

## Quenched asymptotics for survival probabilities in the random saturation process

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**Abstract.** We report on the recent work [3]. There, the asymptotics of the survival probabilities of particles in a random environment of obstacles, are computed. The model is the following: particles are injected at a time dependent rate at the origin of the lattice  $\mathbb{Z}^d$ . Once born, they diffuse among sites which are free of traps. Each trap has a random depth, which decreases by one each time a particle is absorbed. The logarithmic asymptotic decay of the probability that a particle born at some fixed time survives at some later time  $t$  is computed, showing the presence of three injection regimes. Here we report on the quenched version of these results. A key tool for proving this result is the method of enlargement of obstacles developed by Sznitman [9]. © Académie des Sciences/Elsevier, Paris

### *Asymptotiques presque sûres des probabilités de survie dans le processus de saturation aléatoire*

**Résumé.** Nous présentons des résultats de [3], donnant le comportement asymptotique de la probabilité de survie d'une particule dans un milieu aléatoire. Le modèle étudié est le suivant : des particules sont injectées à l'origine du réseau  $\mathbb{Z}^d$  avec un taux qui dépend du temps. Une fois nées, ces particules diffusent parmi les sites qui sont libres de pièges. Chaque piège a une profondeur aléatoire, qui décroît de 1 chaque fois qu'une particule est absorbée. Le logarithme de la probabilité qu'une particule née à un instant donné survive jusqu'à un instant postérieur  $t$ , est calculé pour  $t \rightarrow \infty$ , montrant l'existence de trois régimes d'injection. Ici nous présentons la version presque sûre de ces résultats. Un outil-clé dans la preuve est la méthode d'agrandissement des obstacles développée par Sznitman [9]. © Académie des Sciences/Elsevier, Paris

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### Version française abrégée

Dans cette Note nous présentons un modèle de croissance, diffusion et de piégeage, qui a comme motivation initiale une version simplifiée d'un problème de gestion de déchets nucléaires. Le modèle est construit sur le réseau  $\mathbb{Z}^d$  et a trois propriétés principales. On définit d'une part un milieu aléatoire par un ensemble de variables aléatoires *i.i.d.*,  $\eta(x)$  indexées par le réseau  $\mathbb{Z}^d$ .  $\eta(x)$  représente la profondeur initiale d'un piège au site  $x$ , avec la convention qu'il n'y a pas de piège

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Note présentée par Paul MALLIAVIN.

au site  $x$  si  $\eta(x) = 0$ . D'autre part, des particules sont injectées à l'origine du réseau. On notera  $N(t)$  le nombre de particules nées avant le temps  $t$ . Enfin, le milieu aléatoire et les particules interagissent : quand une particule est née, elle se déplace comme une marche aléatoire simple en temps continu jusqu'à ce qu'elle tombe sur un piège non encore saturé. À cet instant, la particule s'arrête et reste dans le piège, et la profondeur du piège diminue d'une unité. Lorsque la profondeur atteint zéro ce piège est saturé et n'agit plus comme un piège. Notre but ici, est de présenter un résultat qui décrit le comportement asymptotique presque sûr du logarithme de la probabilité de survie d'une particule née à un instant donné. Selon le comportement de  $N(t)$  lorsque  $t \rightarrow \infty$ , nous montrons l'existence de trois régimes principaux. La preuve utilise une adaptation de la dernière version de la méthode d'agrandissement des obstacles développée par Sznitman [9], et une inégalité isopérimétrique pour le Laplacien discret sur  $\mathbb{Z}^d$ .

Dans des contextes similaires, on peut citer des travaux qui comprennent quelques éléments du modèle décrit ci-dessus. Par exemple, A.S. Sznitman [9] et antérieurement Donsker et Varadhan ([5], [6]) (dans le contexte de la saucisse de Wiener) ont étudié un modèle du mouvement brownien dans un milieu aléatoire de pièges poissonniens. Dans le cadre du modèle de cette Note, ces travaux correspondent à un cas où on négligerait le phénomène de saturation des pièges. D'autre part, pour le modèle d'« Internal Diffusion Limited Aggregation » (IDLA) introduit par Diaconis et Fulton [4] (en temps discret), Lawler, Bramson et Griffeath [8] ont démontré un théorème pour la forme de l'ensemble des pièges saturés.

