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ASYMPTOTIC SURVIVAL PROBABILITIES IN THE RANDOM SATURATION PROCESS

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We consider a model of diffusion in random media with a two-way coupling (i.e., a model in which the randomness of the medium influences the diffusing particles and where the diffusing particles change the medium). In this particular model, particles are injected at the origin with a time-dependent rate and diffuse among random traps. Each trap has a finite (random) depth, so that when it has absorbed a finite (random) number of particles it is "saturated," and it no longer acts as a trap. This model comes from a problem of nuclear waste management. However, a very similar model has been studied recently by Gravner and Quastel with different tools (hydrodynamic limits). We compute the asymptotic behavior of the probability of survival of a particle born at some given time, both in the annealed and quenched cases, and show that three different situations occur depending on the injection rate. For weak injection, the typical survival strategy of the particle is as in Sznitman and the asymptotic behavior of this survival probability behaves as if there was no saturation effect. For medium injection rate, the picture is closer to that of internal DLA, as given by Lawler, Bramson and Griffeath. For large injection rates, the picture is less understood except in dimension one.

1. Introduction. We present a model of growth, diffusion and trapping in a random environment. This model has three main features: a random environment, an injection pattern and a two-way coupling between random walks and the random environment (i.e., the random environment acts on the particles by trapping, and the particles act on the environment by saturation of the traps).

We describe rapidly the three ingredients of the model.

1. First, the random environment is given by a collection of i.i.d. integer valued random variables $\eta(x)$ at each site x of the lattice $\mathbb{Z}^d$. Here $\eta(x)$ is the initial depth (or capacity) of the trap at site x, with the convention that the site x is not a trap if the depth $\eta(x)$ is zero.
2. Second, the injection pattern: at the origin of the lattice $\mathbb{Z}^d$ particles are injected (or born) at a time-dependent rate. We will mainly study deterministic injection patterns but most of the results are true with Poissonian

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injections. We will indicate later where random injection does make a difference.

3. Finally, the interaction between the medium and the random walks: when born, the particles perform continuous time simple random walks on the cubic lattice, until they find a nonsaturated trap (i.e., visited by less particles than its initial depth). When meeting such a trap the particle stops and stays forever in this trap. The depth of the trap is then decreased by 1. When a trap is full or saturated, that is, when it has been visited by as many particles as its initial depth, or equivalently when its depth has reached zero, it no longer acts as a trap and particles can walk on it.

Our initial motivation came from a simplified version of a problem of confinement of heavy nucleotides in nuclear waste management by high-performance clay barriers. This context suggested the random injection at one point and the trapping and saturation mechanism. A more complete study would ask for a model with interaction between the particles, and the possibility for "desorption" (i.e., for the particles to leave the traps after a long time). But other various motivations can be proposed. For instance Funaki [8] studies a related model (with a deterministic environment and no injection and with an interactive dynamics) in the context of "melting" (see also [9]).

The model we examine is flexible. For instance, it encompasses two models recently studied. Namely, the Poissonian traps model of A. S. Sznitman (see [17] and references within) and the internal diffusion limited aggregation (IDLA) model introduced by Diaconis and Fulton [5] in a discrete time setting and studied in the continous time setting by Lawler, Bramson and Griffeath (see [12, 13]). The analog of the Poissonian traps model in the discrete context of the cubic lattice corresponds in our model to the situation where saturation of the traps is omitted. For instance, this would be the case where the injection is limited to the injection of only one particle. Or equivalently, the case when the initial depth of the traps is infinite, whatever the injection rate is. IDLA deals with the case in which all sites are initially traps of depth 1 (no randomness of the medium), and the injection has a specific rate (i.e., the total number of particles born at time t is a Poisson process of constant intensity). The trapping and saturation mechanism is sometimes called "noise reduction" in the literature about growth models (see paragraph 4.1 of [11] in the context of the Eden where what corresponds to the depth of the traps is deterministic).

We have chosen to deal with the simplest possible description of the initial randomness of the medium (i.e., i.i.d. distribution of initial depths of traps) in order to use the very powerful machinery developed by Sznitman (see [15, 16, 17]). We had to adapt to the discrete context his latest version of the "enlargement of obstacles" method (see [16, 17]) and this is the subject of an appendix, which might be of independent use. We have limited ourself to the case of bounded depths. In fact what we really had in mind was the situation where $\eta(x)$ could take only the values 0 (the site i is then not a trap) and $m \in \mathbb{N}$. Situations where very deep traps would be present could produce very different behaviors.

After this description of the model we now state some of the natural questions about this model. There are at least four types of such questions (ordered from the simplest to the most difficult one). The first one is about the shape of the saturated set.

1. What is the shape of the set of saturated traps?

The second question is about survival probabilities.

2. What is the proportion of live particles at time t? What is their age distribution?

More precisely, what is the probability of survival of the kth born particle until time t? This question can be asked first when k is fixed and then when both k and t go to infinity.

The third question is about the location of live particles.

3. What is the typical path of the kth born particle if it is conditioned to live until time t?

And finally the last question is about the collective behavior of the live particles.

4. What is the profile of the cloud of live particles at time t?

Gravner and Quastel [9] deal with the fourth question. In the context of IDLA (with zero-range dynamics), they prove among other things that when $d = 2$, under an hydrodynamic scaling limit, the profile of the cloud of live particles converges weakly in probability to the solution of the one-phase Stefan problem with a source at the origin.

The results of this paper concern the first two questions. The third one will be treated elsewhere. Before describing our results, we recall that two main lines of statements are possible: annealed and quenched. The first corresponds to a statement in average and the second to an almost sure statement with respect to the randomness of the medium. We will give results in both situations, though we believe that the most important are the quenched ones. The main result of this paper is the existence of three very different situations depending on the strength of the injection. We naturally call these the low, medium and high injection regimes. Let us call $N(t)$ the number of particles that have been born at time t. The high injection regime is reached when $N(t) \gg t^{d/2}$, both in the annealed and quenched situations. It differs from the other two injection regimes by the fact that most particles will survive (will not be trapped). In this high injection regime, we do not know too much about the answer to question 1 (the shape of the saturated region), except when the lattice has dimension 1 (this is reported elsewhere [3]).

