

A PROOF OF SZNITMAN'S CONJECTURE ABOUT BALLISTIC RWRE

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ABSTRACT. We consider a random walk in a uniformly elliptic i.i.d. random environment in \mathbb{Z}^d for $d \geq 2$. It is believed that whenever the random walk is transient in a given direction it is necessarily ballistic. In order to quantify the gap which would be needed to prove this equivalence, several ballisticity conditions have been introduced. In particular, in [Sz01, Sz02], Sznitman defined the so called conditions (T) and (T') . The first one is the requirement that certain unlikely exit probabilities from a set of slabs decay exponentially fast with their width L . The second one is the requirement that for all $\gamma \in (0, 1)$ condition $(T)_\gamma$ is satisfied, which in turn is defined as the requirement that the decay is like e^{-CL^γ} for some $C > 0$. In this article we prove a conjecture of Sznitman of 2002 [Sz02], stating that (T) and (T') are equivalent. Hence, this closes the circle proving the equivalence of conditions (T) , (T') and $(T)_\gamma$ for some $\gamma \in (0, 1)$ as conjectured in [Sz02], and also of each of these ballisticity conditions with the polynomial condition $(P)_M$ for $M \geq 15d + 5$ introduced by Berger, Drewitz and Ramírez in [BDR14].

1. INTRODUCTION

Random walk in random environment is one of the most fundamental mathematical models of probability theory. It describes the movement of a particle in a disordered landscape and its importance stems due to its relevance as a reliable framework to study phenomena originating from different sciences, including its connection to homogenization theory through the link between the rescaled particle movement and the rescaling of the appropriate differential operators (see [Z04], [Sz04] or [DR14] for more details and further references). Some basic and simple to state questions about it have remained persistently open. An example is the relationship between directional transient behavior, where the random walk drifts away in a certain direction, and ballisticity, where this drifting happens with a non-vanishing velocity. For a random walk defined in the hyper-cubic lattice \mathbb{Z}^d in an environment which satisfies some minimal assumptions, which are uniform ellipticity and which is i.i.d., it is expected that if $d \geq 2$, directional transience implies ballisticity. In order to tackle this question, several intermediate conditions which are stronger than directional transience, but in some sense close to it, have been introduced, with the expectation that they would measure the gap needed to prove (or disprove), the conjectured equivalence between directional transience and ballisticity. Essentially, since directional transience in a given direction $\ell \in \mathbb{S}^{d-1}$ implies that the exit probability of the random walk from a slab perpendicular to ℓ , centered at the origin of width L , through its side in the negative region of space (in ℓ coordinate), decays to 0 as $L \rightarrow \infty$, it is natural to define intermediate conditions, called *ballisticity conditions*, which quantify this decay, with the expectation that they would imply ballisticity. Indeed, in [Sz01] and [Sz02], Sznitman defined the so called conditions (T) , (T') and $(T)_\gamma$ for $\gamma \in (0, 1)$. The first one is defined as the requirement that the above mentioned decay is exponentially fast as $L \rightarrow \infty$ for an open set of directions, $(T)_\gamma$ is defined as the requirement that the decay is like e^{-CL^γ} for some constant $C > 0$, while (T') is defined as the fulfillment of $(T)_\gamma$ for all

2010 *Mathematics Subject Classification.* 60K37, 82D30, 82C41.

Key words and phrases. Random walk in random environment, ballisticity conditions.

Enrique Guerra has been partially supported by Fondo Nacional de Desarrollo Científico y Tecnológico 3180255. Alejandro F. Ramírez has been partially supported by Fondo Nacional de Desarrollo Científico y Tecnológico 1180259 and Iniciativa Científica Milenio.

$\gamma \in (0, 1)$. It was shown in [Sz01, Sz02] and in [BDR14] that all of these conditions imply ballisticity and a functional central limit theorem. Thanks to these results, and to the fact that condition (T') can be in principle checked case by case, it was established in [Sz03], that it is possible to construct perturbations of the simple symmetric random walk with a very small local drift, which are nevertheless ballistic.

In [Sz02], Sznitman conjectured that $(T), (T')$ and $(T)_\gamma$ for any $\gamma \in (0, 1)$ are all equivalent, proving that for $\gamma \in (0.5, 1)$, indeed $(T)_\gamma$ implies (T') . Subsequently Drewitz and Ramírez in [DR11], were able to push this equivalence down to $\gamma \in (\gamma_d, 1)$ for some dimension dependent constant $\gamma_d \in (0.366, 0.388)$. In [DR12], the equivalence between (T) and (T') was established for dimensions $d \geq 4$, while in [BDR14] the full equivalence was proved between these conditions for $d \geq 2$ showing also that both conditions are equivalent to an effective polynomial condition $(P)_M$, for $M \geq 15d + 5$, where instead of exponential or stretched exponential decay for the exit probability through the unlikely side of the slabs, one imposes a polynomial decay of the form $1/L^M$. Nevertheless, although a strong indication that the conjectured equivalence between conditions (T) and (T') was true was given in [GR15], the proof of the equivalence remained open.

In this article we prove that for uniformly elliptic i.i.d. environments conditions (T) and (T') are equivalent, closing the circle and hence finishing the proof of Sznitman's conjecture of [Sz02] that $(T), (T')$ and $(T)_\gamma$ for any $\gamma \in (0, 1)$, are all equivalent.

The proof mimics one-dimensional estimates through a coarse graining method where sites are mapped into growing strips, introducing crucial controls on atypically small probabilities, to decouple the behavior of the random walk in overlapping strips.

In the following section we will define the basic notation and formulate the main result of this article. In Section 3, the proof of this theorem is presented.

2. NOTATION AND RESULTS

Denote by $|\cdot|_1$ and $|\cdot|_2$ the l_1 and l_2 norms respectively, defined on \mathbb{Z}^d and let $U := \{e \in \mathbb{Z}^d : |e|_1 = 1\} = \{e_1, -e_1, \dots, e_d, -e_d\}$. Define $\mathcal{P} := \{p(e), e \in U : p(e) \geq 0, \sum_{e \in U} p(e) = 1\}$ and the *environmental space* $\Omega := \mathcal{P}^{\mathbb{Z}^d}$. We will call an element of $\omega = \{\omega(x) : x \in \mathbb{Z}^d\} \in \Omega$ an *environment* where for each $x \in \mathbb{Z}^d$, $\omega(x) = \{\omega(x, e), e \in U\} \in \mathcal{P}$. A random walk in a fixed environment ω starting from $x \in \mathbb{Z}^d$ is defined as the Markov chain $\{X_n : n \geq 0\}$ with $X_0 = x$ and transition probabilities to jump from a site y to a nearest neighbor $y + e$, $\omega(y, e)$. We denote by $P_{x, \omega}$ the law of this random walk. Whenever a probability measure \mathbb{P} is prescribed on Ω , we call $P_{x, \omega}$ the *quenched law* of the random walk in random environment (RWRE). We define the *averaged* or *annealed* measure of the RWRE as the semidirect product $P_x := \mathbb{P} \times P_{x, \omega}$ defined on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$.

