'_l, \delta'_m, \epsilon'_q\} \tag{7.2}$$

is a symplectic basis for the lattice in X, where $j = 1, \ldots, r$, $k = 1, \ldots, s$, $l = 1, \ldots, t$, $m = 1, \ldots, 2u$, $q = 1, \ldots, 3v$, for which the matrix of h is written as $F = F(r, s, t, u, v)$. Such a basis is said to be *adapted* to h. We calculate

$$I + F + F^2 = 0 * 0 * (3I_t, 0, L^{(v)}, 3I_t, 0, (L^t)^{(v)}),$$

where

$$L = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

On the other hand,

$$2I - F - F^2 = 3I_r * 3I_s * (0, 3I_{2u}, M^{(v)}, 0, 3I_{2u}, (M^t)^{(v)}),$$

where

$$M = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Proposition 7.8. *Suppose h is an automorphism of the form $F(r,s,t,u,v)$ and B is a symplectic basis adapted to h, given by (7.2). With the notation as above, let L_i be the sublattice corresponding to A_i. Then*

$$\begin{aligned} L_1 &= \langle \{\gamma_l, -\epsilon_{3q-2} + \epsilon_{3q-1} + 3\epsilon_{3q}, \gamma_l', \epsilon_{3q}'\}_{l=1,\ldots t,\, q=1,\ldots,v} \rangle_{\mathbb{Z}} \\ L_2 &= \langle \{\alpha_j, \beta_k, \delta_m, \epsilon_{3q-2}, \epsilon_{3q-1}, \alpha_j', \beta_k', \delta_m', \\ &\quad \epsilon_{3q-2}' + \epsilon_{3q-1}', 3\epsilon_{3q-2}' + \epsilon_{3q}'\}_{j=1,\ldots,r,\, k=1,\ldots,s,\, m=1,\ldots,2u,\, q=1,\ldots,v} \rangle_{\mathbb{Z}}. \end{aligned}$$

Proof. L_i is just the intersection of Λ with the image of $3p_i$. Note that if $W = \oplus_j W_j$ is a real subspace of V and Λ can be written as a direct sum $\oplus_j \Lambda_j$ satisfying $\Lambda_j \subset W_j$, then $W \cap \Lambda$ is just $\oplus_j W_j \cap \Lambda_j$. The generators for the sublattices arise directly from this together with the fact that

$$\langle (3,0,1)^t, (0,3,-1)^t \rangle_{\mathbb{R}} \cap \mathbb{Z}^3 = \langle (1,1,0)^t, (3,0,1)^t \rangle_{\mathbb{Z}},$$

which gives us the elements of the forms $\epsilon_{3q-2}' + \epsilon_{3q-1}'$ and $3\epsilon_{3q-2}' + \epsilon_{3q}'$ written in the basis for L_2. This completes the proof. □

Corollary 7.9. *With the same hypotheses as in proposition (7.8), the (complex) dimension of A_1 is $t+v$, while the dimension of A_2 is $r+s+2u+2v$.*

Note that proposition (7.8) gives us a basis for the lattice L_i, in each case. Recall that this is an $\mathbb{R}$-basis for the A_i. Now we have the sublattices, we can find the kernel of the isogeny and the type of the polarization on the subvarieties A_i.

Corollary 7.10. *With the same hypotheses as in proposition (7.8), the kernel of the isogeny $A_1 \times A_2 \to X$ is the subgroup generated by the $2v$ points*

$$\left\{ \left(-\frac{1}{3}\epsilon_{3q-2} + \frac{1}{3}\epsilon_{3q-1} + 2\epsilon_{3q}, \frac{1}{3}\epsilon_{3q-2} + \frac{2}{3}\epsilon_{3q-1} \right), \right.$$
$$\left. \left(\frac{2}{3}\epsilon'_{3q}, \epsilon'_{3q-2} + \frac{1}{3}\epsilon'_{3q} \right) \mid 1 \leq q \leq v \right\}.$$

The order of the kernel is 3^{2v}. In particular, if $v = 0$, the isogeny is in fact an isomorphism.

Proof. Using the bases for A_i over $\mathbb{R}$ given by proposition (7.8), we have that the elements of the kernel are exactly the pairs of the form (z_1, z_2), for

$$z_1 = \sum_q y_q^1(-\epsilon_{3q-2} + \epsilon_{3q-1} + 3\epsilon_{3q}) + y_q^2\epsilon'_{3q}$$
$$z_2 = \sum_q y_q^3\epsilon_{3q-2} + y_q^4\epsilon_{3q-1} + y_q^5(\epsilon'_{3q-2} + \epsilon'_{3q-1}) + y_q^6(3\epsilon'_{3q-2} + \epsilon'_{3q}),$$

with $y_q^i \in \mathbb{R}$ satisfying

$$-y_q^1 + y_q^3 \in \mathbb{Z}$$
$$y_q^1 + y_q^4 \in \mathbb{Z}$$
$$3y_q^1 \in \mathbb{Z}$$

and

$$\begin{aligned} y_q^5 + 3y_q^6 &\in \mathbb{Z} \\ y_q^5 &\in \mathbb{Z} \\ y_q^2 + y_q^6 &\in \mathbb{Z}. \end{aligned}$$

Notice that conditions for $\{y_q^1, y_q^3, y_q^4\}$ are independent from conditions for $\{y_q^2, y_q^5, y_q^6\}$. Putting the latter variables equal to zero gives us the subgroup generated by pairs of the form

$$\left(-\frac{1}{3}\epsilon_{3q-2} + \frac{1}{3}\epsilon_{3q-1} + 2\epsilon_{3q}, \frac{1}{3}\epsilon_{3q-2} + \frac{2}{3}\epsilon_{3q-1}\right).$$

Analogously, putting the first set of variables equal to zero we get the other set of generators. This completes the proof. □

Corollary 7.11. *Keeping the same hypotheses, we have*

(1) The induced polarization in A_1 is of type $(1^t, 3^v)$.