Here we will concentrate on the two other regimes. We will see that in both of them, there is a growing saturated zone, spherical with a radius growing as $\big((1/aw_d)N(t)\big)^{1/d}$, where a is the average depth of each obstacle and w_d the volume of a sphere of unit radius. We will also see that in both these regimes, the survival probability tends to zero, but at very different rates. The reason for this difference can be roughly explained as follows. When the injection rate is too low [$N(t) \ll \ln t$ in the quenched case and $N(t) \ll t^{d/(d+2)}$ in the annealed case], the saturated zone is too small to really matter for survival and the saturation effect is irrelevant, so that the result is essentially given by the survival probability in the Poissonian traps model without saturation. When the injection rate is in the medium range [$\ln t \ll N(t) \ll t^{d/2}$ in the quenched case and $t^{d/(d+2)} \ll N(t) \ll t^{d/2}$ in the annealed case], the saturated zone is large enough to modify the survival probability, which can now heuristically be computed as the probability that a Brownian motion does not cross some spherical moving boundary.

The main difficulty in the proof of the large deviation estimates of Theorems 2 and 3, providing the quenched and annealed logarithmic asymptotics of the survival probability of a single particle, corresponds to the proof of the upper bounds. For the quenched and annealed medium regimes, we do not have a good enough control of the probability that the saturated set of obstacles is not a ball. This means that the shape Theorem 1, answering question 1 and stating that with "high" probability the set of saturated traps corresponds to erasing obstacles within a ball of a certain radius, cannot be used for the upper bounds, and therefore all possible shapes for the saturated set at some given time have to be considered. With the exception of part (ii) of the annealed Theorem 3, an understanding about the asymptotic behavior of the principal Dirichlet eigenvalue of the discrete Laplacian on large sets with random absorbing obstacles is needed. It is the case that the smallest possible value that one can obtain for this principal Dirichlet eigenvalue, after erasing a high enough predetermined amount of obstacles, corresponds to erasing a ball. This is the content of part (ii) of Theorem 6, which is proved by means of an adaptation of the latest version of the enlargement of obstacles technique of Sznitman [16, 17]. An analogous analysis is required for the proof of the upper bound of Theorem 2 in the low regime. We would like to remark that the use of an adaptation of the latest version of the enlargement of obstacle, where different scales are introduced for the so-called bad and density sets, has been crucial to obtain the upper bounds for injection rates close to the critical ones [$N(t) \sim \ln t$ in the quenched case and $N(t) \sim t^{d/(d+2)}$ in the annealed situation]. As part of the proof of Theorem 6 mentioned above, a discrete version of the Faber–Krahn inequality was needed (given in Lemma 13). It might be the case that this precise estimate is known and has already been proved, but we were unable to find the proper references.

The detailed answers to question 1 are given in Theorem 1. Theorems 2 and 3 deal with question 2, in the quenched and annealed situation, respectively. The organization of the paper is as follows. In Section 2 we introduce the model together with the notation that will subsequently be used, and state the results. In the third section the shape Theorem 1 is proved. We first prove a shape theorem for the discrete time version of our model, where a particle is born only after the previous one has been trapped. This is then used to prove Theorem 1. The whole approach is an adaptation of the proof of the shape theorem for IDLA in [13] to a context where the obstacles have a random distribution and the injection pattern is variable. The only part of the proof of the discrete time version model which presents an additional difficulty with respect to [13] is the lower bound in the quenched case (see Section 3.3). In Section 4, some key asymptotic estimates of principal Dirichlet eigenvalues are obtained. Here the main tool is Sznitman's enlargement of obstacles adapted to the discrete context. A secondary ingredient proved here is a version of Faber–Krahn inequality in the cubic lattice. In the last section Theorems 2 and 3, describing the decay of the survival probability, are proved. Both the asymptotic estimates of Section 4 and the shape Theorem 1 form the basis of this proof. In Appendix A, the cubic lattice version for random walk of the enlargement of obstacle method of Sznitman is constructed. Finally, in Appendix B some lemmas used in the proof of Theorems 2 and 3 of Section 5 are proved. This includes Lemma 20 concerning a decay estimate for the survival time of a random walk in a ball with a time-dependent radius.

2. Notation and Results. In what follows we will define a stochastic process corresponding to the dynamics described in the introduction of random walks in a lattice with some absorbing sites or obstacles. Let m be some natural number and define $\mathscr{I} := \{0, \dots, \bar{a}\}$. The state space representing the obstacle configuration endowed with the natural topology will be denoted by $\Gamma := \mathscr{I}^{\mathbb{Z}^d}$. Let $\mathscr{B}$ be the corresponding Borel σ-algebra. Given an element $\eta \in \Gamma$ we denote its xth coordinate by $\eta(x)$. A site x such that $\eta(x) \geq 1$ represents a site with an obstacle present, while $\eta(x) = 0$ means that there is no obstacle. Furthermore, let P be the probability measure on $\Omega := D([0,\infty), \mathbb{Z}^d)^{\mathbb{N}}$ endowed with its Borel σ-algebra $\mathscr{D}$, under which the coordinate process $\{Z_n: n \in \mathbb{N}\}$ represents independent simple random walks on $\mathbb{Z}^d$ each of jump rate 1, and such that $Z_n(0) = 0$. Let $\{T_n \in [0,\infty): n \in \mathbb{N}\}$ be a sequence of strictly increasing times and define random walks $\{Y_n: n \in \mathbb{N}\}$ by $Y_n(t) := 0$ if $0 \leq t \leq T_n$ and $Y_n(t) := Z_n(t - T_n)$ if $t > T_n$. Let us define $N(t) := \sum_{n=1}^{\infty} 1_{[0,t]}(T_n)$, where 1_B is the indicator function of $B \subset \mathbb{R}$, representing the total number of random walks that have been born at time t.

We now proceed to define a collection of probability measures $Q_{N,\eta}$ on the space $(\Omega, \mathscr{D})$, indexed by the set of right continuous increasing functions N from $[0,\infty)$ taking values on $\mathbb{N}$ and by the set of configurations $\eta \in \Gamma$, and which we will call the random saturation process. Under each measure $Q_{N,\eta}$, the coordinate process $\{Z_n: n \in \mathbb{N}\}$ on Ω will represent the dynamics of interacting random walks. In the sequel it is understood that any infimum over an empty subset of $\mathbb{N}$ or $\mathbb{R}$ is infinity.