We will say that \mathbb{P} is *uniformly elliptic* if there is a constant $\kappa > 0$ such that $\mathbb{P}(\omega(x, e) \geq \kappa) = 1$ and that it is i.i.d. if the random variables $\{\omega(x) : x \in \mathbb{Z}^d\}$ are i.i.d. under \mathbb{P} . Throughout the rest of this article we will assume that \mathbb{P} is uniformly elliptic and i.i.d.

Given a direction $\ell \in \mathbb{S}^{d-1}$, we say that the random walk is transient in direction ℓ if

$$\lim_{n \rightarrow \infty} X_n \cdot \ell = \infty.$$

Furthermore, we say that the random walk is ballistic in direction ℓ if

$$\liminf_{n \rightarrow \infty} \frac{1}{n} X_n \cdot \ell > 0.$$

It is believed that for dimensions $d \geq 2$, whenever a random walk in a uniformly elliptic i.i.d. environment is transient in a given direction, it is necessarily ballistic [DR14]. In order to give at least a partial answer to the above question, several conditions which interpolate between directional transience and ballisticity, called *ballisticity conditions*, have been introduced: conditions $(T), (T')$,

$(T)_\gamma$ for $\gamma \in (0,1)$ defined by Sznitman in [Sz01, Sz02] and $(P)_M$ for $M \geq 1$, defined by Berger, Drewitz and Ramírez in [BDR14].

For $L \in \mathbb{R}$ and $\ell \in \mathcal{S}^{d-1}$ define the stopping times

$$T_L^\ell := \inf\{n \geq 0 : X_n \cdot \ell \geq L\} \quad (1)$$

and

$$\tilde{T}_L^\ell := \inf\{n \geq 0 : X_n \cdot \ell \leq L\}. \quad (2)$$

Let $\gamma \in (0,1]$. We say that condition $(T)_\gamma$ in direction ℓ , also denoted by $(T)_\gamma|\ell$, is satisfied if there exists an open subset O of \mathcal{S}^{d-1} containing ℓ such that for all $\ell' \in O$ we have that

$$\limsup_{L \rightarrow \infty} L^{-\gamma} \log P_0 \left[\tilde{T}_{-L}^{\ell'} < T_L^{\ell'} \right] < 0.$$

Condition $(T)|\ell$ is defined as $(T)_1|\ell$, while condition $(T')|\ell$ as the requirement that $(T^\gamma)|\ell$ is fulfilled for all $\gamma \in (0,1)$. For $M \geq 1$, the polynomial condition $(P)_M$, is essentially defined as the requirement that there exists an L_0 such that for some $L \geq L_0$ we have that

$$P_0 \left[\tilde{T}_{-L}^{\ell'} < T_L^{\ell'} \right] \leq \frac{1}{LM}$$

(see [BDR14] for the exact definition). Note the *effective* nature of the polynomial condition as opposed to conditions (T) , (T') and $(T)_\gamma$, in the sense that it is a condition that in principle can be verified for any given uniformly elliptic i.i.d. law \mathbb{P} .

In a series of works [DR11, DR12, BDR14], culminating with the introduction of the polynomial condition in [BDR14], the equivalence between conditions (T') , $(T)_\gamma$ for a given $\gamma \in (0,1)$ and the polynomial condition $(P)_M$ for $M \geq 15d + 5$, was established for all dimensions $d \geq 2$. Nevertheless, the conjectured equivalence between (T) and (T') has remained open. In this article we prove this equivalence.

Theorem 2.1. *Consider a random walk in an i.i.d. uniformly elliptic environment satisfying condition (T') . Then, condition (T) is satisfied.*

An automatic corollary of Theorem 2.1, is the extension of the applicability of Yilmaz large deviation result of [Y11] stating that under condition (T) there is equality between the quenched and annealed large deviation rate functions for random walks in uniformly elliptic i.i.d. environments in $d \geq 4$, to random walks satisfying condition $(P)_M$ for $M \geq 15d + 5$ for which condition (T) had not been proved directly, as the perturbative examples of [Sz03] and the more recent ones in [RS18].

The proof of Theorem 2.1, is based on a new method which captures the independence of events defined in overlapping slabs through a careful use of atypical quenched exit estimates. To explain in more detail this new strategy, we recall how Sznitman in [Sz02] proved that $(T)_\gamma$ for $\gamma \in (0.5,1)$ implies (T') . He introduced the so called *effective criterion*, which somehow mimics the well known criteria proved by Solomon in [So75] for random walks on \mathbb{Z} in an i.i.d. elliptic environment, which says that if the expectation of $\omega(0,-1)/(1-\omega(0,-1))$ is smaller than one, the random walk is ballistic to the right. He then proved that $(T)_\gamma$ for $\gamma \in (0.5,1)$ implies the effective criterion, and that the effective criterion implies (T') . The effective criterion is defined through a quantity ρ analogous to the one dimensional quotient, but defined in a larger scale, and basically it is required that some power of ρ should have a small expectation at some scale. As for one-dimensional random walks, somehow ρ can be used to expand the probability to exit through the left side of large slabs. To obtain an exponential decay of this probability (which would hence prove (T)), it is necessary to compute the expected value of products of ρ_i 's, where each $i \in \mathbb{Z}$ labels a sub-slab of the large slab, and ρ_i is distributed as ρ . On the other hand the sub-slabs

overlap by pairs, so that for a given i , ρ_i and ρ_{i+1} are not independent. This lack of independence is a big complication to obtain adequate estimates for the expectation, and in order to prove that the effective criterion implies condition (T') , Sznitman in [Sz02], decoupled the computation of the expectation of products of the ρ_i 's using Cauchy-Schwarz inequality. The iterative use of this argument caused in the end a decay which was not better than $e^{-L^{\gamma_L}}$ as $L \rightarrow \infty$, where L is the width of the final slab and $\gamma_L = 1 - \frac{C}{(\log L)^{1/2}}$, and hence the proof of (T') . Improving this argument through the use of Cauchy-Schwarz inequality is possible, and produces a decay which is somehow almost exponentially fast (see [GR15] for details), but still not enough to obtain the exponential decay of condition (T) . A key idea introduced in this article to prove Theorem 2.1 to avoid the use of Cauchy-Schwarz inequality is to compare the probability to exit through the left side of a sub-slab, with the probability to exit through the left side of the sub-slab without ever moving to the right of the initial departure point. This comparison is done through the use atypical quenched exit estimates which control the smallness of the quenched exit probability of the random walk through atypical exit points. These estimates have been extensively used in [Sz01, Sz02, DR12, BDR14, FH13] in the i.i.d. case and more recently in [G17] in the case of environments satisfying some kind of mixing condition. Once this comparison is done in a proper way, the computation of the expected value of the ρ_i 's is reduced to expectations of products of independent terms, essentially obtaining the desired exponential decay. These new methods are inspired on Sznitman's effective criterion but do not rely directly on it.

