Asymptotic Shape and Propagation of Fronts for Growth Models in Dynamic Random Environment

Harry Kesten, Alejandro F. Ramírez and Vladas Sidoravicius

Abstract We survey recent rigorous results and open problems related to models of Interacting Particle Systems which describe the autocatalytic type reaction $A+B \to 2B$, with diffusion constants of particles being respectively $D_A \geq 0$ and $D_B \geq 0$. Depending on the choice of the values of D_A and D_B, we cover three distinct cases: so called "rumor or infection spread" model ($D_A > 0$, $D_B > 0$); the Stochastic Combustion process ($D_A = 0$ and $D_B > 0$); and finally the "modified" Diffusion Limited Aggregation, which corresponds to the case $D_A > 0$, $D_B = 0$ with modified transition rule: $A+B \to 2B$ occurs when A- and B-particles become nearest neighbors and A-particle attempts to jump on a vertex where B-particle is located. Then such jump is suppressed, and A-particle becomes B-particle.

1 Introduction

As the basic model we consider the following interacting particle system: There is a "gas" of particles, each of which performs a continuous time simple random walk on $\mathbb{Z}^d$, with jump rate $D_A \geq 0$. These particles are called A-particles and move independently of each other. In the case $D_A \geq 0$ they are regarded as individuals who are ignorant of a rumor or are healthy. We assume that we start the system with

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$N_A(x,0-)$ A-particles at x, and that the $N_A(x,0-)$, $x \in \mathbb{Z}^d$, are i.i.d., mean μ_A Poisson random variables. In addition, there are B-particles which perform continuous time simple random walks with jump rate $D_B \geq 0$. We start with a finite number of B-particles in the system at time 0. In the case $D_B > 0$, B-particles are interpreted as individuals who have heard a certain rumor or who are infected. The B-particles move independently of each other. In the case when $D_B > 0$ the only interaction is that when a B-particle and an A-particle coincide, the latter instantaneously turns into a B-particle. It models the irreversible autocatalytic reaction $A + B \to 2B$, for which corresponding diffusion constants D_A and D_B may differ. In addition to the possible interpretations of such systems mentioned above, in the case of $D_A = 0$ the system can be interpreted as a model for the burning of propellant material, where the B-particles have been interpreted as "packets of energy" which together with A-particles produce more energy, according to the reaction $A + B \to 2B$ (see [24]), and the process is called in Physics literature the Stochastic Combustion process. It is also known as the "frog" model (see [1] and [24]). Finally, in the case $D_A > 0$ but $D_B = 0$, in order to create spatial growth we modify the transition rule: the transition from A- to B-state happens when an A-particle is at a neighboring vertex to B-particle and performs an attempt to jump to the vertex where B-particle is located. Then such jump is suppressed, and an A-particle becomes B-particle and stops moving. In this case we assume that all A-particles which are at the same neighboring site also become B-particles.

The first basic and natural question to ask is how fast B-particles spread. Specifically, if

$$\widetilde{B}(t) = \{x \in \mathbb{Z}^d : \text{ a } B\text{-particle visits } x \text{ during } [0,t]\},$$

$$B(t) = \widetilde{B}(t) + \left[-\frac{1}{2}, \frac{1}{2}\right]^d,$$

then we are interested in the asymptotic behavior of $B(t)$. The first major goal is to prove a full "shape theorem", saying that $t^{-1}B(t)$ converges almost surely to a non-random set B_0, with the origin as an interior point, so that the true growth rate for $B(t)$ is linear in t.

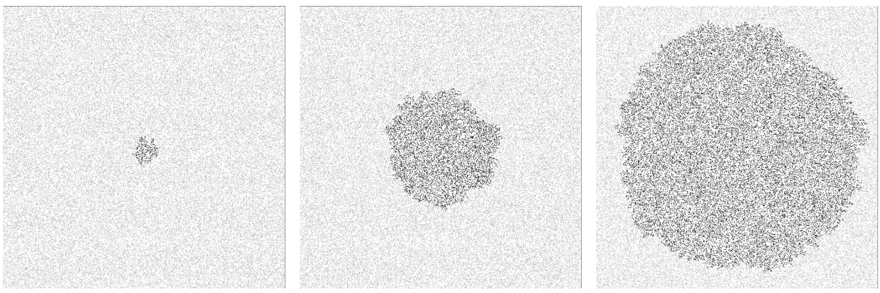

Fig. 1 Three instants of the growth of the cluster of B-particles (dark collor) in the case $D_A = D_B > 0$.

The study of these systems was suggested by Frank Spitzer to the first author around 1980. At that time only the case when the A and B-particles perform the same random walks (i.e., $D_A = D_B > 0$) seems to have been considered.

Shape theorems have a fairly long history and have become the first goal of many investigations of stochastic growth models. To the best of our knowledge Eden (see [7]) was the first one to ask for a shape theorem for his celebrated Eden model. The problem turned out to be a stubborn one. The first real progress was due to Richardson, who proved in [25] a shape theorem not only for the Eden model, but also for a more general class of models, now called Richardson models. In these models one typically thinks of the sites of $\mathbb{Z}^d$ as cells which can be of two types (for instance B and A or infected and susceptible). Cells can change their type to the type of one of their neighbors according to appropriate rules. One starts with all cells off the origin type A and cell of type B at the origin and tries to prove a shape theorem for the set of cells of type B at a large time. An important example of such a model is first-passage percolation, which was introduced in [13] (this includes the Eden model, up to a time change). A quite good shape theorem for first-passage percolation is known (see [22], [8], [16]). In more recent first-passage percolation papers even sharper information has been obtained which gives estimates on the rate at which $(1/t)B(t)$ converges to its limit B_0 (see [14] for a survey of such results).

Shape theorems for quite a few variations of Richardsons model and first-passage percolation have been proven (see for instance [4] and [10]), but as far as we know these are all for models in which the cells do not move over time, with one exception. This exception is the Stochastic Combustion Process (or ”frog model”) which follows the rules given above, but which has $D_A = 0$, i.e., the susceptible or type A cells stand still (see [1] and [24] for this model). The papers [17], [18] may be the first ones which allow both types of particles to move.

In nearly all cases shape theorems are proven by means of Kingman’s subadditive ergodic theorem (see [22]). This is also what is used for the Stochastic Combustion process. For this model one can show that the family of random variables $\{T_{x,y}\}$ is subadditive, were $T_{x,y}$ is a version of the first time a particle at y is infected, if one starts with one infected particle at x and one susceptible at each other site. More precisely, the $T_{x,y}$ can all be defined on one probability space such that

$$T_{x,z} \leq T_{x,y} + T_{y,z}, \quad \text{a.s.} \tag{1}$$

for all $x, y, z \in Z^d$ and such that their joint distribution is invariant under translations. Unfortunately this subadditivity property is no longer valid if one allows both types of particles to move. There is no obvious family of random variables with properties like those of the $T_{x,y}$. Nevertheless, subadditivity methods are still heavily used in the proof of the full shape Theorem in the case $D_A > 0$, $D_B > 0$. However, now subadditivity is used only for certain ”half-space” processes which approximate the true process (see for more details Sect. 2). Moreover, these half-space processes have only approximate superconvolutive properties (in the terminology of [12]), which replaces ”almost sure” subadditivity relation in (1) by relation between distribution functions. The tool of superconvolutivity in other models with no obvious subadditivity in the strict sense goes back to [25], and was used in [4] and [29], and could be stated in the following the way [12]:

Lemma 1. *(H. Kesten, 1974, see [12]) Let $X_s, s \geq 1$, and $Y_{s,t}$, $1 \leq s < t$, be random variables (s,t integer) satisfying the following conditions:*

$$P\{X_{s+t} \leq x\} \geq P\{X_s + X_t' + Y_{s,t} \leq x\} \tag{2}$$

for all real x and $s,t \geq 0$, where X_t' has the same distribution as X_t but is independent of all X_s,

$$EY_{s,t}^2 \leq C, \quad E|X_1|^2 < \infty, \quad E(X_s^-)^2 < C$$

for all $s,t \geq 1$ and some C (independent of s and t). Then, there exists a $0 \leq \gamma < \infty$ such that

$$\sum_{k=0}^{\infty} \{|\frac{X_{m2^k}}{m2^k} - \gamma| > \varepsilon\} < \infty$$

for all $m \geq 1$, $\varepsilon > 0$.

However it is not so easy to use superconvlutive property and find right quantities which satisfy (2), in order to prove shape theorems.