For a given $n \in \mathbb{N}$, we define the stopping time

$$s_n^1 := \inf\{t \geq 0: \eta(Y_n(t)) > 0\},$$

which is the first time that the random walk Y_n visits an obstacle. Now let

$$t_1 := \inf\{s_n^1: n \in \mathbb{N}\}.$$

This is the first time some obstacle has been visited. We now define

$$Y_n^1(t) := \begin{cases} Y_n(t), & \text{if } s_n^1 > t_1, \\ Y_n(t \wedge s_n^1), & \text{if } s_n^1 = t_1. \end{cases} \tag{1}$$

Here we have stopped those random walks which hit a trap for the first time. Let $\mathbb{M}_1 := \{n \in \mathbb{N}: s_n^1 = t_1\}$ be the set of indices where the infimum in the definition of T_1 is attained. Similarly define $\mathbb{Z}_1 := \{x \in \mathbb{Z}^d: x = Y_n(s_n^1)$ for some $n \in \mathbb{M}_1\}$. It is easy to see that P-a.s. the set $\mathbb{M}_1$ has a unique element, which we will denote by n_1, and this is the index of the unique random walk which is stopped in (1). Therefore P-a.s. the set $\mathbb{Z}_1$ has a unique element x_1 such that $x_1 := Y_{n_1}(s_{n_1}^1)$. We now update the obstacle configuration by defining

$$\eta^1(x) := \begin{cases} \eta(x_1) - 1, & \text{if } x = x_1, \\ \eta(x), & \text{if } x \neq x_1. \end{cases}$$

In other words, we decrease by 1 the site which has the trap which has been visited first. Note again, that P-a.s. this site is unique. Now define recursively for $k \geq 2$, $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$ the stopping times s_n^k and t_k, and the processes $\eta_t^k(x)$ and $Y_m^k(t)$, as follows:

$$s_n^k := \inf\{t \geq 0: \eta^{k-1}(Y_n^{k-1}(t)) > 0\},$$

$$t_k := \inf\{s_n^k: n \in \mathbb{N}\backslash(\mathbb{M}_1 \cup \cdots \cup \mathbb{M}_{k-1})\},$$

$$\mathbb{M}_k := \{n \in \mathbb{N}: s_n^k = t_k\},$$

$$\mathbb{Z}_k := \{x \in \mathbb{Z}^d: x = Y_n^k(s_n^k) \text{ for some } n \in \mathbb{M}_k\},$$

$$Y_n^k(t) := \begin{cases} Y_n^k(t), & \text{if } s_n^k > t_k, \\ Y_n^k(t \wedge s_n^k), & \text{if } s_n^k = t_k, \end{cases}$$

$$\eta^k(x) := \begin{cases} \eta^{k-1}(x) - 1, & \text{if } x \in \mathbb{Z}_k, \\ \eta^{k-1}(x), & \text{otherwise.} \end{cases}$$

Note that P-a.s. the sets $\mathbb{M}_k$ and $\mathbb{Z}_k$, for $k \geq 1$, each have a unique element. From the fact that the total number of random walks $\{Y_m: m \in \mathbb{N}\}$ in movement at a given time t is finite, it is not difficult to check that for each $n \in \mathbb{N}$, as $k \to \infty$ the sequence of processes Y_n^k converges P-a.s. on the Skorokhod topology of Ω. Let us call such a limit $X_n^{N,\eta}$. We then define $X^{N,\eta} := \{X_n^{N,\eta}: n \in \mathbb{N}\}$. Note that under the probability measure P, this process represents random walks which move freely until they visit the first site x which has received less than $\eta(x)$ visits, at which time they are frozen. To this process there corresponds a probability measure $Q_{N,\eta}$ under which the coordinate process $Z := \{Z_n: n \in \mathbb{N}\}$ on Ω is distributed as $X^{N,\eta}$ under P. It is defined by $Q_{N,\eta}(A) := P(X^\eta \in A)$ for every set $A \subset \mathscr{D}$. In the sequel, we will say that Z under the probability measure $Q_{N,\eta}$ is a random saturation process on an obstacle configuration η and driven by an injection N. We will denote by $\tau_n := \inf\{t \geq 0: Z_n(s) = Z_n(t) \text{ for } s \geq t\}$, the time at which the random walk Z_n is frozen.

We will now endow the obstacle state space $(\Gamma, \mathscr{B})$ with a product probability measure μ defined by

$$\mu(\eta(x) = \alpha) = p_\alpha,$$

where $\sum_{\alpha \in \mathscr{I}} p_\alpha = 1$. Note that a random saturation process with injection $N(t) = [t]$ (where for $x \in \mathbb{R}$, $[x]$ represents the closest integer greater than or equal to x) and an obstacle configuration with law μ such that $\mathscr{I} = \{0, 1\}$ and $p_0 = 0$, $p_1 = 1$, corresponds closely to the continuous time version of internal diffusion limited aggregation (IDLA) introduced in [13]. In fact, the only difference is that in the continuous time version of IDLA, the birth times $\{T_n: n \in \mathbb{N}\}$ are sums of exponentially distributed random variables. In contrast, a random saturation process with injection $N(t) = 1$ and obstacle configuration with a law given by μ is a random walk on a lattice where sites are absorbing independently of each other with some positive probability. This model was studied in [1, 2] using an adaptation to the lattice of the first version of the enlargement of obstacle method developed by Sznitman [15].

