Mourre theory for unitary operators in two Hilbert spaces and quantum walks on trees

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Santiago, September 2020

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Mourre theory in one Hilbert space

- $\mathcal{H}$, Hilbert space with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$
- $\mathscr{B}(\mathcal{H})$, set of bounded linear operators on $\mathcal{H}$
- $\mathscr{K}(\mathcal{H})$, set of compact operators on $\mathcal{H}$
- U, unitary operator in $\mathcal{H}$ with spectral measure $E^U(\cdot)$ and spectrum

$$\sigma(U) \subset \mathbb{S}^1 := \{ \mathrm{e}^{it} \mid t \in [0, 2\pi) \}$$

- A, self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(A)$

Definition

$U \in C^k(A)$ if the map

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{-itA}\, U\, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H})$$

is strongly of class C^k.

$U \in C^1(A)$ if and only if

$$\big|\langle \varphi, U A\varphi\rangle - \langle A\varphi, U\varphi\rangle\big| \leq \text{Const.}\, \|\varphi\|^2 \quad \text{for all } \varphi \in \mathcal{D}(A).$$

The bounded operator associated to the continuous extension of the quadratic form is written $[U, A]$, and

$$[iU, A] = \text{s-}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{e}^{-itA}\, U\, \mathrm{e}^{itA}\,.$$

Definition

$U \in C^{1+\varepsilon}(A)$ for some $\varepsilon \in (0,1)$ if $U \in C^1(A)$ and

$$\big\| \mathrm{e}^{-itA}[U,A]\,\mathrm{e}^{itA} - [U,A]\big\|_{\mathscr{B}(\mathcal{H})} \le \text{Const. } t^{\varepsilon} \quad \text{for all } t \in (0,1).$$

One has the inclusions:

$$C^2(A) \subset C^{1+\varepsilon}(A) \subset C^1(A) \subset C^0(A) \equiv \mathscr{B}(\mathcal{H}).$$

Theorem (Fernández-Richard-T. 2013)

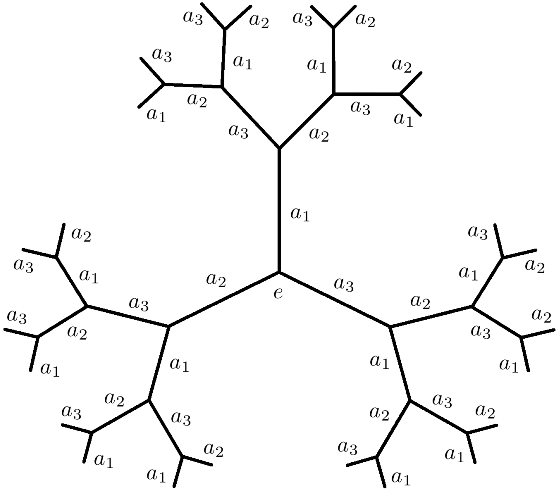
Let $U \in C^{1+\varepsilon}(A)$ *and suppose there exist an open set* $\Theta \subset \mathbb{S}^1$, $a > 0$ *and* $K \in \mathscr{K}(\mathcal{H})$ *such that*

$$E^U(\Theta)\, U^{-1}[A, U]\, E^U(\Theta) \geq a\, E^U(\Theta) + K. \qquad (\star)$$

Then, U *has at most finitely many eigenvalues in* Θ*, each one of finite multiplicity, and no singular continuous spectrum in* Θ*.*

- The inequality ($\star$) is called a Mourre estimate for U on Θ.
- The operator A is called a conjugate operator for U on Θ.
- If $K = 0$, then U has purely a.c. spectrum in $\Theta \cap \sigma(U)$.
- In fact, one obtains a limiting absorption principle (resolvent estimate) under the hypotheses of the theorem.

Quantum walks on homogeneous trees of degree 3



Homogeneous tree $\mathcal{T}$ with neutral element e and generators a_1, a_2, a_3

Even/odd elements of $\mathcal{T}$ defined with the word norm $|\cdot|$:

$$\mathcal{T}_{\mathrm{e}} := \left\{x \in \mathcal{T} \mid |x| \in 2\mathbb{N}\right\} \quad \text{and} \quad \mathcal{T}_{\mathrm{o}} := \left\{x \in \mathcal{T} \mid |x| \in 2\mathbb{N}+1\right\}$$

with characteristic functions χ_{e} and χ_{o}.