3. PROOF OF THEOREM 2.1

The proof of Theorem 2.1 will be done obtaining recursively estimates on the annealed atypical exit probability from appropriate boxes of the walk in an increasing sequence of scales. To do this, each box at a given scale will be subdivided in smaller boxes of a size of the previous scale. A key step here will be to decouple (in the sense of independence) the atypical exit probabilities of overlapping slabs. To implement our recursive argument, we will need a seed inequality, which will enable us to pass from estimates at a given scale to estimates at the next scale. This is the content of Section 3.1. In Section 3.2, the seed estimate is used to implement the recursion, and obtain a final estimate for the decay of the atypical exit probability at a given scale. In Section 3.3, this estimate is used to finish the proof of Theorem 2.1.

3.1. Seed estimate. Here we will derive an estimate for the annealed atypical exit probability of the random walk at a given scale in terms of the annealed atypical exit probability of the random walk at a smaller scale. In order to obtain a useful estimate, we will compare the atypical exit probability of a given box, with the event that the random walk exits atypically without crossing its starting level in the direction opposite to the atypical side. This comparison will be made through the use of classical atypical quenched exit estimates obtained by Sznitman in [Sz01, Sz02]. Once these probabilities are compared, it will be basically possible to argue that the quenched atypical exit probabilities from overlapping slabs at the smaller scale are independent.

Let us introduce some notation. Given a subset $V \subset \mathbb{Z}^d$, we define its boundary by

$$\partial V := \{x \notin V : |x - y|_1 = 1 \text{ for some } y \in V\}.$$

We are assuming that there is a direction $\ell \in \mathbb{S}^{d-1}$ such that $(T')|_\ell$ is satisfied. This means that we have a stretched exponential decay through atypical sides of slabs for $\ell' \in O$, where O is an open subset of \mathbb{S}^{d-1} containing ℓ . Let us fix $\ell' \in O$ and let R be a rotation on \mathbb{R}^d defined by $R(e_1) = \ell'$, $L > 0$ and $\tilde{L} > 0$. To simplify notation, we define the triple $\mathbf{S} := (R, L, \tilde{L})$ and define the box associate to the triple \mathbf{S} by

$$B_{\mathbf{S}} := R \left((-L, L) \times (-\tilde{L}, \tilde{L})^{d-1} \right) \cap \mathbb{Z}^d \quad (3)$$

and its *positive boundary* or *positive side* by

$$\partial^+ B_{\mathbf{S}} := \partial B_{\mathbf{S}} \cap \{z : z \cdot \ell' \geq L, |z \cdot R(e_k)| < \tilde{L}, \text{ for } 2 \leq k \in d\}.$$

We also define the random variable attached to \mathbf{S} ,

$$q_{\mathbf{S}} := P_{0,\omega} \left[X_{T_{B_{\mathbf{S}}}} \notin \partial^+ B_{\mathbf{S}} \right].$$

Let now L_0, \tilde{L}_0, L_1 and \tilde{L}_1 be integers greater than $3\sqrt{d}$ and such that

$$N := \frac{L_1}{L_0} \in \mathbb{N} \cap \{z \in \mathbb{R} : z \geq 2\} \text{ and } \tilde{N} := \frac{\tilde{L}_1}{\tilde{L}_0} \in \{z \in \mathbb{N} : z > N\}.$$

Throughout the rest of this article we will use the fact that there exists a constant

$$c_1 = c_1(d) \tag{4}$$

such that given any pair of points $x, y \in \mathbb{Z}^d$, there exists a nearest neighbor path of length at most $c_1|x - y|_1$ joining them. Furthermore, in general the constants (which might depend on the dimension d and the ellipticity constant κ) will be denoted by $c_1(d, \kappa), c_2(d, \kappa), \dots$, sometimes just writing c_1, c_2, \dots .

Now, consider the corresponding triples $\mathbf{S}_0 := (R, L_0, \tilde{L}_0)$ and $\mathbf{S}_1 := (R, L_1, \tilde{L}_1)$. The following seed estimate provides us with an upper bound for $\mathbb{E}[q_{\mathbf{S}_1}]$ in terms of $\mathbb{E}[q_{\mathbf{S}_0}]$.

Proposition 3.1 (Seed estimate). *Let $d \geq 2$. Consider a random walk in an i.i.d. uniformly elliptic environment. Let $\ell \in \mathbb{S}^{d-1}$ and assume that condition (T') is satisfied and let $\beta \in (1/2, 1)$. Then there exists $c_2(d, \kappa) > 0, \mu > 0$ and an open set $O \subset \mathbb{S}^{d-1}$ which contains ℓ , such that for all $\ell' \in O, L_0 > 3\sqrt{d}, \tilde{L}_0 > 3\sqrt{d}$ and $\tilde{N} \geq 48N$ we have that*

$$\begin{aligned} \mathbb{E}[q_{\mathbf{S}_1}] &\leq (N+2) \left(c_2 \kappa^{-3c_1} \tilde{L}_1^{d-1} e^{3c_1 \log(1/\kappa) L_0^\beta} \right)^{2N+2} \mathbb{E}[q_{\mathbf{S}_0}]^N \\ &\quad + \left(c_2 \tilde{L}_1^{d-2} \frac{L_1^3}{L_0^2} \tilde{L}_0 \mathbb{E}[q_{\mathbf{S}_0}] \right)^{\frac{\tilde{N}}{12N}} + c_2 N \tilde{L}_1^{d-1} e^{-\mu L_0^{d(2\beta-1)}}. \end{aligned} \tag{5}$$

Proof. The strategy will be to divide the box defined by the triple $\mathbf{S}_1 = (R, L_1, \tilde{L}_1)$ into smaller sub-boxes corresponding to $\mathbf{S}_0 = (R, L_0, \tilde{L}_0)$. We should have in mind that the eventually the length \tilde{L}_0 and \tilde{L}_1 will be chosen much larger than L_0 and L_1 respectively, so the boxes will look very much like slabs. We will divide the proof in eight steps: in step 0, we give the necessary definitions to make the above described subdivision; in step 1, we use classical one-dimensional arguments to estimate the probability that the random walk exits through the atypical sides of the large box (the back and lateral sides), in terms of an expansion involving a quantity analogous to the one-dimensional quotient between jumping to the left and to the right; in step 2 we will define an important typical quenched event which will eventually decouple the behavior in different slabs; in step 3 we will use the above definition to make this decoupling; in steps 4 and 5, we will apply the previous decoupling to express the atypical exit probability from the large box, in terms of a product of corresponding probabilities in the small scale; in step 6 we bound the atypical quenched exit event; in step 7, we bound the probability of lateral exit of the random walk; and finally in step 8 we combine the estimates of the previous steps to finish the proof.