In Section 2 we focus on the the full shape theorem for the case $D_A = D_B > 0$. Essentially we were not able to go beyond the law of large numbers for this model and show fluctuations and Large Deviation Principle for the front. One of the aspects which makes these type of problems difficult is that the process as seen from the front does not converge exponentially fast to its equilibrium. Within a certain class of one-dimensional nonlinear diffusion equations having uniformly traveling wave solutions describing the passage from an unstable to a stable state, it has been observed that for certain initial conditions the velocity of the front at a given time has a rate of relaxation towards its asymptotic value which is algebraic. These are the so called pulled fronts, whose speed is determined by a region of the profile linearized about the unstable solution. For the F-KPP equation, Bramson [3] proved that the speed of the front at a given time is below its asymptotic value and that the convergence is algebraic. In general, the slow relaxation is due to a gapless property of a linear operator governing the convergence of the centered front profile towards the stationary state. However for the Stochastic Combustion model in dimension 1 progress was made in [5] and [2], and it is the content of the Sect. 3. Finally in Sect. 4 we focuss on the modified DLA model.

2 Spread of an infection in a moving population ($D_A > 0$, $D_B > 0$)

In this Section we mainly deal with the case $D_A = D_B > 0$. The upper bound for $B(t)$ (see Theorem 1 below) is relatively easy and is proven even for $D_A \neq D_B$. On the other hand the lower bound is much harder. However it turns out that the lower bound for $B(t)$ in Theorem 2, in the case $D_A = D_B$, can be obtained by the methods of [15], which we explain below. It is still an open problem whether $B(t)$ grows linearly with t when $D_A > 0$, but $D_A \neq D_B$. In this case we can only prove that $B(t) \supset \mathscr{C}\big(K_1 t/(\log t)^p\big)$ eventually, for some constants $K_1, p > 0$.

2.1 Shape Theorem

Throughout we shall use $N_A(x,t)$ ($N_B(x,t)$) to denote the number of A-particles (respectively, B-particles) at position x at time t. N_B denotes the total number of B-particles at time 0. We always take $0 < N_B < \infty$ and consider N_B, as well as the positions of the initial B-particles, as non-random. At a site x with a B-particle at time 0 all particles immediately turn to B-particles. We write $N_A(x, 0-)$ for the number of A-particles "just before" the B-particles are added to the system. In accordance with these rules we take $N_A(x,0) = 0$, $N_B(x,0) = N_A(x,0-) + N_B(x,0)$ at a site x to which a B-particle is added at time 0. If x does not have a B-particle at time 0, then $N_A(x,0) = N_A(x,0-)$ and $N_B(x,0) = 0$.

Our first theorem states that the rumor/infection cannot spread from the origin faster than linearly in time.

Theorem 1. *For some constant $C_1 < \infty$, and all sufficiently large t,*

$$E\{\text{number of } B\text{-particles with a position outside } \mathscr{C}(C_1 t) \text{ at time } t\} \leq 2N_B e^{-t}. \quad (3)$$

Consequently it is a.s. the case that

$$B(t) \subset \mathscr{C}(2C_1 t) \text{ eventually.} \quad (4)$$

This result holds for any $D_A, D_B \geq 0$ and probably is even valid if one allows the A and B-particles to perform any random walk with bounded jumps of mean zero.

The proof of Theorem 1 is basically a Peierls argument. It associates to each B-particle, ρ say, present at time t, a so called genealogical path which describes the sequence of B-particles which transmitted the rumor/infection from the initial B-particles to ρ at time t, and also describes the relevant pieces of the paths of these intermediate particles. One proves (3) by taking the expectation of the number of genealogical paths which lead to a B-particle outside $\mathscr{C}(C_1 t)$ at time t.

The next theorem, which is the first important ingredient in proving shape theorem, shows that the rumor/infection spreads at least linearly in time, but only if both the A and B-particles perform simple random walks with the same jumprate.

Theorem 2. *If $D_A = D_B > 0$, then there exists a constant $C_2 > 0$ such that for each constant $K > 0$*

$$P\{\mathscr{C}(C_2 t) \not\subset B(t)\} \leq \frac{1}{t^K} \text{ for all large } t. \quad (5)$$

Consequently, a.s.

$$\mathscr{C}(C_2 t) \subset B(t) \text{ eventually.} \quad (6)$$

For proving a shape theorem we will need a form of Theorem 2 which also gives some information about the possible occurrence of A-particles amid the spreading B-particles. More specifically, the same proof as for Theorem 2 can be used to prove the next proposition.

Proposition 1. *If $D_A = D_B$, then for all K there exists a constant $C_3 = C_3(K)$ such that*

$$P\{\text{there is a vertex } z \text{ and an } A\text{-particle at the space-time point } (z,t) \text{ while} \tag{7}$$

$$\text{there also was a } B\text{-particle at } z \text{ at some time } \leq t - C_3[t \log t]^{1/2}\} \tag{8}$$

$$\leq \frac{1}{t^K} \text{ for all sufficiently large } t. \tag{9}$$

Consequently, for large t,

$$P\{\text{at time } t \text{ there is a site in } \mathscr{C}(C_2 t/2) \text{ which} \tag{10}$$

$$\text{is occupied by an } A\text{-particle}\} \leq \frac{2}{t^K}. \tag{11}$$

Remark. It can be checked that the constants C_1, C_2 do not depend on the number or positions of the initial B-particles. However, the lower bounds for the times for which (3)-(6) are valid do depend on these initial data.

The proof of the Theorem 2 is rather involved. To help the intuition, it is best to think of the one-dimensional case, started with one B-particle at the origin and no other B-particle. All the major difficulties appear already in this special case. Until the last paragraph of these heuristic remarks we therefore take $d = 1$.

In this one-dimensional case, there is for each t a rightmost B-particle, at position $\mathscr{R}(t)$ say, and a leftmost B-particle at position $-\mathscr{L}(t)$. At time t all particles in $[-\mathscr{L}(t), \mathscr{R}(t)]$ are B-particles and all particles outside $[-\mathscr{L}(t), \mathscr{R}(t)]$ are A-particles. Basically we want to show that $\liminf_{t\to\infty} \mathscr{R}(t)/t > 0$ and similarly for $\mathscr{L}(t)$. If there is exactly one particle at $\mathscr{R}(t)$ at time t, then $\mathscr{R}(.)$ behaves like a simple random walk, that is, $P\{\mathscr{R}(t+dt) = \mathscr{R}(t) \pm 1\} = Dt/2 + O(dt2)$, with D standing for the common value of D_A and D_B. However, if there is more than one particle at $\mathscr{R}(t)$ at time t, then the rightmost particle moves one step to the right as soon as one of the particles at $\mathscr{R}(t)$ makes a jump to the right, whereas the rightmost position moves a step to the left only when all particles at $\mathscr{R}(t)$ move to the left. Thus, the rightmost B-particle has a drift to the right at all times when there is more than one particle at $\mathscr{R}(t)$. When there is at least one other particle (of either type) "close to" the rightmost B-particle, then there is a positive probability that in the next time unit another particle will coincide with the rightmost B-particle. This will still provide $\mathscr{R}(.)$ with an upwards drift. By using large deviation estimates for martingales one can see that the only way for $\mathscr{R}(t)/t$ to become small (with a non-negligible probability) is if the particle at $\mathscr{R}(s)$ has for most $s \in [0,t]$ no particle (of any type) nearby. We therefore want to show that the probability of this event goes to 0. One is tempted to try and prove this by studying the environment as seen from the position $\mathscr{R}(t)$. However, this approach seems difficult because the dependence between $\mathscr{R}(t)$ and the particles near $\mathscr{R}(t)$ is very complicated. We have been unable to use this approach. Instead, it turns out to be easier to prove a much stronger property, which uses almost no property of the path $s \mapsto \mathscr{R}(s)$. Roughly speaking we prove that every space-time path $s \mapsto \widehat{\pi}(s)$ with not too many jumps during $[0,t]$ has some particle "near $\widehat{\pi}(s)$ most of the time".

To make this more specific, we introduce some notation. A path $\pi = (x_0,...,x_m)$ is a sequence of integers with $x_{j+1} - x_j = \pm 1$, $1 \le i \le m$. We regard the x_j as the successive positions of a space-time path $\widehat{\pi}$. There are many space-time paths which traverse the same positions in the same order. A space-time path $\widehat{\pi}$ is specified by giving its successive positions x_i and jump times s_i. For $s_1 < s_2 < \dots$ we shall sometimes denote the path which jumps to x_i at time s_i by $\widehat{\pi}(\{s_i,x_i\})$. We make the convention that $s_0 = 0$, and unless stated otherwise, $x_0 = 0$. In addition we are here only discussing space-time paths over the time interval $[0,t]$, so we tacitly take $s_m \le t$. $\widehat{\pi}(\{s_i,x_i\})$ is then the path which is at position x_i during $[s_i,s_{i+1})$ for $0 \le i < m$, and at position x_m during $[s_m,t]$. If it is important that the path has exactly m jump times, then we shall write $\widehat{\pi}(\{s_i,x_i\}_{i \le m})$. Throughout this proof we shall only consider paths which are contained in $\mathscr{C}(t\log t) = [-t\log t, t\log t]$. Of particular interest for us is the following class of paths with exactly ℓ jumps:

$$\Xi(\ell,t) = \{\widehat{\pi}(\{s_i,x_i\}_{0 \le i \le \ell}) \text{ with } 0 = s_0 < s_1 < \cdots < s_\ell < t \text{ and } x_i \in \mathscr{C}(t\log t)\}.$$

Instead of using the path followed by $\mathscr{R}(.)$, we shall construct special paths $\widehat{\pi}$ with the property that there is a B-particle at $(\widehat{\pi}(s),s)$ for all $s \le t$ [so that automatically $R(t) \le \widehat{\pi}(t)$], and such that these paths are with high probability in $\Xi(\ell,t)$ for some $\ell \le 2Dt$, and also have a drift to the right at any time s when there are at least two particles at $\widehat{\pi}(s)$. Thus, it will be sufficient to show that every space-time path $\widehat{\pi} \in \cup_{\ell \le 2Dt} \Xi(\ell,t)$ has some particle "near $\widehat{\pi}(s)$ most of the time".