Now let

$$\zeta(x, t) := \sum_{n \in \mathbb{N}} 1_{Z_n(t)}(x), \tag{2}$$

where for $A \subset \mathbb{Z}^d$, we define $1_A: \mathbb{Z}^d \to \{0, 1\}$ as the indicator function of the set A. Here $\zeta(x, t)$ represents the number of random walks at time t in site x. Then define

$$S_t := \{x \in \mathbb{Z}^d: \zeta(x, t) \geq \eta(x) > 0\}. \tag{3}$$

This set corresponds to the sites x of the cubic lattice $\mathbb{Z}^d$ which have an obstacle, and which have been visited at least $\eta(x)$ times. We will call it the set of saturated obstacles at time t. In the sequel, for $x \in \mathbb{Z}^d$, we define the norm $\|x\| := \sqrt{x_1^2 + \cdots + x_d^2}$, where for $1 \leq i \leq d$, x_i is the ith coordinate of x. Also, given two real valued functions $f_1(t)$ and $f_2(t)$, the notation $f_1(t) \ll f_2(t)$ will mean that $\lim_{t \to \infty}(f_1(t)/f_2(t)) = 0$. The main result of the third section of these notes is the following shape theorem.

THEOREM 1. *Consider a random saturation process on an obstacle configuration η and driven by an injection N. Let η be distributed according to some product measure μ and call $a := \mu(\eta(x))$ the average depth of obstacles at time 0. Define $B_r := \{x \in \mathbb{Z}^d: \eta(x) > 0$ and $\|x\| < r\}$. Assume that $1 \ll N(t) \ll t^{d/2}/\ln t$ and that $a > 0$. Then:*

(i) *For every $\epsilon > 0$, $Q_{N,\mu}$-a.s. there exists a $t_0 \geq 0$ such that*

$$B_{(1-\epsilon)\left((1/aw_d)N(t)\right)^{1/d}} \subset S_t \subset B_{(1+\epsilon)\left((1/aw_d)N(t)\right)^{1/d}},$$

whenever $t \geq t_0$.

(ii) μ-a.s. *the following is true*: *for every* $\epsilon > 0$, $Q_{N,\eta}$-a.s. *there exists a* $t_0 \geq 0$ *such that*

$$B_{(1-\epsilon)\left((1/aw_d)N(t)\right)^{1/d}} \subset S_t \subset B_{(1+\epsilon)\left((1/aw_d)N(t)\right)^{1/d}},$$

whenever $t \geq t_0$.

REMARK 1. It is not difficult to see that if N is distributed according to some Poisson process R of time-dependent rate $\lambda(t)$, a.s. with respect to the distribution of R the annealed and quenched versions of parts (i) and (ii) of Theorem 1, with N replaced by $\int_0^t \lambda(s)\,ds$ in the statement, are true.

Note that in the above shape theorem, at time t, the volume of the limiting sphere times the average depth of obstacles, equals the total number of particles $N(t)$ that have been born. This means that the proportion of frozen random walks converges to 1.

Let $k(t)$: $[0,\infty) \longrightarrow \mathbb{N}$ be an increasing function of time and let $g(t) := T_{k(t)}$ be the birth time of the random walk $Z_{k(t)}$. In the third section of these notes, we will be interested in understanding the asymptotic behavior of the survival probability of the random walk $Z_{k(t)}$ with law given by $Q_{N,\eta}$, both when $k(t)$ is fixed as time goes to infinity and when $k(t)$ goes to infinity together with time. To state the results of this section let us introduce some notation. Let λ_d be the principal Dirichlet eigenvalue of the Laplacian operator divided by $2d$ on the ball of unit radius and w_d the volume of the ball. Define $p := \mu(\eta(x) > 0)$, $a := \mu(\eta(x))$ and denote by $p_c(d)$ the critical probability of site percolation on $\mathbb{Z}^d$. In the sequel we assume that $p > 0$. The first theorem is a quenched version of the asymptotics of the survival probability.

THEOREM 2. *Consider a random saturation process on an obstacle configuration* η *and driven by an injection* N. *Assume that* $0 < N(t) \ll t^{d/2-\epsilon}$ *for some* $\epsilon \in (0,1)$, *that* $\limsup_{t\to\infty} k(t) > \bar{a}$ *and that* $t - g(t) \gg 1$. *Then*:

(i) *Assume that* $1 \ll N(t) \ll (t-g(t))^{d/2}$. *If* $\ln(t-g(t)) \ll N(t)$ *or* $p > 1 - p_c(d)$ *then*

$$\lim_{t\to\infty} \frac{1}{h_M(k,t)} \ln Q_{N,\eta}(\tau_{k(t)} > t) = -1 \quad \mu\text{-a.s.,} \tag{4}$$

where $h_M(k,t) := \lambda_d (aw_d)^{2/d} \int_{g(t)}^t (ds/N(s)^{2/d})$.

(ii) *If* $N(t) \ll \ln(t-g(t))$ *and* $p < 1 - p_c(d)$ *then*

$$\lim_{t\to\infty} \frac{1}{h_L(k,t)} \ln Q_{N,\eta}(\tau_{k(t)} > t) = -1 \quad \mu\text{-a.s.,}$$

where $h_L(k,t) = \lambda_d (w_d |\ln(1-p)|)^{2/d} (t - g(t)/\ln(t-g(t))^{2/d})$.

REMARK 2. The condition $N(t) \ll t^{-\epsilon+d/2}$ for some $\epsilon > 0$, is necessary to ensure the validity of the shape Theorem 1. On the other hand, the less important condition $\liminf_{t\to\infty} k(t) \geq \bar{a}$ is included to rule out the possibility that the random walk born at time $g(t) = T_{k(t)}$ dies at the origin as soon as it is born. This might be the case when $k(t)$ is some constant smaller than $\bar{a}$, which is the maximum value of the obstacle capacity η at each site.