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## 1. Introduction

In this Note we present a model of growth, diffusion and trapping in a random environment, having as initial motivation a simplified version of a problem in nuclear waste management of confinement of heavy nucleotides by high-performance clay barriers. The model is constructed beginning from the lattice  $\mathbb{Z}^d$  and has three main features: there is a random environment given by a collection of *i.i.d.* random variables  $\eta(x)$  at each site  $x$  of the lattice  $\mathbb{Z}^d$ .  $\eta(x)$  represents the initial depth of the trap at site  $x$ , with the convention that there is no trap if  $\eta(x) = 0$ ; at the origin of  $\mathbb{Z}^d$  particles are injected at a time dependent rate given by the number  $N(t)$  of born particles up to time  $t$ ; finally, there is an interaction between the medium ( $\eta$ ) and the particles: when born, particles perform continuous time simple random walks until they find a trap of depth greater than zero. At this point the particle stops and stays forever in the trap, and the depth of the trap is decreased by one. Our aim here, is to present a result describing the quenched logarithmic asymptotics of the survival probability (the probability of not getting trapped) of a particle born at a given time. Depending on the long time behaviour of  $N(t)$ , we show the existence of three main regimes.

There have been related works encompassing some aspects of the model described above. For instance, A.S. Sznitman [9] and previously Donsker and Varadhan ([5], [6]) (in the context of the Wiener sausage) have studied a model of Brownian motion on a random environment of Poissonian traps. In the context of the model of this Note, this corresponds to the absence of saturation of the traps. On the other hand, there is the Internal Diffusion Limited Aggregation (IDLA) model introduced by Diaconis and Fulton [4] in a discrete time setting and studied in a continuous time setting by Lawler, Bramson and Griffeath [8] which prove a shape theorem for the cluster of saturated traps. The continuous version corresponds to the model of this note where initially every site of the lattice is an obstacle of depth one. In another related work in the context of IDLA, Gravner and Quastel [7] prove among other things that, when  $d = 2$  and the injection rate is constant, under an hydrodynamic scaling limit, the profile of live particles converges weakly to the solution of the one phase Stefan problem with a source at the origin.

## 2. Notation and results

Firstly, let  $m$  be some natural number and define  $\mathcal{I} := \{n \in \mathbb{N} : 0 \leq n \leq m\}$ . The state space representing the obstacle configuration endowed with the natural topology will be denoted by  $\Gamma := \mathcal{I}^{\mathbb{Z}^d}$ . Let  $\mathcal{B}$  be the corresponding Borel  $\sigma$ -algebra. Define for each finite set  $F \in \mathbb{Z}$  the continuous projections  $\pi_F$  from  $\Gamma$  to  $\mathcal{I}^F$ . Given  $\eta \in \Gamma$  we define its  $x$ -th coordinate by  $\eta(x) := \pi_{\{x\}}\eta$ . A site  $x$  such that  $\eta(x) \geq 1$  represents a site with an obstacle present, while  $\eta(x) = 0$  means that there is no obstacle. Let  $\{T_n \in [0, \infty) : n \in \mathbb{N}\}$  be a sequence of strictly increasing times which will represent the times at which random walks are introduced at the origin.

Let  $N(t) := \sum_{n=1}^{\infty} \mathbf{1}_{[0,t]}(T_n)$ , where  $\mathbf{1}_B$  is the indicator function of  $B \subset \mathbb{R}$ . This represents the total number of random walks that have been born at time  $t$ .

We can now describe the dynamics of the random saturation process. At time  $T_n$ , a random walk  $Z_n$  is introduced at the origin  $0$  of  $\mathbb{Z}^d$ . Then,  $Z_n$  moves as a simple random walk of total jump rate equal to 1 until the first time it jumps to some site  $x$  which has received less than  $\eta(x)$  visits. After this moment it remains at site  $x$  forever. A rigorous definition of the dynamics of the empirical density of particles  $\sum_{i=1}^{\infty} \mathbf{1}_{Z_n(t)}(x)$  of the random saturation process can be given as

a Markov process on the space  $D([0, \infty), \mathbb{N}^{\mathbb{Z}^d})$ , endowed with its Borel- $\sigma$  field. However, a richer construction is possible on the space  $\Omega := D([0, \infty), \mathbb{Z}^d)^{\mathbb{N}}$ , endowed with its Borel- $\sigma$  field  $\mathcal{D}$ , i.e. in the space describing the dynamics of each random walk  $Z_n$ . In [3] we present such a construction. As it turns out, it defines a probability measure  $Q_{N,\eta}$  on the space  $(\Omega, \mathcal{D})$  such that the canonical coordinate process  $Z := \{Z_n : n \in \mathbb{N}\}$  has the dynamics described above. Also, the construction explicitly shows a coupling between each  $Z_n$  and a free random walk  $Y_n$  such that with probability one under  $Q_{N,\eta}$  one has  $Z_n(t) = Y_n(t)$  for  $t \leq \tau_n$ , where  $\tau_n$  is the first hitting time of  $Z_n$  to a trap, i.e. to a site  $x$  which has received less than  $\eta(x)$ . In the sequel we will say that  $Z$  under  $Q_{N,\eta}$  is a random saturation process on an obstacle configuration  $\eta$  and driven by an injection  $N$ .