Step 0: preliminary definitions. For each $i \in \mathbb{Z}$ define the set

$$\mathcal{H}_i := \{x \in \mathbb{Z}^d : |x - x'|_1 = 1 \text{ and } (x \cdot \ell' - iL_0)(x' \cdot \ell' - iL_0) \leq 0 \text{ for some } x' \in \mathbb{Z}^d\}$$

as well as the function $I : \mathbb{Z}^d \rightarrow \mathbb{Z}$ given by

$$I(x) = i, \text{ whenever } x \cdot \ell \in \left[iL_0 - \frac{L_0}{2}, iL_0 + \frac{L_0}{2} \right).$$

Let $(\theta_n)_{n \geq 0}$ be the canonical shift on $(\mathbb{Z}^d)^\mathbb{N}$. We then define the successive visit times to the different $(\mathcal{H}_i)_{i \in \mathbb{Z}}$ sets as

$$\begin{aligned} V_0 &= 0, \quad V_1 = \inf\{n \geq 0 : X_n \in \mathcal{H}_{I(X_0)+1} \cup \mathcal{H}_{I(X_0)-1}\} \\ &\text{and } V_{i+1} = V_1 \circ \theta_{V_i} + V_i \text{ for } i \geq 1. \end{aligned}$$

Let us also define the first exit time of the random walk from the box B_{S_1} through its lateral sides as

$$\tilde{T} := \inf\{n \geq 0 : |X_n \cdot R(e_j)| \geq \tilde{L}_1 \text{ for some } j \in [2, d]\}.$$

In order to mimic the one-dimensional criteria for ballisticity of random walks in random environment [So75], it will be convenient to consider for each $x \in \mathbb{Z}^d$ and integer i the random variables defined by the equations

$$\begin{aligned} q(x, \omega) &:= P_{x, \omega}[X_{V_1} \in \mathcal{H}_{I(x)-1}] =: 1 - p(x, \omega), \\ \hat{q}(x, \omega) &:= P_{x, \omega}[X_{V_1} \in \mathcal{H}_{I(x)-1}, V_1 \leq \tilde{T}], \\ \hat{p}(x, \omega) &:= P_{x, \omega}[X_{V_1} \in \mathcal{H}_{I(x)+1}, V_1 \leq \tilde{T}] \text{ and} \\ \hat{\rho}(i, \omega) &:= \sup \left\{ \frac{\hat{q}(x, \omega)}{p(x, \omega)} : x \in \mathcal{H}_i, |x \cdot R(e_j)|_2 < \tilde{L}_1 \text{ for all } 2 \leq j \leq d \right\}. \end{aligned} \quad (6)$$

Consider the function $f : \Omega \times \mathbb{Z} \rightarrow \mathbb{R}^+$ defined by

$$\begin{aligned} f(j, \omega) &= 0 \text{ for } j \geq N+2 \text{ and} \\ f(j, \omega) &= \sum_{j \leq m \leq N+1} \prod_{m < i \leq N+1} \hat{\rho}(i, \omega)^{-1} \text{ for } j \leq N+1. \end{aligned} \quad (7)$$

Throughout the rest of the proof we may not write explicitly the dependence on ω of the random variables involved.

Step 1: one-dimensional argument to bound exit probabilities. Here we will use one-dimensional explicit formulas for exit probabilities to obtain the following bound.

$$P_{0, \omega} \left[\tilde{T}_{-L_1}^{\ell'} < \tilde{T} \wedge T_{L_1}^{\ell'} \right] \leq \frac{f(0)}{f(-N)}, \quad (8)$$

[cf. (2) and (1)]. The proof of (8) is similar to the proof of inequality (2.18) of [Sz02], but for completeness we will outline it here. Consider the $(\mathcal{F}_{V_m})_{m \geq 0}$ -stopping time

$$\tau := \inf\{m \geq 0 : X_{V_m} \in \mathcal{H}_{-N} \cup \mathcal{H}_{N+1}\}.$$

We now assert that the random variables

$$E_{0, \omega} [f(I(X_{V_{m \wedge \tau}})), V_{m \wedge \tau} \leq \tilde{T}]$$

are decreasing with m . Indeed, as in (2.20) of ([Sz02]) observe that

$$\begin{aligned} &E_{0, \omega} [f(I(X_{V_{(m+1) \wedge \tau}})), V_{(m+1) \wedge \tau} \leq \tilde{T}] \\ &\leq E_{0, \omega} [f(I(X_{V_{m \wedge \tau}})), V_{m \wedge \tau} \leq \tilde{T}, \tau \leq m] + E_{0, \omega} [f(I(X_{V_{m+1}})), V_{m+1} < \tilde{T}, \tau > m] \\ &= E_{0, \omega} [f(I(X_{V_{m \wedge \tau}})), V_{m \wedge \tau} \leq \tilde{T}, \tau \leq m] + E_{0, \omega} [\tau > m, V_m < \tilde{T}, E_{X_{V_m}, \omega} [f(I(X_{V_1})), V_1 \leq \tilde{T}]]. \end{aligned}$$

On the other hand, on $\{\tau > m, V_m < \tilde{T}\}$ (recall definitions in (6)) we have that $P_{X_{V_m}, \omega}$ -a.s

$$\begin{aligned}
 & E_{X_{V_m}, \omega} [f(I(X_{V_1})), V_1 \leq \tilde{T}] \\
 & \leq f(I(X_{V_m})) + p(X_{V_m}, \omega) (f(I(X_{V_m}) + 1) - f(I(X_{V_m}))) \\
 & \quad + \hat{q}(X_{V_m}, \omega) (f(I(X_{V_m}) - 1) - f(I(X_{V_m}))) \\
 & \leq f(I(X_{V_m})) + \prod_{I(X_{V_m})-1 < j \leq N+1} \hat{\rho}(j)^{-1} (\hat{q}(X_{V_m}, \omega) - p(X_{V_m}, \omega) \hat{\rho}(I(X_{V_m}))), \tag{9}
 \end{aligned}$$

where we have used that $\hat{p}(X_{V_m}, \omega) \leq p(X_{V_m}, \omega)$ and $p(X_{V_m}, \omega) + \hat{q}(X_{V_m}, \omega) \leq 1$ in the first inequality and the explicit expression for $\hat{\rho}$ in (7) to get the last inequality. At the same time, by the definitions (6) and by the fact that $P_{0, \omega}$ -a.s. on the event $\{V_m < \tilde{T}\}$, we have $X_{V_m} \in \mathcal{H}_{I(X_{V_m})} \cap \{z \in \mathbb{Z}^d : |z \cdot R(e_i)| < \tilde{L}_1 \text{ for all } 2 \leq i \leq d\}$ we conclude that the last term in (9) is negative. As a result, using Fatou's lemma, we have

$$E_{0, \omega} [f(I(X_{V_\tau})), V_\tau \leq \tilde{T}, \tilde{T}_{-L_1}^{\ell'} < \tilde{T} \wedge T_{L_1}^{\ell'}] \leq f(0).$$

The claim (8) follows now after noticing $P_{0, \omega}$ -a.s. on the event appearing in (8), that $X_{V_\tau} \in \mathcal{H}_{-N}$ and $V_\tau \leq \tilde{T}$.