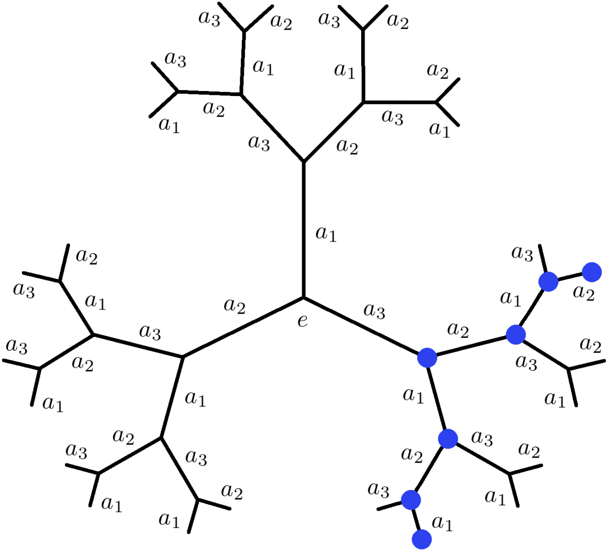
In the Hilbert space $\mathcal{H} := \ell^2(\mathcal{T}, \mathbb{C}^3)$ the evolution operator of the quantum walk is $U := SC$ with[1]

$$S := \begin{pmatrix} S_{23} & 0 & 0 \\ 0 & S_{31} & 0 \\ 0 & 0 & S_{12} \end{pmatrix}, \quad S_{ij} f := \chi_{\mathrm{e}} f(\cdot\, a_i) + \chi_{\mathrm{o}} f(\cdot\, a_j), \quad f \in \ell^2(\mathcal{T}) \quad \text{(shift)}$$

$$(C\varphi)(x) := C(x)\varphi(x), \quad \varphi \in \mathcal{H},\ x \in \mathcal{T},\ C(x) \in \mathrm{U}(3) \quad \text{(coin)}.$$

The operator U is unitary because S and C are unitary.

[1] Definitions of [Hamza-Joye 2014] and [Joye-Marin 2014].



Support of the iterates $S_{12}^n \delta_{a_3}$ $(n \in \mathbb{Z})$

We assume that C is anisotropic; it converges with short-range rate to an asymptotic coin on each branch of $\mathcal{T}$:

Assumption (Short-range)

For $i = 1, 2, 3$, there exist a diagonal matrix $C_i \in \mathrm{U}(3)$ and a scalar $\varepsilon_i > 0$ such that

$$\left\| C(x) - C_i \right\|_{\mathscr{B}(\mathbb{C}^3)} \leq \mathrm{Const.} \langle x \rangle^{-(1+\varepsilon_i)} \quad \text{if } x \in \mathcal{T}_i$$

where $\mathcal{T}_i := \left\{ x \in \mathcal{T} \mid \text{the first letter of } x \in \mathcal{T} \text{ is } a_i \right\}$.

Free evolution operator

The assumption provides three operators $U_i := SC_i$ describing the asymptotic behaviour of U on each branch of $\mathcal{T}$.

It also suggests to define the free evolution operator as

$$U_0 := U_1 \oplus U_2 \oplus U_3 \quad \text{in} \quad \mathcal{H}_0 := \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H},$$

and to define the identification operator $J : \mathcal{H}_0 \to \mathcal{H}$ as

$$J\Phi := \sum_{k=1}^{3} \chi_k \varphi_k, \quad \Phi = (\varphi_1, \varphi_2, \varphi_3) \in \mathcal{H}_0,$$

with

$$\chi_1 := \chi_{\mathcal{T}_1 \cup \{e\}}, \quad \chi_2 := \chi_{\mathcal{T}_2}, \quad \chi_3 := \chi_{\mathcal{T}_3}.$$

For $k = 1, 2, 3$, we define a modified word norm

$$|\cdot|_k : \mathcal{T} \to \mathbb{N}, \quad x \mapsto |x|_k := \left|x_k^{-1} x\right|,$$

where x_k is the longest word in x finishing by a letter a_k.

Example

If $x = a_1 a_2 a_1 a_3 a_2$, then $x_1 = a_1 a_2 a_1$, and if $x = a_1 a_2 a_1 a_2 a_1$, then $x_3 = e$.