Now we endow the obstacle state space  $(\Gamma, \mathcal{B})$  with a product probability measure  $\mu$  given by  $\mu(\eta(x) = \alpha) = p_\alpha$ , where  $\sum_{\alpha \in \mathcal{I}} p_\alpha = 1$ . Let  $S_t := \{x \in \mathbb{Z}^d : \zeta(x, t) \geq \eta(x) > 0\}$ , where for  $A \subset \mathbb{Z}^d$ ,

we define  $\mathbf{1}_A : \mathbb{Z}^d \rightarrow \{0, 1\}$  as the indicator function of the set  $A$ . This set corresponds to the sites  $x$  of the cubic lattice  $\mathbb{Z}^d$  which have an obstacle, and which been visited at least  $\eta(x)$  times. We will call it the set of saturated obstacles at time  $t$ . Let  $k(t) : [0, \infty) \rightarrow \mathbb{N}$  be an increasing function of time and let  $g(t) := T_{k(t)}$  be the birth time of the random walk  $Z_{k(t)}$ . We will be interested in understanding the asymptotic behaviour of the survival probability of the random walk  $Z_{k(t)}$  with law given by  $Q_{N,\eta}$ , both when  $k(t)$  is fixed as time goes to infinity, and when  $k(t)$  goes to infinity together with time. Let  $\lambda_d$  be the principal Dirichlet eigenvalue of the Laplacian operator divided  $2d$  on the ball of unit radius of  $\mathbb{R}^d$  and  $w_d$  its volume. Define  $p := \mu(\eta(x) > 0)$ ,  $a := \mu(\eta(x))$ ,  $\bar{a} := \max\{n \in \mathcal{A}\}$  and denote by  $p_c(d)$  the critical probability of site percolation on  $\mathbb{Z}^d$ . In the sequel we assume that  $p > 0$ .

**THEOREM 1.** – *Consider a random saturation process on an obstacle configuration  $\eta$  and driven by an injection  $N$ . Assume that  $0 < N(t) \ll t^{d/2-\varepsilon}$  for some  $\varepsilon \in (0, 1)$ , that  $\limsup_{t \rightarrow \infty} k(t) > \bar{a}$  and*

*that  $t - g(t) \gg 1$ . Then:*

(i) *assume that  $1 \ll N(t) \ll (t - g(t))^{d/2}$ . If  $\ln(t - g(t)) \ll N(t)$  or  $p > p_c(d)$  then,*

$$\lim_{t \rightarrow \infty} \frac{1}{h_M(k, t)} \ln Q_{N,\eta}(\tau_{k(t)} > t) = -1 \text{ } \mu\text{-a.s.},$$

where  $h_M(k, t) := \lambda_d (aw_d)^{2/d} \int_{g(t)}^t \frac{ds}{N(s)^{2/d}}$  ;

(ii) if  $N(t) \ll \ln(t - g(t))$  and  $p < p_c(d)$  then,

$$\lim_{t \rightarrow \infty} \frac{1}{h_L(k, t)} \ln Q_{N, \eta}(\tau_{k(t)} > t) = -1 \text{ } \mu\text{-a.s.},$$

$$\text{where } h_L(k, t) = \lambda_d(w_d |\ln(1 - p)|)^{2/d} \frac{t - g(t)}{(\ln(t - g(t)))^{2/d}}.$$