Step 2: typical quenched exit event. Here we will define an exit event for the random walk at a given slab, which corresponds somehow to a minimal size the typical quenched exit probability should have. Let us start introducing for each $i \in \mathbb{Z}$, the *frontal* part of the \mathcal{H}_i -boundary,

$$\partial^+ \mathcal{H}_i = \partial \mathcal{H}_i \cap \{z : z \cdot \ell' - iL_0 \geq 0\}.$$

Define also for each $x \in \mathbb{Z}^d$ the quenched probabilities that the random walk starting from x exits the corresponding slab through its atypical side, but without ever visiting the right-hand half of the slab, as

$$\tilde{q}(x, \omega) = P_{x, \omega} [X_{V_1} \in \mathcal{H}_{I(x)-1}, H_{\partial^+ \mathcal{H}_{I(x)}} = \infty]. \tag{10}$$

The above probability will be a key definition in our proof since it will be in a sense comparable to $q(x, \omega)$, but it will enable us to produce enough independence in the products appearing in the right-hand side of (8).

Let us also define the truncation of \mathcal{H}_i as

$$\mathcal{H}'_i := \mathcal{H}_i \cap \{z : |z \cdot R(e_j)| < \tilde{L}_1, \text{ for all } 2 \leq j \leq d\},$$

its frontal boundary by

$$\partial^+ \mathcal{H}'_i := \partial^+ \mathcal{H}_i \cap \{z : |z \cdot R(e_j)| < \tilde{L}_1, \text{ for all } 2 \leq j \leq d\}$$

and for $\beta \in (0, 1)$ define

$$\begin{aligned}
 \mathcal{H}_{i, \beta} := & \left\{ x \in \mathbb{Z}^d : \exists x' \in \mathbb{Z}^d, |x - x'|_1 = 1 \left(x \cdot \ell' - i(L_0 + 1 + L_0^\beta) \right) \times \right. \\
 & \left. \left(x' \cdot \ell' - i(L_0 + 1 + L_0^\beta) \leq 0 \right) \right\} \cap \{z : |z \cdot R(e_j)| < \tilde{L}_1, \text{ for all } 2 \leq j \leq d\}.
 \end{aligned}$$

Keeping in mind the above remark let

$$\tilde{c} := c_1 \log \left(\frac{1}{\kappa} \right) \tag{11}$$

and the asymmetric slab

$$U_{\beta, L_0} := \{x \in \mathbb{R}^d : x \cdot \ell' \in (-L_0^\beta, L_0)\}.$$

We can now define the *typical quenched exit event* as

$$\mathfrak{T} := \left\{ \omega \in \Omega : \inf_{\substack{z \in \mathcal{H}_{i,\beta} \\ -N \leq i \leq N+2}} P_{z,\omega} \left[\left(X_{T_{z+U_{\beta,L_0}}} - z \right) \cdot \ell' > 0 \right] > e^{-\tilde{c}L_0^\beta}, \right. \\ \left. \inf_{\substack{z \in \mathcal{H}'_i \\ -N \leq i \leq N+2}} P_{z,\omega} \left[\left(X_{T_{z+U_{\beta,L_0}}} - z \right) \cdot \ell' > 0 \right] > e^{-\tilde{c}L_0^\beta} \right\}. \quad (12)$$

Step 3: comparing \hat{q} with \tilde{q} . Here we will prove that whenever $\omega \in \mathfrak{T}$, for all $i \in \mathbb{Z}$ and $x \in \mathcal{H}_i$,

$$\hat{q}(x, \omega) \leq e^{2\tilde{c}L^\beta} \sup_{y \in \mathcal{H}'_i} \tilde{q}(y, \omega). \quad (13)$$

To prove (13) note that on the event $\{X_{V_1} \in \mathcal{H}_{I(x)-1}, V_1 \leq \tilde{T}\}$, the number of excursions from the set \mathcal{H}_i to its frontal boundary $\partial^+ \mathcal{H}_i$ before the time V_1 ,

$$\mathfrak{E}_i := \sum_{n=0}^{V_1-1} \mathbb{1}_{\{X_{n-1} \in \mathcal{H}_i, X_n \in \partial^+ \mathcal{H}_i\}} \quad (14)$$

is $P_{x,\omega}$ -a.s. finite. Before using the finiteness of the random variables defined in (14), we define $U_0 := 0$ and sequences of $(\mathcal{F}_n)_{n \geq 0}$ -stopping times corresponding to the moments of the consecutive excursions defined above, as

$$U_1 := \inf\{n \geq 0 : X_n \in \partial^+ \mathcal{H}_i\}, W_1 := H_{\mathcal{H}_i} \circ \theta_{U_1} + U_1,$$

and by recursion in $k \geq 1$ define

$$U_{k+1} := U_1 \circ \theta_{W_k} + W_k, W_{k+1} := H_{\mathcal{H}_i} \circ \theta_{U_{k+1}} + U_{k+1}.$$

Then, for each $i \in \mathbb{Z}$ and $x \in \mathcal{H}'_i$, we have

$$P_{x,\omega} \left[X_{V_1} \in \mathcal{H}_{I(x)-1}, V_1 \leq \tilde{T} \right] = \sum_{j=0}^{\infty} P_{x,\omega} \left[X_{V_1} \in \mathcal{H}_{I(x)-1}, V_1 \leq \tilde{T}, \mathfrak{E}_i = j \right] \leq \\ P_{x,\omega} \left[X_{V_1} \in \mathcal{H}_{I(x)-1}, H_{\partial^+ \mathcal{H}_i} = \infty \right] + \sum_{j=1}^{\infty} P_{x,\omega} \left[X_{V_1} \in \mathcal{H}_{I(x)-1}, V_1 \leq \tilde{T}, \mathfrak{E}_i = j \right]. \quad (15)$$