To this end we choose a large integer C_0 and partition space-time $\mathbb{Z} \times [0,\infty)$ into the following blocks of size $\Delta_r := C_0^{6r}$:

$$\mathscr{B}_r(i,k) = [i\Delta_r, (i+1)\Delta_r) \times [k\Delta_r, (k+1)\Delta_r).$$

We call these intervals r-blocks. Next we will define "good" and "bad" r-blocks. There is a standard percolation argument which also partitions space into large blocks which can be good or bad, and then shows that on the one hand the bad blocks do not percolate, and on the other hand that no percolation of bad blocks implies a desired property. In our case the desired property would be that any space-time path $\widehat{\pi} \in \Xi(\ell,t)$ intersects at most εt bad r-blocks for a suitable ℓ and for a small ε. This is indeed the desired property we are after, but we have not succeeded in simply working with r-blocks for one fixed r, because of the complicated dependence of the configurations in different r-blocks. Instead we work with r-blocks for all r. This is why we say that our proof is based on a multiscale argument.

The notion of being "good" or "bad" depends on the presence of particles in certain sets, so we need to count numbers of particles. We define $N^*(x,t)$ as the number of particles at the space-time point (x,t) in the system which evolves freely, without any B-particles. In this system we start off with $N_A(x,0) = N_A(x,0-)$ particles at x at time 0 and let all these particles perform independent random walks without any interaction. Note that $N^*(x,t) \le N_A(x,t) + N_B(x,t)$. Let $Q_r(x) := [x, x + C_0^r)$. The important counts are

$$U_r(x,v) = \sum_{y \in Q_r(x)} N^*(y,v) = \sum_{y: x \le y < x + C_0^r} N^*(y,v).$$

We call the r-block $B_r(i,k)$ bad if

$$U_r(x,v) < \gamma_r \mu_A C_0^r \tag{12}$$

for some (x,v) with integer v for which $Q_r(x) \times v$ is contained in $\widetilde{\mathscr{B}}_r(i,k)$, and where $\widetilde{\mathscr{B}}_r(i,k) := [(i-3)\Delta_r, (i+4)\Delta_r) \times [(k-1)\Delta_r, (k+1)\Delta_r) \supset \mathscr{B}_r(i,k)$.

For the time being the only important properties are that the γ_r are strictly increasing (but slowly) and satisfy

$$0 < \gamma_0 < \gamma_r < \gamma_\infty \le 1/2, \quad r > 0.$$

Roughly speaking, the bad blocks are blocks in which the number of A-particles in some only-space-like cube of specified size and which is nearby in space-time, is less than half the expected amount. Indeed, it is well known that in our setup each $U_r(x,v)$ has a Poisson distribution of mean $\mu_A C_0^r$. A block is called good if it is not bad.

If a space-time path $\widehat{\pi}$ is in a good r-block at a given time s, then there are a reasonable number of particles within distance C_0^r of $\widehat{\pi}(s)$ at time s, by definition of a good block. We therefore would like to show that "most" space-time paths intersect "few" bad blocks during $[0,t]$. To quantify this statement we define

$$\phi_r(\widehat{\pi}) = \text{ number of bad } r\text{-blocks which intersect the space-time path } \widehat{\pi},$$
$$\Phi_r(\ell) = \sup_{\widehat{\pi} \in \Xi(\ell,t)} \phi_r(\widehat{\pi}).$$

The principal part of the proof is to show that for any choice of $K > 0$ and $\varepsilon_0 > 0$ there exists an r_0 such that

$$P\{\Phi_r(\ell) \ge \varepsilon_0 C_0^{-6r}(t+\ell) \text{ for some } r \ge r_0, \ell \ge 0\} \le \frac{2}{t^K} \tag{13}$$

for all large t. This result has the desired form, because any path $\widehat{\pi}$ spends at most C_0^{6r} time units in a given r-block, and therefore at most $C_0^{6r}\phi_r(\widehat{\pi})$ time units in bad blocks during $[0,t]$. Moreover, as we stated before, we only need to consider space-time paths in $\sum_{\ell \le 2Dt} \Xi(\ell,t)$. Thus if the property in braces in (13) holds, then any $\widehat{\pi} \in \sum_{\ell \le 2Dt} \Xi(\ell,t)$ satisfies

$$C_0^{6r}\phi_r(\widehat{\pi}) \le C_0^{6r} \sup_{\ell \le zDt} \Phi_r(\ell) \le \varepsilon_0(1+2D)t,$$

and spends at most $\varepsilon_0(1+2D)t$ time units in bad blocks (for $r \geq r_0$). For $\varepsilon = \varepsilon_0(1+2D) < 1/2$ this shows that the paths of interest to us have a drift to the right for at least $t/2$ time units.

This leaves us with the problem of proving (13). This is done by means of a recurrence relation (with random terms) for the Φ_r. Note that each bad r-block has to lie either in a good $(r+1)$-block or in a bad $(r+1)$-block. Since any $(r+1)$-block contains exactly C_0^{12} r-blocks, the number of bad r-blocks which intersect a path $\widehat{\pi}$, and which are contained in a bad $(r+1)$-block (and which necessarily intersects $\widehat{\pi}$) is at most $C_0^{12}\phi_{r+1}(\widehat{\pi}) \leq C_0^{12}\Phi_{r+1}$. A similar estimate holds for the number of bad r-blocks which intersect $\widehat{\pi}$ and which are contained in a good $(r+1)$-block.

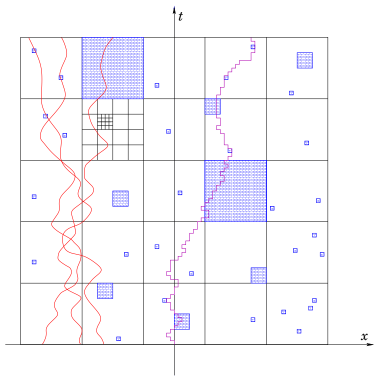

Fig. 2 Bad boxes are gray. Good (= white) boxes of scale r may contain bad boxes of scale $r+1$.

At this stage it may be useful to say a few words about the case of dimension greater than 1. There is no clear analogue of $\mathscr{R}(t)$, or at least none that is helpful. Instead of constructing paths which have a drift to the right at least half the time one now fixes an $x \in \mathscr{C}(C_2t) \cap Z^d$ and tries to construct a space-time path $\lambda(.) = \lambda(.,x)$ which has a B-particle at $\lambda(s)$ for all s, and which has a tendency to move toward x. In fact, our $\lambda(s)$ behaves like a (d-dimensional) simple random walk at times s when there is only one particle at $\widehat{\pi}(s)$, but if there are at least two particles at $\lambda(s)$ and a particle jumps away from $\lambda(s.)$ at time s, then the conditional expectation of $\|\lambda(s)-x\|_2$ is smaller than $\|\lambda(s-)-x\|_2$. This will give us a path which with high probability reaches x during $[0,t]$, provided the path has at least two particles "near $\lambda(s)$" at least a positive fraction of the time. In this way all points $x \in \mathscr{C}(C_2t) \cap Z^d$ are reached by the infection during $[0,t]$. From there on there are only minor differences between the cases $d=1$ and $d>1$ for Theorem 2.

Remark. In recent work [11] proved an improved upper bound for the velocity for the class of models where A- and B-particles respectively evolve with a diffusion constant $D_B = 1$ and a possibly time dependent jump rate $D_A \geq 0$. Even more generally, A-particles follow some independent stochastic process which also includes long range random walks with drift and various deterministic processes. assume that the density of A-particles is ρ. Then they get in all dimensions an upper bound of order $\max(\rho, \sqrt{\rho})$ that depends only on ρ and d and not on the specific process followed by A-particles, in particular that does not depend on D_A. They also argue that for $d \geq 2$ or $\rho \geq 1$ this bound can be optimal (in ρ). This leads us to another challenging open problem:

Open Problem. There is numerical evidence and some theoretical arguments in Physics literature (see [23] and references therein) which claim that for given fixed density of A-particles which are moving with the jump rate $D_A > 0$ the limiting velocity depends solely on D_B and not on D_A. Prove or disprove it!