Let us briefly discuss the meaning of the above result. For the sake of clarity, let us consider the case in which $k(t)$ is some constant greater than $\bar{a}$. When $p < 1 - p_c$, we know that with μ-a.s. there exists a unique trap free cluster on the lattice $\mathbb{Z}^d$. The above theorem shows that for $N(t) \ll t^{-\epsilon+d/2}$, for some $\epsilon > 0$, there appear to be two different injection regimes when $p < 1 - p_c$. There is the regime which we will denote by *quenched low regime*, when $N(t) \ll \ln t$, given by part (ii). Here subscript L in h_L stands for *low*. The survival strategy for random walks in this regime consists essentially of traveling fast to a distance of order t to some region of the lattice free of obstacles and of radius of the order of $(\ln t)^{1/d}$. This is exactly the survival strategy of a Brownian motion on $\mathbb{R}^d$ with Poissonian obstacles (see [17]) or of a simple random walk on the lattice with site obstacles distributed according to some product measure (see [2]). There is a second injection regime for $\ln t \ll N(t) \ll t^{-\epsilon+d/2}$, which we call *quenched medium regime*, given by part (i) of Theorem 2. The subscript M in h_M stands for *medium*. Here random walks are provided with a better survival strategy than going far to find natural clearings, as in the low regime. Namely, by the shape Theorem 1, the high enough injection produces a central clearing larger than those that can be found far away. Thus, the typical survival strategy of a particle is to stay in this central region. When $p > 1 - p_c$, so that μ-a.s. there is no infinite trap free cluster, Theorem 2 states that fort any injection rate satisfying the condition $N(t) \ll t^{-\epsilon+d/2}$, the decay of the survival probability is as in the medium regime. For the purpose of illustrating the above description, let us consider the special situation in which the injection rate is of the form

$$N(t) = (\ln t)^{\alpha}$$

for $\alpha \geq 0$. In this case, the logarithm of the probability that a random walk born at some fixed time survives up to time t is going to decay like some function $h(t) = t/(\ln t)^{\beta}$, where β is a function of α. Figure 1 shows the dependence of β with respect to α. Note that when $\alpha > 1$, we are in the medium regime and $h = h_M$. On the other hand, for $0 < \alpha < 1$ we are in the low regime, and, depending on the value of the percolation parameter p, the decay function h takes the values h_M (for $p > 1 - p_c$) or h_L (for $p < 1 - p_c$). The second result of Section 4 is an annealed version of Theorem 2. As before, we are assuming that $p > 0$.

THEOREM 3. *Consider a random saturation process in an obstacle configuration η and driven by an injection N. Assume that $g(t) < t$. Assume that $N(t) \ll t^{d/2-\epsilon}$ for some $\epsilon \in (0, 1)$, that $\limsup_{t\to\infty} k(t) > \bar{a}$ and that $t - g(t) \gg 1$. Then*:

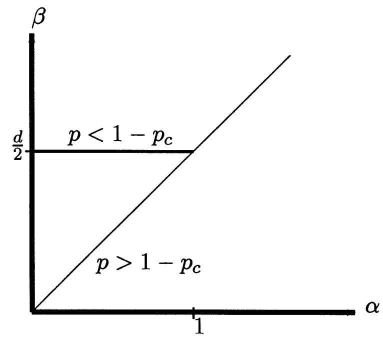

FIG. 1. *Dependence of β with respect to α.*

(i) *Assume that* $1 \ll N(t) \ll (t - g(t))^{d/2}$. *If* $(t - g(t))^{d/(d+2)} \ll N(t)$ *or* $p = 1$, *then*

$$\lim_{t\to\infty} \frac{1}{h_M(k,t)} \ln Q_{N,\mu}(\tau_{k(t)} > t) = -1,$$

where $h_M(k,t) := \lambda_d (a w_d)^{2/d} \int_{g(t)}^{t} (ds/N(s)^{2/d})$.

(ii) *If* $N(t) \ll (t - g(t))^{d/(d+2)}$ *and* $p < 1$, *then*

$$\lim_{t\to\infty} \frac{1}{\tilde{h}_L(k,t)} \ln Q_{N,\mu}(\tau_{k(t)} > t) = -1,$$

where $\tilde{h}_L(k,t) = (w_d |\ln(1-p)|)^{2/(d+2)} \big((d+2)/2\big)\big(2\lambda_d/d\big)^{d/(d+2)} (t - g(t))^{d/(d+2)}$.

The main feature of the quenched Theorem 2 is still in this annealed theorem, namely the presence of two injection regimes for $N(t)$ satisfying $N(t) \ll t^{-\epsilon+d/2}$ for some $\epsilon > 0$. The role of p_c is here played by $p = 1$. Furthermore, this time the transition between the two regimes occurs at the injection rate $N(t) \sim t^{d/(d+2)}$. What we call here *low regime*, when $N(t) \ll t^{d/(d+2)}$, corresponds to the studying the decay properties of the annealed survival probability of a Brownian motion in $\mathbb{R}^d$ with Poissonian obstacles [6] or a simple random walk on the lattice with each site absorbing independently of the others [6, 1]. This means that the survival strategy of a practice in the low regime is to stay in a natural clearing of radius $t^{1/(d+2)}$ produced at the origin. What we call *annealed medium regime*, when $t^{d/(d+2)} \ll N(t) \ll t^{-\epsilon+d/2}$, gives the same decay as the quenched medium regime with particles taking advantage of the central clearing produced by saturation.

3. Shape theorems. The object of this section is to prove the shape Theorem 1. We will follow [13] closely. As there, we will first consider a discrete time version of the random saturation process. This will be a process on the set of obstacles, representing the saturated ones. We will prove a quenched and annealed shape theorem for this process which will subsequently be used to prove Theorem 1. Except for the lower bound of the quenched theorem, the whole proof follows the arguments given in [13].

3.1. *Definition of discrete time version of random saturation.* As in the previous section, we define a probability measure μ on $\Gamma = \mathscr{I}^{\mathbb{Z}^d}$ as the product measure μ such that

$$\mu(\eta(x) = \alpha) = p_\alpha,$$

where $\sum_{\alpha \in \mathscr{I}} p_\alpha = 1$ and $\eta \in \Gamma$. For every $\eta \in \Gamma$ we can define the random subset of $\mathbb{Z}^d$,

$$O(\eta) = \{x \in \mathbb{Z}^d\colon \eta(x) > 0\},$$

which represents the sites with "active" obstacles. Now consider a discrete time random walk on the lattice $\mathbb{Z}^d$ starting at the origin and which is killed upon touching an obstacle in $O(\eta)$. Furthermore, assume that when the random walk is killed at the site x, the state $\eta \in \Gamma$ suffers a transition to a state η^x where

$$\eta^x(y) = \begin{cases} \eta(y), & \text{if } y \neq x, \\ \eta(x) - 1, & \text{if } y = x. \end{cases}$$