On the other hand, for each $j \geq 1$ we can use successively the strong Markov property to see that

$$P_{x,\omega} \left[X_{V_1} \in \mathcal{H}_{I(x)-1}, V_1 \leq \tilde{T}, \mathfrak{E}_i = j \right] \leq \\ P_{x,\omega} \left[U_1 < \tilde{T} \wedge \mathcal{H}_{I(x)-1}, P_{X_{U_1}, \omega} \left[W_1 < \tilde{T} \wedge H_{\mathcal{H}_{I(x)+1}}, \dots \right. \right. \\ \left. \dots, P_{X_{W_{j-1}}, \omega} \left[U_1 < \tilde{T} \wedge H_{\mathcal{H}_{I(x)-1}}, P_{X_{U_j}, \omega} \left[W_1 < \tilde{T} \wedge \mathcal{H}_{I(x)+1}, \right. \right. \right. \\ \left. \left. P_{X_{W_j}, \omega} \left[X_{V_1} \in \mathcal{H}_{I(x)-1}, V_1 \leq \tilde{T}, H_{\partial^+ \mathcal{H}_i} = \infty \right] \dots \right] \right]. \quad (16)$$

Moreover, notice that the last expression in (16) is less than or equal to

$$\sup_{x \in \mathcal{H}'_i} P_{x,\omega} \left[X_{V_1} \in \mathcal{H}_{I(x)-1}, H_{\partial^+ \mathcal{H}_i} = \infty \right] \sup_{y \in \partial^+ \mathcal{H}'_i} P_{y,\omega} \left[W_1 < \tilde{T} \wedge H_{\mathcal{H}_{I(x)+1}} \right]^j. \quad (17)$$

Therefore, applying (17) to inequality (15) we get that

$$\begin{aligned}
 P_{x,\omega} \left[X_{V_1} \in \mathcal{H}_{I(x)-1}, V_1 \leq \tilde{T} \right] &\leq \sup_{x \in \mathcal{H}'_i} P_{x,\omega} \left[X_{V_1} \in \mathcal{H}_{I(x)-1}, H_{\partial^+ \mathcal{H}_i} = \infty \right] \\
 &\times \sum_{j=0}^{\infty} \sup_{y \in \partial^+ \mathcal{H}'_i} P_{y,\omega} \left[W_1 < \tilde{T} \wedge H_{\mathcal{H}_{i+1}} \right]^j.
 \end{aligned} \tag{18}$$

Note that for each $-N \leq i \leq N$ and each $y \in \partial^+ \mathcal{H}'_i$, we know that there is another point $y' \in H_{i,\beta}$ which can be joined to y using a nearest neighbour path inside of box B_1 of length less than or equal to $c_1 L_0^\beta$ [cf. (4)]. Thus, since $\omega \in \mathfrak{T}$,

$$\begin{aligned}
 P_{y,\omega} \left[W_1 < \tilde{T} \wedge H_{\mathcal{H}_{i+1}} \right] &\leq 1 - P_{y,\omega} \left[W_1 \geq H_{\mathcal{H}_{i+1}} \right] \\
 &\leq 1 - \kappa^{c_1 L_0^\beta} P_{y',\omega} \left[\left(X_{T_{y'+u_{\beta,L_0}}} - y' \right) \cdot \ell' > 0 \right] \\
 &\stackrel{(12)}{<} 1 - \kappa^{-c_1 L_0^\beta} e^{-\tilde{c} L_0^\beta} \stackrel{(11)}{=} 1 - e^{-2\tilde{c} L_0^\beta}.
 \end{aligned}$$

Using the fact that $\sum_{j=0}^{\infty} (1 - e^{-2\tilde{c} L_0^\beta})^j = e^{2\tilde{c} L_0^\beta}$, this finishes the proof of (13).

Step 4: bound on the atypical exit probability in the typical quenched exit event. Here we will prove that

$$\mathbb{E} \left[P_{0,\omega} \left[\tilde{T}_{-L_1}^{\ell'} \leq T_{L_1}^{\ell'} \wedge \tilde{T} \right], \mathfrak{T} \right] \leq \sum_{m=0}^{N+1} \prod_{-N < i \leq m} \left(e^{3\tilde{c} L_0^\beta} \mathbb{E} [\tilde{q}(i, \omega)] \right). \tag{19}$$

Notice that by inequality (8), we find that

$$P_{0,\omega} \left[\tilde{T}_{-L_1}^{\ell'} \leq T_{L_1}^{\ell'} \wedge \tilde{T} \right] \leq \frac{\sum_{m=0}^{N+1} \prod_{m < i \leq N+1} \rho_i^{-1}}{\prod_{-N < j \leq N+1} \rho_j^{-1}} = \sum_{m=0}^{N+1} \prod_{-N < i \leq m} \rho_i. \tag{20}$$

Now, on the typical quenched exit event \mathfrak{T} we have that

$$\sum_{m=0}^{N+1} \prod_{-N < i \leq m} \rho_i \leq \sum_{m=0}^{N+1} \prod_{-N < i \leq m} \left(e^{\tilde{c} L_0^\beta} \hat{q}(i, \omega) \right). \tag{21}$$

Combining the bounds (13) with (21) and (20), we conclude that

$$\mathbb{E} \left[P_{0,\omega} \left[\tilde{T}_{-L_1}^{\ell'} \leq T_{L_1}^{\ell'} \wedge \tilde{T} \right], \mathfrak{T} \right] \leq \sum_{m=0}^{N+1} \mathbb{E} \left[\prod_{-N < i \leq m} \left(e^{3\tilde{c} L_0^\beta} \tilde{q}(i, \omega) \right) \right]. \tag{22}$$

Now we use the crucial observation that as $-N \leq i \leq N$, the random variables $\tilde{q}(i, \omega)$ are independent, which finishes the proof of (19).

Step 5: refined bound on the atypical exit probability in the typical quenched exit event. Here we will refine the bound (19), showing that there exists a constant c_3 , such that

$$\mathbb{E} \left[P_{0,\omega} \left[\tilde{T}_{-L_1}^{\ell'} \leq \tilde{T} \wedge T_{L_1}^{\ell'} \right], \mathfrak{T} \right] \leq \sum_{m=0}^{N+1} \left(c_3 e^{3\tilde{c} L_0^\beta} \tilde{L}_1^{d-1} \kappa^{-3c_1} \mathbb{E} [q_{\mathbf{s}_0}] \right)^{N+m} \tag{23}$$

Observe that for each $i \in \mathbb{Z}$ and $y \in \mathcal{H}'_i$, there exist a point $y' \in \{z : |z \cdot R(e_k)| < \tilde{L}_1\} \cap \mathbb{Z}^d$ such that $|y + 3\ell' - y'|_1 \leq 1$ and a self-avoiding nearest neighbour path of length at most $3c_1$ connecting y with y' . Therefore,

$$\begin{aligned} q_{\mathbf{s}_0}(\omega) \circ \theta_{y'} &= P_{y', \omega} \left[X_{T_{B_{\mathbf{s}_0+y'}}} \notin \partial^+ B_{\mathbf{s}_0+y'} \right] \\ &\geq \kappa^{3c_1} P_{y', \omega} \left[X_{V_1} \in \mathcal{H}_{I(y)-1}, H_{\partial^+ \mathcal{H}_i} = \infty \right]. \end{aligned}$$