Finally, we turn to the "full" shape theorem. Theorem 2, in particular inclusion (6) is a crucial tool for proving the shape theorem. We do not know of shortcut which proves the shape theorem without much of the development of [17] for (6).

Theorem 3. *If $D_A = D_B > 0$, then there exists a non-random, compact, convex set B_0 such that for all $\varepsilon > 0$ almost surely*

$$(1-\varepsilon)B_0 \subset \frac{1}{t}B(t) \subset (1+\varepsilon)B_0 \text{ for all large } t. \tag{14}$$

The origin is an interior point of B_0, and B_0 is invariant under reflections in coordinate hyperplanes and under permutations of the coordinates.

As we already mentioned before, the present model does not have the strict subadditivity properties. But subadditivity still remains the principal ingredient used for the proof of Theorem 3. However, we now use subadditivity only for certain "half-space" processes which approximate the true process. The "half-space" processes are evolving exactly in the same way as the original "full-space" process, but the starting initial conditions are different. We assume that initially particles are located only in half-spaces determined by a chosen direction.

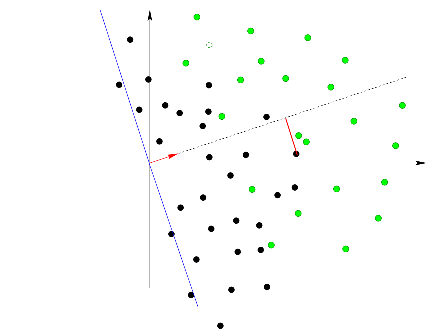

Fig. 3 "Half-space" process starts with the initial configuration of particles restricted to the half-plane. During the dynamics particles move freely and are not restricted to the half-plane. B-particles are black, and projection of the position of "most advanced" B-particle at time t on to the line parallel to chosen vector u determines $H(t,u)$.

However even these half-space processes have only approximate superconvolutive properties, but these properties are strong enough to show that for each unit vector u there exists a constant $\lambda(u)$ such that almost surely

$$\lim_{n\to\infty} \frac{1}{t} H(t,u) = \lambda(u), \tag{15}$$

where $H(t,u)$ is basically the maximum of $\langle x,u\rangle$ over all x which have been reached by a B-particle by time t ($\langle x,u\rangle$ is the inner product of x and u; for technical reasons $H(t,u)$ will be calculated in a process in which the starting conditions are slightly different from our original process). Thus the B-particles reach in time t half-spaces in a fixed direction u at a distances which grow linearly in t. Except in dimension 1, it then still requires a considerable amount of technical work to go from this result about the linear growth of the distances of reached half-spaces to the full asymptotic shape result. In particular, the coupling between the two half-space processes clearly relies heavily on the assumption $D_A = D_B$, so that we can assign the same path to a particle in the two processes, even though the types of the particle in the two processes may be different.
Remark. Our proof in [17] shows that the right hand inclusion in (6) remains valid for arbitrary jump-rates of the A and the B-particles. However, it is still not known whether the left hand inclusion holds in general. The lower bound for $B(t)$ is known only when $D_A = D_B$, or when $D_A = 0$, that is, when the A and B-particles move according to the same random walk (see [17]).
Open problem. Another interesting open problem is related to renormalization scheme itself. In spite of apparent robustness our argument requires not only that $D_A = D_B$, but also that the initial distribution of particles is product of i.i.d. Poisson with parameter μ (which is the invariant measure for the system of independent walkers). We do not know how to prove neither lower bound nor full shape theorem if started from different initial conditions.

If $D_A \neq D_B$, then we can only prove the upper bound $B(t) \subset \mathscr{C}(C_1 t)$ eventually.

2.2 Phase transition

In this subsection we will briefly discuss the situation when an "infected" B-particle can recover, i.e. all B-particles recuperate (that is, turn back into A-particles) independently of each other at a rate λ. As before, we assume that we start the system with $N_A(x,0-)$ A-particles at x, and that the $N_A(x,0-)$, $x \in \mathbb{Z}^d$, are i.i.d., mean μ_A Poisson random variables. In addition we start with one additional B-particle at the origin. We show that there is a critical recuperation rate $\lambda_c > 0$ such that the B-particles survive (globally) with positive probability if $\lambda < \lambda_c$ and die out with probability 1 if $\lambda > \lambda_c$.

Before formally stating our theorem we make some comments about the precise formulation of the model, and introduce some notation. First we define for $\eta = A$ or B

$$N_\eta(x,t) = \text{number of } \eta\text{-particles at the space-time point } (x,t).$$

Throughout we write $\mathbf{0}$ for the origin. As stated in the abstract we put $N_A(x,0-)$ A-particles at x just before we start. We then introduce a B-particle at the origin and turn some of the particles at the origin instantaneously to B-particles, so that at time 0 we start with $N_A(x,0) = N_A(x,0-)$ A-particles at $x \neq \mathbf{0}$ and $N_B(\mathbf{0},0) \in [1, N_A(\mathbf{0},0-)+1]$ B-particles at $\mathbf{0}$. However, at any time $t > 0$ an A-particle can turn into a B-particle only if the A-particle itself jumps at t or if some B-particle jumps to the position of the A-particle at time t. Thus, we are not saying that an A-particle turns into a B-particle whenever it coincides with a B-particle. We adopted the rule that a jump is required for the following reason. If we did not make this requirement, then B-particles could effectively not recover at a space time point (x,t) with several B-particles present. Indeed, if one of them tried to turn back into an A-particle at time t it would immediately become of type B again because it coincided with another B-particle. This creates some sort of singularity in the model which we are unable to handle at the moment (see, however, Remark 3 below). This is the reason for the requirement of a jump for a change from type A to type B at all strictly positive times t. Only at $t = 0$ did we change some A-particles at $\mathbf{0}$ to B-particles because they coincided with a B-particle (even though no jump occurred). The choice of the set of A-particles at $\mathbf{0}$ which is turned into B-particles at time 0 will not influence our arguments. Note that because of the jump requirement there may be particles of both types at a single space-time point.

We say that the infection *survives* if

$$P\{\text{there are some } B\text{-particles at all times}\} > 0. \tag{16}$$

Since there cannot be any B-particles after time t if there are no B-particles at t, it follows that (4) is equivalent to

$$\lim_{t\to\infty} P\{\text{there are some } B\text{-particles at time } t\} > 0. \tag{17}$$

One may even replace $\lim_t$ by $\liminf_{t\to\infty}$ in (17). Note that the survival in (4) or (17) is only global survival. Local survival in its strongest form would say that

$$\liminf_{t\to\infty} P\{N_B(\mathbf{0},t) > 0\} > 0. \tag{18}$$

A weaker form of local survival would be that

$$P\{N_B(\mathbf{0},t) \text{ for some arbitrily large } t\} > 0. \tag{19}$$

Clearly (18) implies (19), and this, in turn implies (4). We do not know how to prove that either of the forms (18) or (19) of local survival holds if λ is small enough.

The infection is said to *die out* or to *become extinct* if it does not survive, i.e., if

$$P\{\text{there is some (random) } t \text{ such that there are no } B\text{-particles after } t\} = 1. \tag{20}$$

Here is our principal result.

Theorem 4. *There exists a* $0 < \lambda_c < \infty$ *such that the infection survives if* $\lambda < \lambda_c$ *and dies out if* $\lambda > \lambda_c$.

Remark. The restriction to only one B-particle at time 0 is for convenience only. The theorem remains valid if we start with any finite number of B-particles at (non-random) positions.

Remark. We already remarked that the theorem does not give local survival if λ is sufficiently small. Neither does it tell us anything about the location of the B-particles as a function of t on the event that the B-particles survive forever.

By a special argument one can show that (19) holds for $d = 1$ and $\lambda < \lambda_c$ on the event that the B-particles survive forever.

Remark. The proof that there is survival for small $\lambda > 0$ works even in the case in which an A-particle turns into a B-particle whenever it coincides with a B-particle, that is, if we do not require that the A or B-particle jumps before reinfection can occur after recuperation of a B-particle.

3 The Stochastic Combustion Process ($D_A = 0$, $D_B > 0$)

In this Section we review results concerning the Stochastic Combustion Process. As it was mentioned in the Introduction, this is an interacting particle system which can be interpreted as a model for the burning of a propellant material, where particles represent heat and move as simple random walks which can branch and annihilate. In this Section we consider the process being defined on the lattice $\mathbb{Z}^d$. The dynamics is now defined in terms of B-particles (heat particles) and A-particles (propellant material): (i) B-particles move independently as simple symmetric continuous time random walks; (ii) A-particles are inert, so they do not move at all; (iii) each time an B-particle jumps to a site where there is a A-particle, the A-particle is transformed into an B one and begins to move as a simple symmetric random walk.

There is also a very interesting and important connection of the Combustion Process and Activated Random Walks (ARW) model broadly studied in physics literature in the context of absorbing state phase transition. If we assume that in the Combustion process B-particles can recover, i.e. again become A-particle, at rate 1, independently of anything else, then we obtain well known ARW model. We briefly discuss this case in the last subsection.