Lemma (Conjugate operator for S_{ij})

Take $i, j, k \in \{1, 2, 3\}$ distinct. Then, the operator

$$A_{ij} f := S_{ij}^{-1} \big[|\cdot|_k^2, S_{ij} \big] f, \quad f \in C_c(\mathcal{T}),$$

is essentially self-adjoint in $\ell^2(\mathcal{T})$, with closure also denoted by A_{ij}. Furthermore, one has $S_{ij} \in C^\infty(A_{ij})$ with

$$S_{ij}^{-1}[A_{ij}, S_{ij}] = 2 \quad \textit{(Mourre estimate for } S_{ij} \textit{ on } \mathbb{S}^1\textit{)}.$$

Idea of the proof.

Direct calculations on the dense set $C_c(\mathcal{T})$.[a] □

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[a] Reminiscent of the relations

$$\mathtt{A} := \mathtt{S}^{-1}[\mathtt{X}^2, \mathtt{S}] \quad \text{and} \quad \mathtt{S}^{-1}[\mathtt{A}, \mathtt{S}] = 2,$$

with $\mathtt{S}$ the shift on $\ell^2(\mathbb{Z})$ and $\mathtt{X}$ the position operator on $\ell^2(\mathbb{Z})$.

[Fernández-Richard-T. 2013] then implies that S_{ij} has purely a.c. spectrum. But more can be said.

The relation $S_{ij}^{-1}[A_{ij}, S_{ij}] = 2$ implies by functional calculus the imprimitivity relation

$$\mathrm{e}^{isA_{ij}}\,\gamma(S_{ij})\,\mathrm{e}^{-isA_{ij}} = \gamma(\mathrm{e}^{2is}\,S_{ij}), \quad s \in \mathbb{R},\ \gamma \in C(\mathbb{S}^1).$$

Together with Mackey's imprimitivity theorem, this implies that S_{ij} is unitarily equivalent to a multiplication operator with purely a.c. spectrum covering the whole unit circle $\mathbb{S}^1$.

(Think of a version of Stone-von Neumann theorem on $\mathbb{S}^1$...)

What precedes + direct sums + the fact that the diagonal matrices C_i commute with the other operators, implies:

Proposition (Spectral properties of U_0)

Let

$$A_0\Phi := (\widetilde{A}\oplus\widetilde{A}\oplus\widetilde{A})\Phi, \quad \widetilde{A} := \begin{pmatrix} A_{23} & 0 & 0 \\ 0 & A_{31} & 0 \\ 0 & 0 & A_{12} \end{pmatrix}, \quad \Phi\in C_c(\mathcal{T},\mathbb{C}^3)^3.$$

(a) *A_0 is essentially self-adjoint in $\mathcal{H}_0$, with closure also denoted by A_0.*

(b) *$U_0\in C^\infty(A_0)$ with $U_0^{-1}[A_0,U_0]=2$, and U_0 satisfies the imprimitivity relation*

$$\mathrm{e}^{isA_0}\,\gamma(U_0)\,\mathrm{e}^{-isA_0}=\gamma(\mathrm{e}^{2is}\,U_0), \quad s\in\mathbb{R},\ \gamma\in C(\mathbb{S}^1).$$

(c) *U_0 is unitarily equivalent to a multiplication operator with purely a.c. spectrum covering the whole unit circle $\mathbb{S}^1$.*

Mourre theory in two Hilbert spaces

By refining some results of [Richard-Suzuki-T. 2018], we get the perturbation result:

Theorem

Let U_0, U be unitary operators in Hilbert spaces $\mathcal{H}_0, \mathcal{H}$, let A_0 be a self-adjoint operator in $\mathcal{H}_0$, let $J \in \mathscr{B}(\mathcal{H}_0, \mathcal{H})$, and assume that

- **(i)** *there exists a set $\mathscr{D} \subset \mathcal{D}(A_0 J^*) \subset \mathcal{H}$ such that $J A_0 J^* \restriction \mathscr{D}$ is essentially self-adjoint, with closure denoted by A,*
- **(ii)** $U_0 \in C^1(A_0)$,
- **(iii)** $V := J U_0 - U J \in \mathscr{K}(\mathcal{H}_0, \mathcal{H})$,
- **(iv)** $\overline{V A_0 \restriction \mathcal{D}(A_0)} \in \mathscr{K}(\mathcal{H}_0, \mathcal{H})$
- **(v)** *for each $\eta \in C(\mathbb{S}^1, \mathbb{R})$, one has $\eta(U)(JJ^* - 1_{\mathcal{H}})\eta(U) \in \mathscr{K}(\mathcal{H})$.*

Then, $U \in C^1(A)$ and A is a conjugate operator for U on an open set $\Theta \subset \mathbb{S}^1$ if the operator A_0 is a conjugate operator for U_0 on Θ.