Call $\eta_1 = \eta^x$ and adopt the convention that $\eta_1 = \eta$ whenever the random walk is never killed. Similarly define $\eta_2 = \eta_1^x$. Continuing in this way we can define a discrete time Markov chain η_n with state space Γ and with initial condition $\eta_0 \in \Gamma$. Call P_{η_0} the corresponding probability measure on the path space $\Gamma^{\mathbb{N}}$ with the product topology. Note that this Markov process has δ_0 as unique invariant measure, where $\mathbf{0} \in \Gamma$ is such that $\mathbf{0}(x) = 0\ \forall\ x \in \mathbb{Z}^d$. Also note that this Markov process has transition probabilities $p(n, \zeta)$, $\eta, \zeta \in \Gamma$ given by

$$p(n, \zeta) = \begin{cases} 0, & \text{if } \zeta \neq \eta \text{ and } \zeta \neq \eta^x, \\ h_0(O(\eta), \infty), & \text{if } \zeta = \eta, \\ h_0(O(\eta), x), & \text{if } \zeta = \eta^x. \end{cases}$$

Here $h_y(A, \infty)$ is the probability that a random walk starting at $y \in \mathbb{Z}^d$ never hits $O(\eta)$ and $h_y(A, B)$ represents the probability that the same random walk hits A at $B \subset A$.

In the next subsections we want to prove the following theorems.

THEOREM 4. *Let $a = \mu(\eta(x))$ be the average depth of obstacles. For each $\eta \in \Gamma$, let $B_r = \{x \in O(\eta(0))\colon \|x\| \le r\}$. Then for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that*

$$P_\mu(B_{n(1-\epsilon)} \subset D_{aw_d n^d} \subset B_{n(1+\epsilon)} \textit{ for } n \ge n_0) \ge 1 - \epsilon,$$

where w_d is the volume of the unit ball and $D_k := O(\eta_0) - O(\eta_k)$.

THEOREM 5. *Let* $a = \mu(\eta(x))$ *be the average depth of obstacles. For each* $\eta \in \Gamma$, *let* $B_r = \{x \in O(\eta(0)) \colon \|x\| \le r\}$. *Then* μ-a.s. *the following statement is true: for every* $\epsilon > 0$ *there exists an* $n_0 \in \mathbb{N}$ *such that*

$$P_\eta(B_{n(1-\epsilon)} \subset D_{aw_d n^d} \subset B_{n(1+\epsilon)} \text{ for } n \ge n_0) \ge 1 - \epsilon,$$

where w_d *is the volume of the unit ball and* $D_k := O(\eta_0) - O(\eta_k)$.

REMARK 3. From the above theorem we can conclude that

$$P_\eta\Big(\lim_{n\to\infty} \eta_n = 0\Big) = 1 \quad \mu\text{-a.s.}$$

In what follows we prove Theorems 4 and 5, separating each proof in to an upper and a lower bound part. Since the upper bound part of Theorem 4 is very similar to that of Theorem 5, we omit it. The lower bound part of the annealed Theorem 4 is a straightforward adaptation of the method of [13]. This is the content of the next subsection. On the other hand, the lower bound part of the proof of Theorem 5 requires more careful estimates. This is presented in Section 3.3.

3.2. *Proof of the discrete time annealed lower bound.* Using [13] here we will show the following.

LEMMA 1. *Let* a *and* B_r *be defined as in Theorem* 1. *Then for every* $\epsilon > 0$ *there is an* $n_0 \in \mathbb{N}$ *such that*

$$P_\mu(B_{n(1-\epsilon)} \subset D_{aw_d n^d(1+\epsilon)} \quad \text{for } n \ge n_0) \ge 1 - \epsilon.$$

PROOF. We adopt the same notation as in [13]. Let us call $X_i(t)$ the random walk which produces the transition $\eta_{i-1} \to \eta_i$ and let us remove the killing. Now, for $z \in O(\eta_0)$, define the following stopping times:

$$\sigma_i = \inf_{t\ge 0}\{t \colon X_i(t) \in O(\eta_{i-1})\},$$

$$\tau_{i,z} = \inf_{t\ge 0}\{t \colon X_i(t) = z\},$$

$$\tau_n = \inf_{t\ge 0}\{t \colon X_i(t) \notin B_n\}.$$

First, note that $\{B_{n(1-\epsilon)} \not\subseteq D_{aw_d n^d(1+\epsilon)}\}$. On the other hand the event that site z does not belong to D_k can be written as

$$F_z(k) = \bigcap_{i=1}^{k} \{\sigma_i < \tau_{i,z}\}.$$

Furthermore, the event that at time $aw_d n^d(1+\epsilon)$ a set A is not a subset of $D_{aw_d n^d(1+\epsilon)}$ is contained in

$$\bigcup_{z\in A} F_z \tag{5}$$

Therefore, to prove the lemma it is enough to show that

$$\sum_{n=1}^{\infty} P_\mu\left(\bigcup_{z\in B_{n(1-\epsilon)}(\eta_0)} F_z(aw_d n^d(1+\epsilon))\right) < \infty.$$

At this point let us define the following random variables:

$$N_z(w) = \sum_{i=1}^{aw_d n^d(1+\epsilon)} \theta_{\tau_{i,z}\le\sigma_i}(w),$$

$$L_z(w) = \sum_{i=1}^{aw_d n^d(1+\epsilon)} \theta_{\sigma_i\le\tau_{i,z}<\tau_n}(w),$$

$$M_z(w) = \sum_{i=1}^{aw_d n^d(1+\epsilon)} \theta_{\tau_{i,z}<\tau_n}(w).$$

Here $w \in \mathscr{I}^{\mathbb{Z}^d}$, $\theta_A(w)$ equals 1 if $w \in A$ and 0 otherwise and $z \in B_{n(1-\epsilon)}$. Then, to estimate the probability of the event (5) it is enough to estimate $P_\mu(N_z \le \bar{a})$. But $N_z \ge M_z - L_z$, and

$$P_\mu(N_z \le a) \le P_\mu(M_z \le \alpha + \bar{a}) + P_\mu(L_z \ge \alpha)$$

for any $\alpha \ge 0$. Now clearly,

$$E_\mu(M_z) = aw_d n^d(1+\epsilon) p_0(\tau_z < \tau_n),$$

where p_x is the probability distribution associated to a random walk starting from x, τ_z is the hitting time of this random walk to site z and $\tau_n = \min\{t\colon X_t \notin \overline{B}_n\}$ is the exit time of the ball $\overline{B}_n = \{x\colon \|X\| \le n\}$. On the other hand we can bound L_z by $\widetilde{L}_z$, where