3.1 Shape theorem

For each site $x \in \mathbb{Z}^d$ and time $t \geq 0$, if there is no A-particle at site x at time t, we define $\eta_t(x)$ as the number of B-particles at site x, while $\eta_t(x) := -1$ if there is one A-particle at site x at time t. Consider the stochastic combustion process with an initial condition η_0 where there is one A-particle at each site $x \neq 0$ while one B-particle at site 0. A natural question is to describe the evolution of the set,

$$B_d(t) := \{y \in \mathbb{Z}^d : \eta_t(y) \geq 0\}, \quad t \geq 0.$$

Note that $B_d(0) = \{0\}$. The following result provides a partial answer to such a question. Here, we use the notation $[D] := D \cap \mathbb{Z}^d$ for subsets $D \subset \mathbb{R}^d$.

Theorem 5. *Assume that $B_d(0) = \{0\}$. Then, there is a closed convex bounded subset $B_d^0 \subset R^d$, symmetric under permutations of the coordinate axis and with non-empty interior, such that for every $\varepsilon > 0$, a.s. eventually in t one has that*

$$[B_d^0 t(1-\varepsilon)] \subset B_d(t) \subset [B_d^0 t(1+\varepsilon)].$$

This result was proved in [24] (see also [1]). Let us define the first time site y is visited from the initial configuration $B_d(0) := \{x\}$ (in other words, there is only one B-particle at site x while one A-particle per site $y \neq x$),

$$T_{x,y} := \inf\{t \geq 0 : y \in B_d(t)\}.$$

The proof of Theorem 5 relies on the following crucial subadditivity property,

$$T_{x,y} \leq T_{x,z} + T_{z,y}, \qquad x, y, z \in \mathbb{Z}^d,$$

combined with Kingman's subadditive ergodic theorem. It is however, difficult to prove the required condition $E[T_{0,1}] < \infty$ to apply Kingman's theorem, and most of the work in [24] and [1] is to prove such a condition using different methods, though.

3.2 The stochastic combustion process in dimension $d = 1$

Fix some $r \in \mathbb{Z}$. We assume that initially at each site x with $x \geq r+1$ there is one A particle and no B particle, while at each site $x \leq r$, there are $\eta(x) \geq 0$ particles and no A particles. We are interested in the asymptotic behavior of r_t, the position of the right-most visited site by an B particle at time t. We can prescribe the state space of this system as

$$\mathbb{S}_0 := \{(r, \eta) : r \in \mathbb{Z}, \eta \in \mathbb{N}^{\{\dots, r-1, r\}}\}.$$

In order to avoid pathological situations (involving for example a super-ballistic behavior of the front), we will consider initial conditions which belong to the set

$$\mathbb{S} := \{(r, \eta) : r \in \mathbb{Z}, \eta \in \mathbb{N}^{\{\dots, r-1, r\}}, \sum_{x \leq r} e^{\theta(x-r)} \eta(x) < \infty\},$$

where $\theta > 0$ is chosen sufficiently small. Throughout, given an initial condition $\omega \in \mathbb{S}$, we will denote by $\mathbb{P}_\omega$ the law of the stochastic combustion process starting from ω.

3.2.1 The law of large numbers

Consider an initial condition in $\mathbb{S}$ which has at least one B particle. In [24], it was proven that the there is a deterministic $v > 0$, independent of the initial conditions in $\mathbb{S}$, such that a.s. the following law of large numbers is satisfied,

$$\lim_{t\to\infty} \frac{r_t}{t} = v.$$

Furthermore, in [2] it is proven that condition (21) is optimal in the following sense: there is a $\theta' > 0$ and a $v' > v > 0$ such that when

$$\sum_{x\le 0} e^{\theta' x}\eta(x) = \infty, \tag{21}$$

one has that a.s.

$$\liminf_{t\to\infty} \frac{r_t}{t} \ge v'.$$

3.2.2 Fluctuations and the regeneration time structure

The following theorem was established in [5].

Theorem 6. *There exists $\sigma^2 \in (0,\infty)$ non-random and independent of initial conditions in $\mathbb{S}$ such that*

$$\varepsilon^{-1}(r_{\varepsilon^{-1}t} - \varepsilon^{-1}tv), \qquad t \ge 0,$$

converges in law as $\varepsilon \to 0$ to a Brownian motion with variance σ^2.

The proof of Theorem 6 is based on the construction of a regeneration time sequence with useful properties, which we now describe. For each $r \in \mathbb{Z}$, we define $\bar{r}_t^r$ as the process whose initial condition is (r, δ_r), where δ_r denotes the configuration with one B particle at site r, none elsewhere and the right-most visited site by B particles equal to r. We now construct for each $r \ge 0$, a coupling between the process r_{T_r+t} and a combustion process $\bar{r}_t^r$ such that $r_o = r$ and the configuration of B particles at sites $x \le r$ consists of only one particle at site r and none at $x < r$, so that $\bar{r}_t^r \le r_{T_r+t}$ for $t \ge 0$. Let

$$U := \inf\{t \ge 0 : \bar{r}_t^r - r < \lfloor \alpha_2 t \rfloor\}.$$

Define also the *exponential density norm* of particles by

$$\phi_z(t) := \sum_{x\in\mathbb{Z}} e^{\theta(x-r_t)}\eta_z(t,x),$$

where $\eta_z(x,t)$ is the number of particles at time t and site x which originated from a branching at some site $y \le z$. Now let $L \in \mathbb{N}$ be fixed and

$$W := \inf\{t \ge 0 : \phi_{r-L}(t,r,\eta(0)) \ge e^{\theta(\lfloor \alpha_1 t\rfloor - (r_t - r))}\}.$$

Note that W is the first time that the exponential density norm of particles which originated from a branching at a site at a distance larger than or equal to L from the initial position of the front, increases beyond $e^{\theta(\lfloor \alpha_1 t\rfloor - (r_t - r))}\}$. It is possible to perform a construction of the stochastic combustion process so that each X particle originating from a given site z is represented by a continuous time simple symmetric random walk $Z_{z,i}(t)$, where the index i labels all the particles which originate from site z. We say that the X particle originated from site z if initially the particle was at site z, or if the B particle is created at some time from the branching of a A particle which was initially at site z. We now define

$$V := \inf\{t \ge 0 : \max_{r-L<z<r}\max_{1\le i\le a} Z_{z,i}(t) > \lfloor \alpha_1 t\rfloor + r\},$$

the first time some of the particles originating from a branching at a site at a distance smaller than L from the initial position of the front, hit the line $\lfloor \alpha_1 t\rfloor + r$.

Let now

$$D := \min\{U,V,W\}.$$

Define $U\circ\theta_s, V\circ\theta_s$ and $W\circ\theta_s$ as the first times U,V or W happen after time $s \ge 0$, and $D\circ\theta_s := \min\{U\circ\theta_s, V\circ\theta_s, W\circ\theta_s\}$. For each $y \in \mathbb{Z}$, let

$$T_y := \inf\{t \ge 0 : r_t \ge y\},$$

and fix $p \in (0,1)$. Define for $x \ge r$,

$$J_x := \inf\{j \ge 1 : \phi_{x+(j-1)L}(T_{x+jL}) \le p \text{ and } m_{x+jL-L^{1/4},x+jL}(T_{x+jL}) \ge aL^{1/4}/2\}, \tag{22}$$

the first trial after the front visits site x, such that the exponential density norm of particles originating at sites at a distance larger than L from the front, decreases to a quantity smaller than p and such that there are sufficiently many particles originating from sites close to the front which are again there at time T_{x+jL} when the front advances L steps.

Define sequences of $\mathscr{F}_t$-stopping times, $\{S_k : k \ge 0\}$ and $\{D_k : k \ge 1\}$ as follows. $S_0 := 0$, $R_0 := r$, and for $k \ge 0$,

$$S_{k+1} := T_{R_k + J_{R_k} L} \qquad D_{k+1} := D\circ\theta_{S_{k+1}} + S_{k+1}, \qquad R_{k+1} := r_{D_{k+1}}.$$

The $S_k, k \ge 1$ are good times when there is control on the cloud of particles originating from sites far from the front, in the sense that the exponential norm is small enough, and there are enough particles originating from sites close to the front.

For $k \geq 1$, define $U_k := U \circ \theta_{S_k} + S_k$, $V_k := V \circ \theta_{S_k} + S_k$ and $W_k := W \circ \theta_{S_k} + S_k$. Let

$$K := \inf\{k \geq 1 : S_k < \infty, D_k = \infty\},$$

and define the *regeneration time*

$$\kappa := S_K, \tag{23}$$

if $K < \infty$ and $\kappa = \infty$ otherwise. κ is *not* a stopping time.

$\mathscr{G}$, the information up to time κ, is the completion with respect to $\mathbb{P}_w$ of the smallest σ-algebra containing all sets of the form $\{\kappa \leq t\} \cap A, A \in \mathscr{F}_t$.