(2) The induced polarization in A_2 is of type $(1^{r+s+2u+v}, 3^v)$,

where, by d^x we mean $(d, \ldots, d)$, d appearing x times.

Proof. Note that the basis given for L_1 in proposition (7.8) is already symplectic, with

$$\begin{aligned} \omega(\gamma_l, \gamma_l') &= -1 \\ \omega(-\epsilon_{3q-2} + \epsilon_{3q-1} + 3\epsilon_{3q}, \epsilon_{3q}') &= -3. \end{aligned}$$

For L_2, we slightly change the basis given in proposition (7.8), by replacing $\{\epsilon_{3q-2}, \epsilon_{3q-1}\}$ by $\{\epsilon_{3q-1}, \epsilon_{3q-2} - \epsilon_{3q-1}\}$. The basis for L_2 so obtained is now symplectic, with

$$\omega(\epsilon_{3q-1}, \epsilon'_{3q-2} + \epsilon'_{3q-1}) = -1$$
$$\omega(\epsilon_{3q-2} - \epsilon_{3q-1}, 3\epsilon'_{3q-2} + \epsilon'_{3q}) = -3.$$

Now the argument follows as in the case for L_1. □

Finally, we try to work out how far we are from describing all the order-three automorphisms of abelian varieties, this is, how far we are from finding all the symplectic order-three matrices up to conjugacy in $Sp(2n, \mathbb{Z})$. First, we have an easy result for the case $n = 1$.

Lemma 7.12. *Every 2×2 symplectic matrix R satisfying $R^3 = I$ is conjugated in $Sp(2, \mathbb{Z})$ to a matrix of the form $F(r, s, t, 0, 0)$, for $r + s + t = 1$.*

Proof. For $n = 1$, symplectic matrices are just integer matrices with determinant 1. Let $R \in Sp(2, \mathbb{Z})$. By corollary (7.7), if $R \neq I$, then there exists $U \in \Gamma_2$ such that $URU^{-1} = S = P_1$. Call

$$V = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then either U or VU belongs to $Sp(2, \mathbb{Z})$. As $VP_1V^{-1} = Q_1$, the lemma follows. □

Lemma 7.13. *Let*

$$R = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in Sp(2n, \mathbb{Z})$$

satisfy $R^3 = I$. Then R is conjugated in $Sp(2n, \mathbb{Z})$ to a matrix of the form $F(0, 0, t, u, v)$.

Proof. The hypothesis implies that $A^3 = I$, so A is conjugated in Γ_n to a block matrix $W = (I_t, S^{(u)}, T^{(v)})$, say $UAU^{-1} = W$. Note that D is just A^{2t}. Call

$$V = \begin{bmatrix} W & 0 \\ 0 & W^{2t} \end{bmatrix}.$$

Then $V \in Sp(2n, \mathbb{Z})$ and $VRV^{-1} = F(0, 0, t, u, v)$. $\square$

Lemma 7.14. *Let R be a symplectic matrix given by (5.1). If either B or C are equal to zero, R is not conjugated in $Sp(2n, \mathbb{Z})$ to P_n nor to Q_n.*

Proof. Let

$$R_0 = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} \in Sp(2n, \mathbb{Z})$$

and suppose $R_0 P_n R_0^{-1} = R$. Then $B = -(B_0 B_0^t + A_0 A_0^t + B_0 A_0^t)$ and $C = D_0 D_0^t + C_0 C_0^t + D_0 C_0^t$. Assume $B = 0$. Then $(B_0 + A_0)(B_0 + A_0)^t = B_0 A_0^t$, since $A_0 B_0^t = B_0 A_0^t$. This implies that $B_0 A_0^t$ is positive as a quadratic form, so $-B$ is equal to the sum of three positive quadratic forms. But $B = 0$ implies that all of those are equal to zero, in particular, $A_0 = B_0 = 0$. This is a contradiction as R_0 cannot be singular. Analogously, C cannot be zero. The case for Q_n works in the same way. $\square$

The previous lemma makes it explicit the need of considering matrices of the form P_n and Q_n when talking about order-three symplectic matrices, as they cannot be brought to a matrix of the form $F(0, 0, t, u, v)$ by symplectic conjugation. Also, P_n and Q_n define indeed different transformations, as the following lemma states.

Lemma 7.15. *The matrices P_n and Q_n are not conjugated to each other in $Sp(2n, \mathbb{Z})$.*

Proof. Suppose $RP_nR^{-1} = Q_n$, for R as in (5.1). Comparing the $(1,2)$-block of this equality, we have $BA^t = I + (A+B)(A+B)^t$, so BA^t is positive. The same equality writes $-(BB^t + AA^t + BA^t) = I$. But this is a contradiction because the left hand side is negative. $\square$

To this end, we have described a whole family of order-three group actions on abelian varieties and worked towards answering the question about all the possible symplectic order-three matrices. Our hope is that every symplectic matrix R satisfying $R^3 = I$ is conjugated to some of the form $F(r,s,t,u,v)$, which is of course true for $n = 1$, but we do not know in general. Further interesting work would be to describe other finite groups acting on abelian varieties, which is far from being trivial.

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