$$\widetilde{L}_z = \sum_{y\in\overline{B}_n} \eta(y)\theta^y_{\tau_z<\tau_n}$$

and θ^y_A is the indicator function of event A for a random walk starting from point y. Note that

$$E_\mu(\widetilde{L}_z) = a\sum_{y\in\overline{B}_n} p_y(\tau_z < \tau_n).$$

Now define $G_n(y,z)$, the Green's function of a random walk stopped upon leaving $\overline{B}_n$, by

$$G_n(y,z) = E_y\left[\sum_{t=0}^{\tau_n-1} \theta_{\{X_t=z\}}\right], \qquad z \in \overline{B}_n.$$

Now since $p_0(\tau_z < \tau_n) = G_n(0,z)/G_n(z,z)$ and $p_y(\tau_z < \tau_n) = G_n(y,z)/G_n(y,y)$, we have

$$E_\mu(M_z) = aw_d n^d(1+\epsilon)\frac{G_n(0,z)}{G_n(z,z)},$$
$$E_\mu(\widetilde{L}_z) = a\sum_{y\in\overline{B}_n}\frac{G_n(y,z)}{G_n(z,z)}.$$

At this point we invoke the following lemmas proved in [13] and [12].

LEMMA 2. *Fix $\varepsilon > 0$. For n sufficiently large and $z \in \overline{B}_{n(1-\epsilon)}$,*

$$\sum_{y\in\overline{B}_n} G_n(y,z) \le w_d n^d G_n(0,z).$$

LEMMA 3. *Let $z \in \overline{B}_n$ and $\{z\} = \max\{\|z\|, 1\}$. Then*

$$\begin{aligned} G_n(0,z) &= \frac{2}{\pi}\ln\frac{n}{\{z\}} + o\left(\frac{1}{\{z\}}\right) + O\left(\frac{1}{n}\right), \qquad d=2 \\ &= \frac{2}{d-2}\frac{1}{w_d}(\{z\}^{2-d} - n^{2-d}) + O(\{z\}^{1-d}), \qquad d\ge 3. \end{aligned} \tag{6}$$

Moreover, if $z \in \overline{B}_{n(1-\epsilon)}$, where $\epsilon > 0$, we have

$$G_{\epsilon n}(0,0) \le G_n(z,z) \le G_{2n}(0,0). \tag{7}$$

LEMMA 4. *If $z \in \overline{B}_n$, then*

$$n^2 - \|z\|^2 \le E_z(\tau_n) \le (n+1)^2 - \|z\|^2.$$

LEMMA 5. *Let S be a finite sum of independent indicator random variables with mean μ. For any $0 < \gamma < 1/2$, and for all sufficiently large μ,*

$$P(|S-\mu| \ge \mu^{1/2+\gamma}) \le 2e^{-(1/4)\mu^{2\gamma}}.$$

Now, from Lemma 2 it follows that $E_\mu(M_z) \ge (1+\frac{\epsilon}{2})E_\mu(\widetilde{L}_z)$. But note from inequalities (6) and (7) of Lemma 3 and from Lemma 4 that

$$\begin{aligned} E_\mu(\widetilde{L}_z) &= a\frac{E_z(\tau_n)}{G_n(y,y)} \ge \frac{aE_z(\tau_n)}{G_{2n}(0,0)} \\ &\ge \frac{a}{G_{2n}(0,0)}(n^2 - \|z\|^2). \end{aligned} \tag{8}$$

Therefore,

$$E_\mu(\widetilde{L}_z) \ge \begin{cases} aw_d(d-2)\epsilon n^2, & \text{if } d\ge 3, \\ a\pi\epsilon\frac{n^2}{\ln n}, & \text{if } d=2. \end{cases} \tag{9}$$

Next, from Lemma 5 we have

$$P_\mu(M_z - E_\mu(M_z) \le -E(M_z)^{5/6}) \le 2e^{-(1/4)E_\mu(M_z)^{2/3}}$$
$$\le 2e^{-(1/4)C_d n},$$

where $C_d = (aw_d(d-2)\epsilon)^{2/3}$ if $d \ge 3$ and $C_d = (a\pi\epsilon)^{2/3}$ if $d = 2$. But for n big enough it is true that $E_\mu(M_z) - E_\mu(M_z)^{5/6} \ge (1+\frac{\epsilon}{4})E_\mu(\widetilde{L}_z) + \bar{a}$. Therefore,

$$P_\mu\left(M_z \le \left(1+\frac{\epsilon}{4}\right)E_\mu(\widetilde{L}_z) + \bar{a}\right) \le 2e^{-C_d n/4}. \tag{10}$$

Similarly we have $P_\mu(\widetilde{L}_z \ge E_\mu(\widetilde{L}_z) + E_\mu(\widetilde{L}_z)^{5/6}) \le 2e^{-(1/4)E_\mu(\widetilde{L}_z)^{2/3}} \le 2e^{-C_d n/4}$. From which we conclude that

$$P_\mu\left(\widetilde{L}_z \ge \left(1+\frac{\epsilon}{4}\right)E_\mu(\widetilde{L}_z)\right) \le 2e^{-C_d n/4}. \tag{11}$$

As in [13], the lower bound of Lemma 1 now follows easily from inequalities (10) and (11). □

3.3. *Proof of the discrete time quenched lower bound.* Here we will prove the following lemma.