Define the sequence of regeneration times $\kappa_1 \leq \kappa_2 \leq \cdots$ by $\kappa_1 := \kappa$ and for $n \geq 1$

$$\kappa_{n+1} := \kappa_n + \kappa(w_{\kappa_n + \cdot}),$$

where $\kappa(w_{\kappa_n+\cdot})$ is the regeneration time starting from $w_{\kappa_n+\cdot}$ and we set $\kappa_{n+1} = \infty$ on $\kappa_n = \infty$ for $n \geq 1$. κ_1 is the *first regeneration time* and κ_n is the *n-th regeneration time*. A crucial property of the regeneration structure that has been defined is the following corollary presented in [5], whose proof is relatively straightforward.

Corollary 1. *Let $w \in \mathbb{S}$. (i) Under $\mathbb{P}_w$, $\kappa_1, \kappa_2 - \kappa_1, \kappa_3 - \kappa_2, \ldots$ are independent, and $\kappa_2 - \kappa_1, \kappa_3 - \kappa_2, \ldots$ are identically distributed with law identical to that of κ_1 under $\mathbb{P}_{a\delta_0}[\cdot|U = \infty]$. (ii) Under $\mathbb{P}_w$, $r_{\cdot \wedge \kappa_1}, r_{(\kappa_1+\cdot)\wedge\kappa_2} - r_{\kappa_1}, r_{(\kappa_2+\cdot)\wedge\kappa_3} - r_{\kappa_2}, \ldots$ are independent, and $r_{(\kappa_1+\cdot)\wedge\kappa_2} - r_{\kappa_1}, r_{(\kappa_2+\cdot)\wedge\kappa_3} - r_{\kappa_2}, \ldots$ are identically distributed with law identical to that of r_{κ_1} under $\mathbb{P}_{a\delta_0}[\cdot|U = \infty]$.*

On the other hand, the most challenging step which renders useful the previous corollary, is the following proposition also proved in [5].

Proposition 2. *For every initial data $w \in \mathbb{S}$,*

$$\kappa < \infty, \qquad \mathbb{P}_w - \text{a.s.} \tag{24}$$

Let $a\delta_0$ denote initial data with $r = 0$, $\eta(r) = a$ and $\eta(x) = 0$, $x < 0$. Then

$$\mathbb{E}_{a\delta_0}\left[\kappa^2 | U = \infty\right] < \infty \qquad \textit{and} \qquad \mathbb{E}_{a\delta_0}\left[r_\kappa^2 | U = \infty\right] < \infty. \tag{25}$$

Now, a combination of Corollary 1 and Proposition 2, provides an argument to prove Theorem 6.

3.2.3 Large deviations

In [2], the regeneration time structure presented in the previous subsection, was applied to prove a large deviations principle for the front r_t. First we need to introduce a condition stronger than the one needed to prove the law of large numbers.

Assumption (G). For all $\theta > 0$

$$\sum_{x\le 0}\exp(\theta x)\eta(x)<+\infty. \tag{26}$$

We then have the following result.

Theorem 7. Large Deviations Principle *There exists a rate function* $I:[0,+\infty)\to[0,+\infty)$ *such that, for every initial* ω *condition satisfying* **(G)**,

$$\limsup_{t\to+\infty}\frac{1}{t}\log\mathbb{P}_\omega\left[\frac{r_t}{t}\in C\right]\le-\inf_{b\in C}I(b),\qquad \text{for}\quad C\subset[0,+\infty)\quad\text{closed},$$

and

$$\liminf_{t\to+\infty}\frac{1}{t}\log\mathbb{P}_\omega\left[\frac{r_t}{t}\in G\right]\ge-\inf_{b\in G}I(b),\qquad \text{for}\quad G\subset[0,+\infty)\quad\text{open}.$$

Furthermore, I *is identically zero on* $[0,v]$*, positive, convex and increasing on* $(v,+\infty)$.

The proof of the existence of the rate function is based on a straightforward subadditivity argument. Nevertheless, to prove that the zero set of the rate function I is the interval $[0,v]$ it is necessary to use a sophisticated argument which is based on the use of regeneration times. Since the regeneration positions of the stochastic combustion process do not have exponential moments, it is crucial to perform a coupling with a variation of the stochastic combustion process where the random walks have a fixed but small drift towards the right. For this process the regeneration positions do have exponential moments.

In [2], precise estimates for the probability of the slowdown deviations where obtained. Let

$$U(\eta):=\limsup_{x\to-\infty}\frac{1}{\log|x|}\log\left(\sum_{y=0}^{x}\eta(y)\right),\ u(\eta):=\liminf_{x\to-\infty}\frac{1}{\log|x|}\log\left(\sum_{y=0}^{x}\eta(y)\right),$$

and

$$s(\eta):=\min(1,U(\eta)).$$

For the statement of the following theorem we will write U,u,s instead of $U(\eta)$, $u(\eta)$, $s(\eta)$.

Theorem 8. Slowdown deviations estimates. *Let* η *be an initial condition satisfying* **(G)**. *Then the following statements are satisfied.*

(a)For all $0\le c<b<v$*, as* t *goes to infinity,*

$$\mathbb{P}\left[c\le\frac{r_t}{t}\le b\right]\ge\exp\left(-t^{s/2+o(1)}\right). \tag{27}$$

(b)In the special case where $\eta(x)\ge a$ *for all* $x\le 0$*, one has that, for every* $0\le b<v$*, as* t *goes to infinity,*

$$\mathbb{P}\left[\frac{r_t}{t} \leq b\right] \leq \exp\left(-t^{1/3+o(1)}\right). \tag{28}$$

(c) When $u < +\infty$, as t goes to infinity,

$$\exp\left(-t^{U/2+o(1)}\right) \leq \mathbb{P}\left[r_t = 0\right] \leq \exp\left(-t^{u/2+o(1)}\right). \tag{29}$$

In the case of a homogeneous initial configuration, like $d_- \leq \eta(y) \leq d_+$ for all $y \leq 0$, with $1 \leq d_- \leq d_+ < +\infty$, or when $(\eta(y))_{y \leq 0}$ forms a realization of an i.i.d. family of random variables with positive finite expectation, the above results take a simpler form since $u = U = s = 1$. As a consequence, $\exp(-t^{1/2})$ turns out to be the actual order of magnitude for $\mathbb{P}\left[r_t = 0\right]$, and a lower bound for $\mathbb{P}\left[c \leq \frac{r_t}{t} \leq b\right]$ when $0 \leq c < b < v$.

Open Problem. Extend above results to higher dimensions. dimensions.

3.3 Activated Random Walks model and absorbing state phase transition

Here we briefly discuss the case when B-particles can recuperate. If, as in the Sect. 2.2, we assume that B-particles recuperate at rate $\lambda > 0$, it brings us to the phenomena of absorbing state phase transition, and this process in Physics literature is known broadly as Activated Random Walk (ARW) model.

Numerical analysis and some general theoretical arguments suggest that the ARW model exhibits a phase transition in the parameters λ– the recuperation rate and μ – the initial density of particles in the system, and that there should be two distinct regimes:

i) Low particle density. There is a phase transition in λ in this case, namely if λ is large enough, then system locally fixates, i.e. for any finite volume Λ there is almost surely a finite time t_Λ such that after this time there are no B particles within Λ. If λ is small enough there is no fixation, and we expect that there is a limiting density of active particles in the long-time limit.

ii) High particle density. In this case there is no phase transition. For any $\lambda > 0$, the system does not fixate.

In spite of its intuitive transparency, it is remarkably difficult to prove existence of the phase transition. We start with the first basic fact, proved in [26] using the Diaconis-Fulton representation of the model. This representation provides an Abelian property for the dynamics of the system with finitely many particles, and – what is particularly important – provides monotonicity for the occupation times in μ as well as in λ.

Theorem 9. *[[26]] For $d \geqslant 1$ and any translation-invariant random walk and $\lambda > 0$, there exists $\mu_c \equiv \mu_c(\lambda) \in [0, \infty]$, such that if the initial distribution is i.i.d Poisson with density μ then*

$$P(\textit{system locally fixates}) = \begin{cases} 1, & \mu < \mu_c \\ 0, & \mu > \mu_c. \end{cases}$$

Moreover, μ_c is non-decreasing in λ.

For fixed λ the value of $\mu_c(\lambda)$ is not known, however some theoretical arguments suggest, and numerical simulations support, that the following holds:
Conjecture. For any dimension, any random walk, and any $\lambda > 0$,

$$0 < \mu_c(\lambda) < 1.$$

Using Peierls type argument one can show that $\mu_c(\lambda) < +\infty$:

Theorem 10 ([19]). *Consider simple symmetric random walks on $\mathbb{Z}^d, d \geqslant 1$. There exists $\mu_0 < \infty$ such that $\zeta_c(\lambda) < \mu_0$ for all λ.*

Recently E. Shellef improved this estimate:

Theorem 11. *Under the same hypotheses,*

$$\mu_c(\lambda) \leqslant 1. \tag{30}$$

Open problem. It remains a challenging problem to prove the analog of the Theorem 9 in dimensions two and more.