Let us briefly discuss the meaning of the above result. For the sake of clarity, let us consider the case in which  $k(t)$  is some constant greater than  $\bar{a}$ . When  $p < p_c$ , we know that  $\mu$ -a.s. there exists a unique obstacle free cluster on the lattice  $\mathbb{Z}^d$ . The above theorem shows that when  $N(t) \ll t^{-\varepsilon+d/2}$ , for some  $\varepsilon > 0$ , there appear to be two different injection regimes when  $p < p_c$ . There is a regime which we will denote by *quenched low regime*, when  $N(t) \ll \ln t$ , given by part (ii). The subscript L in  $h_L$  stands for *low*. The survival strategy for random walks in this regime consists essentially in travelling fast to a distance of order  $t/(\ln t)^{2/d}$  to some region of the lattice free of obstacles and of radius of the order of  $(\ln t)^{1/d}$ . This is exactly the survival strategy of a Brownian motion on  $\mathbb{R}^d$  with Poissonian obstacles (see [9]) or of a simple random walk on the lattice with obstacles on sites or bonds distributed according to some product measure (see [2]). There is a second injection regime for  $\ln t \ll N(t) \ll t^{-\varepsilon+d/2}$ , which we call *quenched medium regime*, given by part (i) of Theorem 1. The subscript M in  $h_M$  stands for *medium*. Here random walks are provided with a better survival strategy than going far to find natural clearings, as in the low regime. In fact, it is possible to prove that  $Q_{N, \eta}$ -a.s. eventually for  $t$  large enough, the set of saturated obstacles produces a central clearing (without obstacles) which at time  $t$  is a ball of radius  $\left(\frac{1}{aw_d} N(t)\right)^{1/d}$ . Thus, the typical survival strategy of a particle is to stay in this central region. When  $p > p_c$ , so that  $\mu$ -a.s. there is no infinite trap free cluster, Theorem 1 states that for injection rates satisfying  $N(t) \ll t^{-\varepsilon+d/2}$ , the decay of the survival probability is as in the medium regime.

Finally, let us remark that an annealed version of Theorem 1 has been proved in [3]. As above, one can distinguish different injection regimes, but the transition between the low and medium regime occurs at a higher injection rate, given by the radius of an “annealed” natural central clearing produced as described in Donsker and Varadhan [5].

### 3. Main elements in the proof of Theorem 1

The following fact is central in the appearance of the two different injection regimes (low and medium) in Theorem 1:  $\mu$ -a.s., with  $Q_{\eta, N}$  probability one, eventually in  $t$ , the set of saturated obstacles  $S_t$  at time  $t$  is a ball of radius  $\left(\frac{1}{aw_d} N(t)\right)^{1/d}$  intersected with the original set of traps. We will call the obstacle free region produced in this way, the central clearing. The proof of this shape theorem [3], requires a small modification of the methods used in [8] to prove the corresponding theorem for IDLA.

There are two main survival strategies for a random walk that determine the behaviour for long times, of the probability to survive up to time  $t$ . The first strategy is based on the presence of a central clearing as described above. Then, to survive up to time  $t$  a particle tries to spend all the time in a ball of time dependent radius corresponding to such a clearing. The second survival strategy is to go very fast to a distance of order  $t/(\ln t)^{2/d}$ , to find a natural clearing of the obstacles of size of order  $(\ln t)^{1/d}$ , and to spend the rest of the time up to time  $t$  in this clearing. We will call such clearings, the natural clearings. This is precisely the survival strategy of a single random walk on the lattice with random obstacles on the sites having a Bernoulli product distribution [2].

There is a competition between these two strategies. Depending on the injection rate and on the value of the percolation parameter  $p$ , one dominates the other, fact which is reflected on the

different regimes appearing in the statement of Theorem 1: part (i) corresponds to a random walk following the survival strategy of staying in the central clearing created by the saturated obstacles, while part (ii) corresponds to a random walk which survives by travelling fast to a distance of order  $t/(\ln t)^{2/d}$  to some natural clearing of size of order  $(\ln t)^{1/d}$ , staying there up to time  $t$ .

To illustrate these two situations, in what follows we take a look at the case  $k(t) = 1$ , corresponding to the behaviour of the first particle born. For high injections,  $N(t) \gg \ln t$ , the central clearing produced by saturation dominates in size the natural clearings that can be found within a box of side  $t$ . Similarly, if the percolation parameter is higher than  $p_c$ , there is no infinite cluster of sites free of obstacles, and the possibility of travelling far away to find natural clearings is denied. In both cases, the central clearing determines the behaviour of the decay of the survival probability up to time  $t$  and the hypothesis of part (i) of Theorem 1 are satisfied. More precisely, the logarithm of such probability diverges like  $-\int_0^t \lambda(s) ds$ , where  $\lambda(s)$  is the principal Dirichlet eigenvalue of the discrete Laplacian operator divided by  $2d$  on the set of sites free of obstacles at time  $s$ . For long times, such an eigenvalue decays as the principal Dirichlet eigenvalue of the continuous Laplacian on a ball of radius  $\left(\frac{1}{aw_d} N(s)\right)^{2/d}$ . Such an argument via an application of the shape theorem provides the lower bound on the asymptotics of part (i) of Theorem 1. On the other hand at low injection rates  $N(t) \ll \ln t$  and  $p < p_c$ , it is easy to see that the probability of survival of the the first born particle is bounded below by the survival probability of a single simple random walk on a random environment of absorbing obstacles distributed according to  $\mu$ . Now, by the logarithmic asymptotics proved by Antal [1], [2], this provides the lower bound of part (ii) of Theorem 1.