Hence defining

$$\begin{aligned} \mathcal{H}_{i,3} &:= \left\{ z \in \mathbb{Z}^d : \exists y \in \mathbb{Z}^d |z - y|_1 = 1, (z - iL_0 - 3)(y - iL_0 - 3) \leq 0 \right\} \\ &\quad \cap \{z : |z \cdot R(e_k)| < \tilde{L}_1, \text{ for all } 2 \leq k \leq d\} \end{aligned}$$

we see that

$$\tilde{q}(i, \omega) \leq \kappa^{-3c_1} \sup_{z \in \mathcal{H}_{i,3}} P_{z, \omega} \left[X_{T_{B_{\mathbf{s}_0+z}}} \notin \partial^+ B_{\mathbf{s}_0+z} \right] = \kappa^{-3c_1} \sup_{z \in \mathcal{H}_{i,3}} q_{\mathbf{s}_0}(\omega) \circ \theta_z.$$

Using the bound $\sup_{z \in \mathcal{H}_{i,3}} q_{\mathbf{s}_0}(\omega) \circ \theta_z \leq \sum_{z \in \mathcal{H}_{i,3}} q_{\mathbf{s}_0}(\omega) \circ \theta_z$, we finish the proof of inequality (23).

Step 6: bound on the atypical exit event. Here we will show that there exists a constant $\mu > 0$ such that

$$\mathbb{P}[\mathfrak{I}^c] \leq c_4 N \tilde{L}_1^{d-1} e^{-\mu L_0^{d(2\beta-1)}} \quad (24)$$

holds, for some suitable constant c_4 . Indeed, by Theorem 4.4 of [Sz04] (see also [Sz01, Sz02]), we know that there exist constants $\mu > 0$ (not depending on $\ell' \in O$) and L' such that for all $L \geq L'$ one has that

$$\mathbb{P} \left[P_{0, \omega} \left[X_{T_{U_{\beta, L}}} \cdot \ell' > 0 \right] \leq e^{-\tilde{c}L^\beta} \right] \leq e^{-\mu L^{d(2\beta-1)}}. \quad (25)$$

Choosing $L_0 \geq L'$, using the bound (25) for all the points which are in some \mathcal{H}'_i or $\mathcal{H}'_{i, \beta}$ for some $N \leq i \leq N+2$, and the fact that the cardinality of these points is $c_4 N \tilde{L}_1^{d-1}$ for some constant c_4 , we obtain (24).

Step 7: upper bound on the lateral exit probability. Using exactly the same argument as the one presented in pages 524-526 of the proof of Proposition 2.1 of [Sz02], we see that there is a constant c_5 such that

$$P_0 \left[\tilde{T} \leq \tilde{T}'_{-L_1} \wedge T'_{L_1} \right] \leq (2d-2) \left(c_5 \tilde{L}_1^{(d-2)} \frac{L_1^3}{L_0^2} \tilde{L}_0 \mathbb{E} [q_{\mathbf{s}_0}(\omega)] \right)^{\frac{\tilde{N}}{12N}}, \quad (26)$$

where we used that $\tilde{N} \geq 48N$ and that $N \geq 3$.

Step 8: conclusion. In view of (23)-(24) and (26), we see that there exist positive constants c_3, c_4, c_5 and c_6 such that

$$\begin{aligned} \mathbb{E} [q_{\mathbf{s}_1}(\omega)] &\leq P_0 \left[\tilde{T}'_{-L_1} \leq \tilde{T} \wedge T'_{L_1} \right] + P_0 \left[\tilde{T} \leq \tilde{T}'_{-L_1} \wedge T'_{L_1} \right] \\ &\leq \mathbb{E} \left[P_{0, \omega} \left[\tilde{T}'_{-L_1} \leq \tilde{T} \wedge T'_{L_1} \right], \mathfrak{I} \right] + \mathbb{P}[\mathfrak{I}^c] + P_0 \left[\tilde{T} \leq \tilde{T}'_{-L_1} \wedge T'_{L_1} \right] \\ &\stackrel{(23)-(24)-(26)}{\leq} \sum_{m=0}^{N+1} \left(c_3 \kappa^{-3c_1} \tilde{L}_1^{d-1} e^{3\tilde{c}L_0^\beta} \mathbb{E} [q_{\mathbf{s}_0}(\omega)] \right)^{m+N} \\ &\quad + c_4 N \tilde{L}_1^{d-1} e^{-\mu L_0^{d(2\beta-1)}} + c_6 \left(c_5 \tilde{L}_1^{(d-2)} \frac{L_1^3}{L_0^2} \tilde{L}_0 \mathbb{E} [q_{\mathbf{s}_0}(\omega)] \right)^{\frac{\tilde{N}}{12N}}. \end{aligned} \quad (27)$$

□

3.2. Recursion. Here we will use the result of Proposition 3.1, to inductively derive a bound for the exit probability through atypical sides of boxes at an increasing sequence of scales. Since we assume $(T')|_\ell$, we know that there is an open set $O \subset \mathbb{S}^{d-1}$ containing ℓ , such that for $\ell' \in O$, the atypical exit probabilities from slabs decay like stretched exponentials. Now, given $\ell' \in O$, we choose a rotation R , with $R(e_1) = \ell'$. Let $\nu > 0$. We next consider sequences of scales $(L_k)_{k \geq 0}$ and $(\tilde{L}_k)_{k \geq 0}$, defining a sequence of triples $(\mathbf{S}_k)_{k \geq 0}$ associated to the corresponding boxes

$$B_k := B_{\mathbf{S}_k} := R \left((-L_k, L_k) \times (-\tilde{L}_k, \tilde{L}_k)^{d-1} \right) \cap \mathbb{Z}^d$$

according to the notation (3), that satisfy

$$L_0 > 3\sqrt{d} \tag{28}$$

$$L_0^3 > \tilde{L}_0 > L_0, \tag{29}$$

and

$$L_k = \nu L_{k-1}, \text{ for } k \geq 1, \tag{30}$$

$$\tilde{L}_k = \nu^3 \tilde{L}_{k-1}, \text{ for } k \geq 1. \tag{31}$$

Throughout we write q_k and B_k in place of $q_{\mathbf{S}_k}$ and $B_{\mathbf{S}_k}$. We will also need to use the fact that $(T')|_\ell$ implies that for $\beta \in (3/4, 1)$, there is a constant $c_7(d, \kappa)$ and an $L'' > 0$ such that whenever we choose $L_0 \geq L''$ one has that

$$\mathbb{E}[q_0] \leq e^{-c_7 L_0^{\frac{\beta+1}{2}}}. \tag{32}$$

The next lemma will be instrumental for the final proof.