For general account on this process and other interesting open problems we refer to [9].

4 Modified Diffusion Limited Aggregation ($D_A > 0, D_B = 0$)

To get a better feeling for the problem which we want to present, one could think of investigating the other extreme case, namely when $D_B = 0$. Taken literally, this is not an interesting case. In this case the infected particles stand still and act as traps for the healthy particles. All that happens with any given A-particle is that it walks around till it coincides with one of the B-particles after which it stands still as well. The infected set $\widetilde{B}(t)$ equals $\widetilde{B}(0)$ at all $t \geq 0$ and the speed at which the infection spreads is 0. To obtain something interesting we have to allow the B-particles to move, at least at some times. The simplest situation is the one-dimensional one, i.e. when $d = 1$. We chose to let the B-particle move one unit to the right, when an A-particle jumps on top of it. According to our rules all A-particles which were one unit to the right of the B-particle are turned into B-particles at the time of this jump. This leads to the model we will describe now.

We consider the following problem in one-dimensional DLA. At time t we have an "aggregate" consisting of $\mathbb{Z} \cap [0, R(t)]$ (with $R(t)$ a positive integer). We also have $N(i,t)$ particles at $i, i > R(t)$. All these particles perform independent continuous time symmetric simple random walks until the first time $t' > t$ at which some particle tries to jump from $R(t)+1$ to $R(t)$. The aggregate is then increased to the integers in $[0,R(t')] = [0,R(t)+1]$ (so that $R(t') = R(t)+1$) and all particles which were at $R(t)+1$ at time $t'-$ are removed from the system. The problem is to determine how fast $R(t)$ grows as a function of t if we start at time 0 with $R(0) = 0$ and the $N(i,0)$ i.i.d. Poisson variables with mean $\mu > 0$. What makes this model particularly attractive is that it is conjectured that there is a phase transition for the growth of $R(t)$: we show that if $\mu < 1$, then $R(t)$ is of order $\sqrt{t}$ in a sense which is made precise below, and it is expected that it will grow linearly in t if μ is large enough.

This model is of further interest because it is a one-dimensional version of the celebrated DLA model of Witten and Sander (1981). In this model on $\mathbb{Z}^d$ one again has a growing aggregate $A(t) \subset \mathbb{Z}^d$ and one starts with $A(1) = \{\mathbf{0}\} =$ the origin. Usually t is taken to run through the integers and $A(t)$ has cardinality t. $A(t+1)$ is obtained from $A(t)$ by adding one point of $\mathbb{Z}^d$. This added point is the first point of the boundary of $A(t)$ which is reached by a random walker which starts at infinity (see Kesten (1987) for a more precise description). The main difference between the modified DLA model and the DLA model of Witten and Sander is that the latter adds one A-particle to the system at a time, while in the former there are infinitely many A-particles from the start. However, there have been various investigations for related models in which new A-particles are added to the system before all previously released A-particles have reached the boundary of the aggregate and are removed from the system;see for instance Lawler, Bramson and Griffeath (1992). In the physics literature, almost the same model as we discuss here was already studied by simulations in Voss (1984). However, in Voss' paper the A-particles do not perform independent random walks, but the system of A-particles evolves as an exclusion process; moreover, Voss (1984) considers the two-dimensional case. Also, Chayes and Swindle (1996) investigated hydrodynamic limits for the one-dimensional case in which the A particles follow exclusion dynamics. We remark that the particle density in an exclusion process is necessarily at most 1. As we shall see, in our model the case when the particle density μ is less than 1 can be handled much better than the case with $\mu \geq 1$. We have few results in the latter case.

As a side remark we point out that DLA is usually considered in dimension $d > 1$ in which there is a whole new level of difficulty because we do not know how to describe the "shape" of $A(t)$.

Let us now turn to the question about the rate at which $R(t)$ grows. We take $\tau_0 = 0$. As stated we take $R(0) = R(\tau_0) = 0$ and $N(i,0)$, $i \geq 1$, an i.i.d. sequence of mean μ Poisson random variables. All particles perform independent continuous time simple random walks until they are absorbed by the aggregate. Unless otherwise stated we mean by "simple random walk" a symmetric simple random walk. It is convenient to let the particles continue as a simple random walk even after absorption, by giving the particles also a color, white or black. We start with all particles white, but absorption of the particle by the aggregate is now represented by changing the color of the particle from white to black at the time of its absorption. However, the particle's path is not influenced by its color. After a particle turns black it continues with a continuous time simple random walk path. A black particle has no interaction with any other particle, nor does it influence the motion of $R(\cdot)$. Thus,

R is not increased at a time t when a black particle jumps to $R(t)$. In the sequel we shall always use this description of the system with colored particles.

$N(i,t)$ denotes the number of white particles at the space-time point (i,t). We successively define stopping times τ_k and take $R(t) = k$ on the time interval $[\tau_k, \tau_{k+1})$. Moreover, it will follow by induction on k that

$$\text{at time } \tau_k \text{ there are no white particles in } [0, R(\tau_k)] = [0,k]. \tag{31}$$

We take $\tau_0 = \mathbf{0}$. If τ_k and the $N(i,\tau_k)$ have been determined, and $R(\tau_k) = k$ and 31 holds, then we take

$$\tau_{k+1} = \inf\{t > \tau_k : \text{ some white particle jumps to position } R(\tau_k) = k\}. \tag{32}$$

Since the particles perform simple random walk and 31 holds, only white particles at position $k+1$ at time $\tau_{k+1}-$ can jump to k at time τ_{k+1}. If such a jump occurs, we take $R(\tau_{k+1}) = k+1$ (that is, $R(\cdot)$ jumps up by 1 at time τ_{k+1}), and we change to black the color of all white particles which were at $R(\tau_k)+1 = k+1$ at time $\tau_{k+1}-$ (this includes the particle which jumped to k at τ_{k+1}). It is clear that then 31 with k replaced by $k+1$ holds, so that we can now define τ_{k+2} etc. It also follows from this description that

$$R(t) = k \text{ for } \tau_k \le t < \tau_{k+1}. \tag{33}$$

Remark 1. We will not discuss here how our process can be constructed as a Markov process with the strong Markov property. See [20], [21] for details.

Let us now state our results. Throughout $\mathbf{0}$ denotes the origin and $\{S(t)\}_{t\ge 0}$ is a continuous time simple symmetric random walk on $\mathbb{Z}$ with jump rate D. Unless otherwise stated $S(0) = \mathbf{0}$. C_i will denote a constant with value in $(0,\infty)$. Its value may vary from formula to formula. Our first theorem states that for any value of μ, the common expectation of the $N(k,0)$, it is the case that

$$\limsup_{t\to\infty} \frac{1}{t} R(t) < \infty \text{ a.s.} \tag{34}$$

Theorem 12. *Assume that* $R(0) = \mathbf{0}$ *and that the* $N(i,0), i \ge 1$, *are i.i.d. mean* μ *Poisson variables. Then 34 holds. In fact, there exist constants* $0 < C_i < \infty$ *such that*

$$P\{R(t) > C_1 t\} \le C_2 \exp[-C_3 t]. \tag{35}$$

Remark 2. Theorem 12 remains valid if the particles perform an asymmetric simple random walk, that is, each jump of the random walk is $+1$ or -1 with probability p_+ and $p_- = 1 - p_+$, respectively. No change in the proof is required for this more general case.

In view of Theorem 12 it is reasonable to conjecture that $\lim_{t\to\infty}(1/t)R(t)$ exists and is constant a.s. One might even assume that this limit is strictly positive, but a quick and quite general argument in the next theorem shows that if $\mu < 1$ 'there are not enough particles around' to make $R(t)$ grow linearly with time.

Theorem 13. *Assume that* $\{N(i,0)\}_{i\geq 1}$ *is a stationary ergodic sequence and* $E\{N(i,0)\} = \mu$. *If* $0 < \mu < 1$, *then*

$$\lim_{t\to\infty} \frac{R(t)}{(\log t)^2\sqrt{t}} = 0 \text{ a.s.} \tag{36}$$

Moreover $R(t)/\sqrt{t}$, $t \geq 1$, *is a tight family, i.e.,*

$$P\{R(t) \geq x\sqrt{t}\} \to 0 \text{ as } x \to \infty, \text{ uniformly in } t \geq 1. \tag{37}$$

If we assume that the initially particles are distributed as i.i.d. Poisson random variables with mean μ, *then can be strengthened to*

$$\limsup_{t\to\infty} \frac{R(t)}{\sqrt{t}} < \infty \text{ a.s.} \tag{38}$$

Remark 3. One can formulate a d-dimensional analogue of our model and of Theorem 2. In this version one works on $\mathbb{Z}^d$ and at time 0 the aggregate consists of the origin only, while at the site $x \neq \mathbf{0}$ there are $N(x,0)$ particles, with the $N(x,0)$, $x \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ i.i.d. Poisson variables of mean μ. Again all particles perform independent continuous time simple random walks. They all start out as white particles. We denote the aggregate at time t by $A(t)$. If at some time t, a white particle jumps from a site $x \notin A(t-)$ onto the aggregate, then we set $A(t) = A(t-) \cup \{x\}$, and all particles which were at x at time $t-$ are changed to black at time t.