The proof of the upper bounds of Theorem 1 turn out to be more difficult. For part (i) (high injection or percolating obstacles case), the main problem is that there is no good control on the probability that the central clearing is a ball. Thus, the central clearing shape theorem is useless. We therefore have to consider all possible shapes for the saturated set of obstacles at a given time and not only “balls”. It is the case that the smallest possible value that one can obtain for the principal Dirichlet eigenvalue of the discrete Laplacian on the obstacle free sites in a box after erasing a predetermined large enough amount of obstacles, corresponds to erasing a sphere. Part (i) of Theorem 2 is a statement of this fact. To state it, we need to introduce some notation. For a given configuration of obstacle depth  $\eta \in \mathcal{I}^{\mathbb{Z}^d}$ , we will denote by  $\mathcal{N}_n(\eta)$  the set of configurations obtained from  $\eta$  after deleting  $n$  obstacles. Thus, for every  $\varsigma \in \mathcal{N}_n(\eta)$  we have  $\sum_{x \in \mathbb{Z}^d} (\eta(x) - \varsigma(x)) = n$ . Now consider the space  $\Upsilon := \{0, 1\}^{\mathbb{Z}^d}$ . This represents a space of site configurations on the lattice: sites in state 1 have an obstacle and are absorbing, and those in state 0 are empty and non-absorbing. Next, given  $\xi \in \Upsilon$ , call the subset of  $\mathbb{Z}^d$  without obstacles  $\mathcal{E}(\xi) := \{x \in \mathbb{Z}^d : \xi(x) = 0\}$ . Now, given an open subset  $U$  of  $\mathbb{R}^d$ , denote by  $\lambda_\xi(U)$  the principal Dirichlet eigenvalue of the discrete Laplacian on  $U \cap \mathbb{E}(\xi)$ . We also define a mapping  $\sigma : \mathcal{I}^{\mathbb{Z}^d} \rightarrow \Upsilon$  by  $\sigma(\eta)(x) = 1$  if  $\eta(x) \geq 1$  and  $\sigma(\eta)(x) = 0$  if  $\eta(x) = 0$ . Finally, for given  $\eta \in \mathcal{I}^{\mathbb{Z}^d}$ , and an open subset  $U \subset \mathbb{R}^d$  we adopt the convention  $\lambda_\eta(U) := \lambda_{\sigma(\eta)}(U)$ .

**THEOREM 2.** – *On  $\mathcal{I}^{\mathbb{Z}^d}$  consider a product measure  $\mu$  such that  $\mu(\eta(x) \geq 1) = p$ , where  $\eta \in \mathcal{I}^{\mathbb{Z}^d}$  and  $0 < p < 1$ . Let  $f(t) : [0, \infty) \rightarrow [0, \infty)$  be an increasing function such that  $f(t) \ll t$ ,  $w_d$  the volume of a ball on  $\mathbb{R}^d$  of unit radius and  $\lambda_d$  the principal Dirichlet eigenvalue of the Laplacian operator on this ball times  $\frac{1}{2d}$ . Then if  $a := \mu(\eta)$ , the following statements are true:*

(i) *suppose that  $f(t) \gg (\ln t)^{1/d}$ . Then,*

$$\lim_{t \rightarrow \infty} f(t)^2 \inf_{\varsigma \in \mathcal{N}_{aw_d f^d(t)}(\eta)} \lambda_\varsigma((-t, t)^d) = \lambda_d \quad \mu\text{-a.s.} ; \tag{1}$$

(ii) suppose that  $f(t) \ll (\ln t)^{1/d}$ . Then,

$$\lim_{t \rightarrow \infty} (\ln t)^{2/d} \inf_{\varsigma \in \mathcal{N}_{aw_d f^d(t)}(\eta)} \lambda_\varsigma((-t, t)^d) = \lambda_d (w_d |\ln(1-p)|)^{2/d} \quad \mu\text{-a.s.},$$