Lemma 3.2. *Let $d \geq 2$. Consider a random walk in random environment satisfying condition $(T')|_\ell$. Then, there exists $\nu = \nu(d, \kappa) > 0$, $L_0 = L_0(d, \kappa) > 0$, $\tilde{L}_0 = \tilde{L}_0(d, \kappa) > 0$ satisfying (28) and (29), and a constant $c_8(d, \kappa) > 0$ such that for all $\ell' \in O$ and $k \geq 0$ we have that for $(L_k)_{k \geq 1}$ defined as in (30) and $(\tilde{L}_k)_{k \geq 1}$ as in (31),*

$$\mathbb{E}[q_k] \leq e^{-c_8 L_k}. \tag{33}$$

Proof. Note that by (32), we have that

$$\mathbb{E}[q_0] \leq e^{-d_0 L_0}, \tag{34}$$

with

$$d_0 := \frac{c_7}{L_0^{\frac{1-\beta}{2}}},$$

for some fixed $\beta \in (3/4, 1)$. Let us now define recursively for $k \geq 0$,

$$d_{k+1} := d_k - \left(\left(1 + 3c_1 \log \frac{1}{\kappa} \right) L_0^\beta + 3 \right) \frac{1}{\nu^{(1-\beta)k}}.$$

We will first prove by induction on $k \geq 0$, that

$$\mathbb{E}[q_k] \leq e^{-d_k L_k}. \tag{35}$$

Note that (35) is satisfied for $k = 0$ (which is (34)). Let us now assume that (35) is satisfied for $k \geq 0$. We will prove that it is then also satisfied by $k + 1$. Note that $L_k = \nu^k L_0$ while $\tilde{L}_k = \nu^{3k} \tilde{L}_0$. Now, by inequality (5) of Proposition 3.1 we have that

$$\begin{aligned}
\mathbb{E}[q_{k+1}] &\leq (\nu + 2) \left(c_2 \kappa^{-3c_1} \tilde{L}_{k+1}^{d-1} e^{3c_1(\log(1/\kappa))L_k^\beta} \right)^{2\nu+2} \mathbb{E}[q_k]^\nu \\
&\quad + \left(c_2 \tilde{L}_{k+1}^{d-2} \frac{L_{k+1}^3}{L_k^2} \tilde{L}_k \mathbb{E}[q_k] \right)^{\frac{\nu^3}{12\nu}} + c_2 \nu \tilde{L}_{k+1}^{d-1} e^{-\mu L_k^{d(2\beta-1)}} \\
&= (\nu + 2) \left(c_2 \kappa^{-3c_1} \nu^{3(k+1)(d-1)} \tilde{L}_0^{d-1} e^{3c_1(\log(1/\kappa))\nu^{\beta k} L_0^\beta} \right)^{2\nu+2} \mathbb{E}[q_k]^\nu \\
&\quad + \left(c_2 \nu^{3(d-2)(k+1)-1} \tilde{L}_0^{d-1} L_0 \mathbb{E}[q_k] \right)^{\frac{\nu^2}{12}} + c_2 \nu^{3(d-1)(k+1)} \tilde{L}_0^{d-1} e^{-\mu \nu^{dk(2\beta-1)} L_0^{d(2\beta-1)}}. \tag{36}
\end{aligned}$$

We will analyze each of the terms of (36) separately. Note that the third term can be written as

$$\exp \left\{ -\mu \nu^{kd(2\beta-1)} L_0^{d(2\beta-1)} + \nu^k L_0 d_0 + 3(d-1)(k+1)\nu + \log(c_2 \tilde{L}_0^{d-1}) \right\} e^{-d_0 L_{k+1}}.$$

Now, the exponent of the first exponential of the above expression is bounded from above by

$$-(\mu \nu^{d(2\beta-1)-1} L_0^{d(2\beta-1)} - L_0 d_0) \nu.$$

Hence, using the fact that for $\beta \in (3/4, 1)$, one has that $d(2\beta-1) > 1$, we can see that there is a $\nu_0 = \nu_0(d, \kappa)$ (also depending on the choice of L_0) such that for $\nu \geq \nu_0$, the above expression is bounded from above by $\log \frac{1}{3}$ and hence the third term of (36) is bounded from above by

$$\frac{1}{3} e^{-d_0 L_{k+1}}. \tag{37}$$

A similar analysis lets us conclude that there is a $\nu_1 = \nu_1(d, \kappa) > \nu_0$ (also depending on the choice of L_0) such that the second term of the right-hand side of (36) is bounded from above also by (37). Let us now write the first term of the right-hand side of (36) as

$$\begin{aligned}
&\exp \left\{ \left(\log \left((\nu + 2) \nu^{3(k+1)(d-1)} \right) + \log \frac{c_2 \tilde{L}_0^{d-1}}{\kappa^{3c_1}} \right. \right. \\
&\quad \left. \left. + 3c_1 \left(\log \frac{1}{\kappa} \right) L_0^\beta \nu^{\beta k} \right) (2\nu + 2) - d_k \nu^{k+1} L_0 \right\}.
\end{aligned}$$

Now, note that there is a $\nu_2 = \nu_2(d, \kappa) \geq \nu_1$ (also depending on the choice of L_0) such that for $\nu \geq \nu_2$, the above expression is bounded from above by

$$\left(\left(\left(1 + 3c_1 \log \frac{1}{\kappa} \right) L_0^\beta + 1 \right) \frac{1}{\nu^{(1-\beta)k}} - d_k \right) L_{k+1}$$

which proves that the first term of the right-hand side of (36) is bounded from above by

$$\frac{1}{3} e^{-d_{k+1} L_{k+1}}.$$

Combining this estimate with (37), we see that (35) is satisfied for $k+1$. Finally, note that

$$d_k \geq d_0 - \sum_{k=1}^{\infty} \left(\left(1 + 3c_1 \log \frac{1}{\kappa} \right) L_0^\beta + 3 \right) \frac{1}{\nu^{(1-\beta)k}} =: c_8 > 0,$$

where the last inequality is satisfied whenever $\nu \geq \nu_3$ for some $\nu_3(d, \kappa)$ (also depending on the choice of L_0). \square

3.3. Final step in the proof of Theorem 2.1. The same argument as the one presented in the proof of Proposition 2.3 leading to (2.57) of [Sz02], shows that (33) of Lemma 3.2, implies that there is an open set $O \subset \mathbb{S}^{d-1}$ such that for all $\ell' \in O$ one has that

$$\lim_{L \rightarrow \infty} \frac{1}{L} \log P_0 \left[\tilde{T}_{-L}^{\ell'} < T_L^{\ell'} \right] < 0.$$

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