Define an outer radius of the aggregate by

$$R^{(o,d)} := \sup\{\|x\|_2 : x \in A(t)\},$$

and define an inner radius as

$$R^{(i,d)}(t) := \inf\{\|x\|_2 : x \notin A(t)\}.$$

The latter is the distance from the origin to the nearest vertex outside $A(t)$. For this model Theorem 12 remains valid. More precisely, 35 and 34 with $R(t)$ replaced by $R^{(o,d)}(t)$ still hold. Theorem 4 has the following analogue: If $\mu < 1$, then

$$\limsup_{t\to\infty} \frac{R^{(i,d)}(t)}{\sqrt{t}} < \infty \text{ a.s.} \tag{39}$$

(Note that 39 is trivially true if there exists a site x_0 which never is occupied by $A(t)$.) We shall not give the proofs of these results here. They are essentially the same as for Theorem 1 and for 38. If we strengthen our assumptions on the $N(i,0)$,

then we can show that in the one-dimensional model $R(t)/\sqrt{t}$ is actually bounded away from 0 in distribution. This holds for all $\mu > 0$.

Theorem 14. *Assume that the* $N(i,0), i \geq 1$, *are i.i.d. with finite second moment* $\mu_2 > 0$. *Then for all* $\varepsilon > 0$ *there exists an* $\eta = \eta(\varepsilon) > 0$ *and a* $t_0 = t_0(\varepsilon)$ *such that*

$$P\Big\{\frac{R(t)}{\sqrt{t}} > \eta\Big\} \geq 1-\varepsilon \textit{ for all } t \geq t_0. \tag{40}$$

Unfortunately the simple proof of () breaks down when $\mu > 1$ and we therefore conjecture that there exists a critical value $\mu_c \geq 1$ such that

$$\lim_{t\to\infty} \frac{1}{t} R(t) \text{ exists and is a.s. a constant which is } \begin{cases} > 0 & \text{if } \mu > \mu_c \\ = 0 & \text{if } \mu < \mu_c. \end{cases} \tag{41}$$

A stronger conjecture would be that

$$\mu_c = 1. \tag{42}$$

Simulations certainly indicate that this is the case, however we have made only little progress towards proving (41), so we pose this as a problem.
Open problem 1. Prove (41), and if this holds, determine μ_c. If one becomes even more ambitious one can ask whether power laws exist as $\mu \downarrow \mu_c$, and what the critical exponents are. To formulate this problem we have to assume that $\lim_{t\to\infty}(1/t)R(t)$ exists. Let us write $S(\mu)$ for this limit.
Open problem 2. Does

$$\lim_{\mu\downarrow\mu_c} \frac{\log S(\mu)}{\log\big(\mu-\mu_c\big)}$$

exist and if so, what is its value? One more problem about the DLA model. This one is motivated by Theorems and .
Open problem 3. Does $t^{-1/2}R(t)$ have a limit distribution as $t \to \infty$ when $\mu < 1$?

The obvious approach to proving that $R(t)$ grows linearly in t is to study our system as seen from the right edge of the aggregate. Indeed the collection of positions of the white particles relative to $R(t)$ form a Markov process. Does this Markov process have a non-trivial invariant probability distribution, and if so is the invariant distribution unique? (By non-trivial we mean that we exclude the distribution which puts no particles at all to the right of the aggregate.) On an intuitive level one would like to say that the invariant measure puts at position $R(t)+x$ roughly a Poisson number of particles with mean equal to μ times the probability that a particle at $R(t)+x$ is white. That is, the mean number of particles at $R(t)+x$ should be $\lim_{t\to\infty}\mu\nu(x,t)$, where

$$\nu(x,t) = P\{R(t)+x-S(s) > R(t-s) \text{ for } 0 \leq s \leq t\}.$$

Actually, all we want to know in first instance is that the density of white particles right in front of $R(t)$ is bounded away from 0 as $t \to \infty$. We want to show that the system does not develop large holes without white particles in front of $R(t)$. To obtain such a result we need some a priori control of $R(t)-R(t-s)$, which we do not know how to control. Tom Kurtz (private communication) showed us that conditionally on the σ-field generated by $\{R(s) : s \leq t\}$, the $N(R(t)+x,t)$ have a Poisson distribution with a mean $\mu_i(t)$, and even derived a system of differential equations for the μ_i. Unfortunately this system still involves the unknown random function $R(\cdot)$ in boundary conditions and we have been unable to make use of these differential equations.

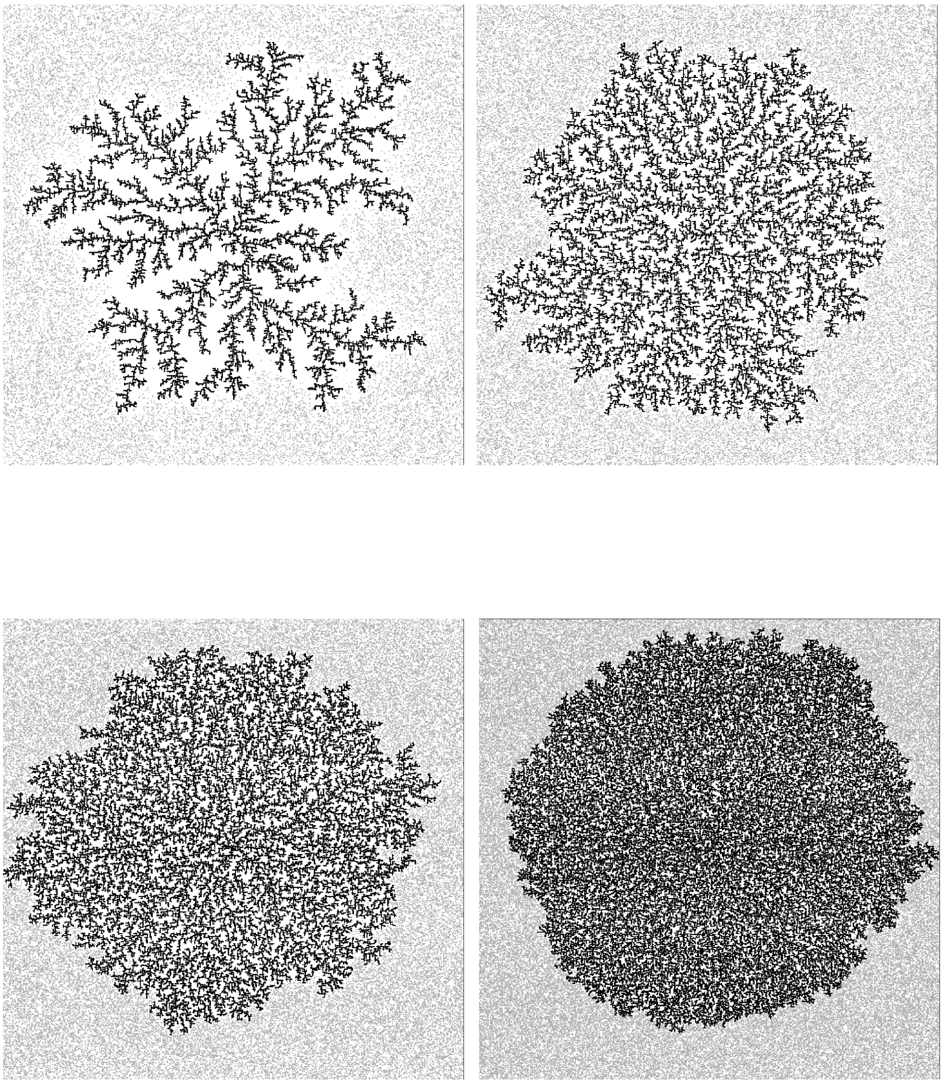

Fig. 4 DLA clusters with the initial density μ of particles being 0.3, 0.4, 0.6 and 0.68. In the two last cases numerical simulations indicate linear growth.

In conclusion we must say that since we were unsuccessful in proving the existence of a non-trivial invariant probability measure for the Markov process of the last paragraph, we designed some caricatures of the model. We hope that these caricatures can be regarded as 'approximations' to the true model and will help us treat the true model. These caricatures have some built-in mechanism that makes it more difficult for a large hole to form in front of the aggregate (see [20], [21]).
Open problem. Show that in dimension larger or equal to two there is linear growth and asymptotic shape for μ large enough. Actually we conjecture that in dimension ≥ 2 the critical density $\mu_c < 1$.

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