Let us first explain how does Theorem 2 complete the proof of parts (i) and (ii) of Theorem 1. If  $P$  is the probability that a simple random walk survives up to time  $t$  in a subset  $G \subset \mathbb{Z}^d$ , then  $P \leq c((\lambda t)^{d/2} + 1) e^{-\lambda t}$ , where  $\lambda$  is the principal Dirichlet eigenvalue of the discrete Laplacian on the set  $G$  and  $c$  is a constant. A combination of this fact with part (i) of the Theorem 2 enables us to prove the upper bound of the logarithmic asymptotics of part (i) of Theorem 1. For part (ii) (low injection regime without percolation of obstacles) the difficulty in proving the upper bound lies in showing that the principal Dirichlet eigenvalue of the discrete Laplacian on the obstacle free sites of a box does not change if we erase a low enough amount of obstacles. This is the content of part (ii) of Theorem 2, which is enough to prove the upper bound of part (ii) of Theorem 1.

We finally comment briefly on the proof of the eigenvalue estimate of part (i) of Theorem 2. We first show that  $\lim_{t \rightarrow \infty} f(t)^2 \inf_{\varsigma \in \mathcal{N}_{aw_d f^d(t)}(\eta)} \lambda_\varsigma((-t, t)^d) \leq \lambda_d \mu\text{-a.s.}$  This is a consequence of the fact

that the left hand side in display (1) can essentially be bounded by  $\lambda_{\varsigma_1}((-t, t)^d)$ , where  $\varsigma_1$  is such that  $\varsigma_1(x) = 0$  if  $x$  is a sphere centred at the origin of volume  $w_d$ , and  $\varsigma_1(x) = 1$  otherwise. The final step is to show that  $\lim_{t \rightarrow \infty} f(t)^2 \inf_{\varsigma \in \mathcal{N}_{aw_d f^d(t)}(\eta)} \lambda_\varsigma((-t, t)^d) \geq \lambda_d \mu\text{-a.s.}$  The proof of this part is

more involved and requires an adaptation to the lattice of the second version of the enlargement of obstacle method of Sznitman [9]. We first suppose that the above lower bound is not satisfied. Then we subdivide the box  $(-t, t)^d$  in small boxes of side  $o(t)$ . The enlargement of obstacle method enables us to estimate the number of such boxes which have a small number of traps. An important ingredient here is the following isoperimetric inequality:

LEMMA 1. – Let  $\varepsilon > 0$ ; for each  $K \subset \varepsilon\mathbb{Z}^d$  define  $\lambda^\varepsilon(K)$  as the principal Dirichlet eigenvalue of the discrete Laplacian divided by  $2d\varepsilon^2$  on  $K$  with Dirichlet boundary conditions. Define  $\overline{K} := \{x \in \mathbb{Z}^d : |x - y| = \varepsilon \text{ for some } y \in K\}$ . Then,

$$\lambda^\varepsilon(K) \geq \lambda_d \left( \frac{w_d}{\varepsilon^d |\overline{K}|_\varepsilon} \right)^{2/d} \frac{1}{1 + C_d \varepsilon^2 \lambda^\varepsilon(K)}, \quad (2)$$

where  $C_d := 3d^2 2^{d-1}$

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## References

- [1] Antal P., Enlargement of obstacles for the simple random walk, Ann. Probab. 23 (1995) 1061–1101.
- [2] Antal P., Trapping problems for the simple random walk, Thesis, ETH, Zürich, 1994.
- [3] Ben Arous G., Ramírez A.F., Asymptotic survival probabilities in the random saturation process, Preprint, 1999.
- [4] Diaconis P., Fulton W., A growth model, a game, an algebra, Lagrange inversion, and characteristic classes, Rend. Sem. Mat. Univ. Pol. Torino 49 (1991) 95–119.
- [5] Donsker M., Varadhan S.R.S., Asymptotics for the Wiener sausage, Commun. Pure Appl. Math. 28 (1975) 525–565.
- [6] Donsker M., Varadhan S.R.S., On the number of distinct sites visited by a random walk, Commun. Pure Appl. Math. 32 (1979) 721–747.
- [7] Gravner J., Quastel J., Internal DLA and the Stefan problem, Preprint, 1999.
- [8] Lawler G., Bramson M., Griffeath D., Internal diffusion limited aggregation, Ann. Probab. 20 (1992) 2117–2140.
- [9] Sznitman A.S., Brownian motion obstacles and random media, Springer-Verlag, Berlin, 1998.