Full evolution operator

Using the short-range assumption, we can verify that V is trace class, that $U \in C^{1+\varepsilon}(A)$, and the hypotheses of the last theorem. We thus get:

Theorem (Spectral properties of U)

The operator U has at most finitely many eigenvalues, each one of finite multiplicity, and no singular continuous spectrum.

This is the first result of this type for quantum walks on homogeneous trees of degree 3 with position-dependent coin.

Wave operators

Theorem (Completeness, version 1)

The wave operators $W_\pm(U, U_0, J) : \mathcal{H}_0 \to \mathcal{H}$ *given by*

$$W_\pm(U, U_0, J) := \operatorname*{s-lim}_{n\to\pm\infty} U^{-n} J U_0^n$$

exist and are complete, that is, $\mathrm{Ran}\,\big(W_\pm(U, U_0, J)\big) = \mathcal{H}_{\mathrm{ac}}(U)$.

Idea of the proof.

$W_\pm(U, U_0, J)$ exist because V is trace class. For the completeness, the idea is to note that $\mathcal{H}_{\mathrm{ac}}(U) \supset \mathrm{Ran}\,\big(W_\pm(U, U_0, J)\big) \supset \mathrm{Ran}(a_\pm)$ with

$$a_\pm \varphi := W_\pm(U, U_0, J)(\varphi, \varphi, \varphi), \quad \varphi \in \mathcal{H},$$

and show that $\mathrm{Ran}(a_\pm) = \mathcal{H}_{\mathrm{ac}}(U)$. □

A direct calculation shows that $a_\pm = \underset{n\to\pm\infty}{\text{s-lim}}\, U^{-n}\big(\sum_{k=1}^{3}\chi_k C_k\big)^n S^n$. Thus:

Corollary (Completeness, version 2)

The operators $a_\pm : \mathcal{H} \to \mathcal{H}$ *given by*

$$a_\pm := \underset{n\to\pm\infty}{\text{s-lim}}\, U^{-n} J_n S^n, \quad J_n := \big(\sum_{k=1}^{3}\chi_k C_k\big)^n,$$

exist, are isometric, and satisfy $\mathrm{Ran}(a_\pm) = \mathcal{H}_{\mathrm{ac}}(U)$.

In other words: If one uses the matrix powers $\big(\sum_{k=1}^{3}\chi_k C_k\big)^n$ as time-dependent identification operators, then the basic shift S can be used as a free evolution operator for U.

Corollary

One has $\sigma_{\mathrm{ac}}(U) = \mathbb{S}^1$*.*

Idea of the proof.

Follows in part from the fact that $a_\pm$ are unitary from $\mathcal{H} = \mathcal{H}_{\mathrm{ac}}(S)$ to $\mathcal{H}_{\mathrm{ac}}(U)$. □

Combining what precedes, we infer that the spectrum of U covers the whole unit circle and is purely absolutely continuous, outside possibly a finite set where U may have eigenvalues of finite multiplicity.

Finally:

Theorem (Completeness, version 3)

The wave operators $W_\pm(U, \widetilde{U}_0) : \mathcal{H} \to \mathcal{H}$ given by

$$W_\pm(U, \widetilde{U}_0) := \underset{n\to\pm\infty}{\text{s-lim}}\, U^{-n}(\widetilde{U}_0)^n, \quad \widetilde{U}_0 := S \sum_{k=1}^{3} \chi_k C_k,$$

exist, are isometric, and are complete, that is, $\mathrm{Ran}(W_\pm(U, \widetilde{U}_0)) = \mathcal{H}_{\mathrm{ac}}(U)$.

Idea of the proof.

Follows from the facts that $\sum_{k=1}^{3} \chi_k C_k$ is diagonal (but not constant) and $\widetilde{U}_0 - U$ trace class. □

Open problems

1. What are the initial subspaces of $W_{\pm}(U, U_0, J)$?
2. What are the asymptotic velocity operators of $U_i = SC_i$?
3. The case of homogenous trees of arbitrary degree ?
4. The case of rooted trees ?
5. The case of coin operators that converge at infinity to arbitrary constant unitary matrices ?

Thank you

References

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