LEMMA 6. *Let a and B_r be defined as in the theorem. Then μ-a.s. the following statement is true: for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that*

$$P_\eta(B_{n(1-\epsilon)} \subset D_{aw_d n^d(1+\epsilon)} \ \textit{for } n \ge n_0) \ge 1-\epsilon.$$

In the sequel we will follow the notation of the previous section. First, note that it is enough to prove that for every $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P_\eta\left(\bigcup_{z\in B_{n(1-\epsilon)}(\eta_0)} F_z(aw_d n^d(1+\epsilon))\right) < \infty \quad \mu\text{-a.s.} \tag{12}$$

Now, the left-hand side is bounded by $\sum_{z\in\overline{B}_{n(1-\epsilon)}} P_\eta(N_z = 0)$. Therefore, it is enough to prove that there is an $\alpha \ge 0$ such that for every $\epsilon > 0$,

$$\sum_{z\in\overline{B}_{n(1-\epsilon)}} (P_\eta(M_z \le \alpha + \bar{a}) + P_\eta(L_z \ge \alpha)) < \infty \quad \mu\text{-a.s.}$$

Now, since M_z does not depend on the obstacle configuration, we conclude in analogy with (10) that for every $\epsilon > 0$,

$$P_\eta\left(M_z \le \left(1+\frac{\epsilon}{4}\right)E_\mu(\widetilde{L}_z) + \bar{a}\right) \le 2e^{-C_d n},$$

where C_d is a positive constant depending on the dimension and ϵ. Thus, the proof of the convergence of the series in (12) will be completed once we show the following.

LEMMA 7.

$$\sup_{z\in \overline{B}_{n(1-\epsilon)}} P_\eta(\widetilde{L}_z \geq (1+\epsilon)E_\mu(\widetilde{L}_z)) \leq f(n,\eta)e^{-(\epsilon^2 w_d(d-2)/4K_\epsilon va)n^2} \quad \mu\text{-a.s.}, \tag{13}$$

where $a := \mu(\eta(y))$, $v := \mu(\eta^2(y))$, $K_\epsilon := \sup_{z:|z|\leq 1-\epsilon} \int_{y:|y|\leq 1}(dy/|y-z|^{d-2})$, *and*

$$\lim_{n\to\infty} \frac{\log f(n,\eta)}{n^2} = 0.$$

PROOF. Define the random variable

$$Y_{n,z} = \widetilde{L}_z - E_\mu(\widetilde{L}_z).$$

Now let z_n be a sequence such that $z_n \in \overline{B}_{n(1-\delta)}$, where $\delta > 0$. Then, by Chebyshev's inequality, we have

$$\limsup_{n\to\infty} \frac{1}{n^2} \log\ P_\eta\big(Y_{n,z_n} \geq \epsilon n^2\big) \leq \limsup_{n\to\infty} \frac{1}{n^2} \log\ E_\eta\big(e^{\lambda Y_{n,z_n}}\big) - \epsilon\lambda, \tag{14}$$

where $\lambda > 0$. To complete the proof of Lemma 7 we will need the following two auxiliary lemmas.

LEMMA 8. *Let* $z_n \in \overline{B}_{n(1-\delta)}$, *where* $\delta > 0$. *Then*

$$\limsup_{n\to\infty} \frac{1}{n^2} \log\ E_\eta(e^{\lambda Y_{n,z_n}}) = (M_\lambda - 1 - a\lambda) \limsup_{n\to\infty} \frac{1}{n^2} \times \sum_{y\in \overline{B}_n} p_y(\tau_{z_n} < \tau_n) \quad \mu\text{-a.s.},$$

where $a = \mu(\eta(y))$ *and* $M_\lambda = E_\mu(e^{\lambda Y_{n,z}})$.

LEMMA 9. *Let* $z_n \in \overline{B}_{n(1-\delta)}$, *where* $\delta > 0$. *Then*

$$\limsup_{n\to\infty} \frac{1}{n^2} \sum_{y\in \overline{B}_n} p_y(\tau_{z_n} < \tau_n) \leq K_\delta,$$

where $K_\delta := \sup_{z:|z|\leq 1-\delta} \int_{y:|y|\leq 1}(dy/|y-z|^{d-2})$.

Before proving these lemmas we show how Lemma 7 is implied by them and the inequality (14). First, note that we can conclude that for $z_n \in \overline{B}_{n(1-\delta)}$ and $\lambda > 0$ (the case in which $M_\lambda - 1 - a\lambda \geq 0$),

$$\limsup_{n\to\infty} \frac{1}{n^2} \log\ P_\eta(Y_{n,z_n} \geq \epsilon n^2) \leq (M_\lambda - 1 - a\lambda)K_\delta - \epsilon\lambda.$$

However, for λ small enough we have $|M_\lambda - 1 - a\lambda| \leq \lambda^2 v$. Thus,

$$\limsup_{n\to\infty} \frac{1}{n^2} \log\ P_\eta(Y_{n,z_n} \geq \epsilon n^2) \leq \lambda^2 v K_\delta - \epsilon\lambda.$$

Choosing $\lambda = \epsilon/2K_\delta v$, we conclude that whenever $z_n \in \overline{B}_{n(1-\delta)}$ and $\epsilon > 0$, one has that

$$\limsup_{n\to\infty} \frac{1}{n^2} \log P_\eta(Y_{n,z_n} \geq \epsilon n^2) \leq -\frac{\epsilon^2}{4K_\delta v} \quad \mu\text{-a.s.}$$

Now, from the inequality (8) we conclude that

$$\limsup_{n\to\infty} \frac{1}{n^2} \log P_\eta\left(\widetilde{L}_z \geq \left(1 + \epsilon \frac{w_d(d-2)}{a}\right) E_\mu(\widetilde{L}_z)\right) \leq -\frac{\epsilon^2}{4K_\delta v} \quad \mu\text{-a.s.}$$

Clearly this implies Lemma 7. □

It remains to prove the auxiliary Lemmas 8 and 9.

PROOF OF LEMMA 8. First note that

$$\log E_\eta(e^{\lambda Y_{n,z_n}}) = \sum_{y\in\overline{B}_n} \big(\log(1 + p_y(\tau_{z_n} < \tau_n)(e^{\lambda\eta(y)} - 1)) \\ - a\lambda p_y(\tau_{z_n} < \tau_n)\big).$$

But since $|x - \log(1+x)| \leq x^2$, we have

$$\left|\sum_{y\in\overline{B}_n} \big(\log(1 + p_y(\tau_{z_n} < \tau_n)(e^{\lambda\eta(y)} - 1)) - p_y(\tau_{z_n} < \tau_n)(e^{\lambda\eta(y)} - 1)\big)\right| \\ \leq C \sum_{y\in\overline{B}_n} p_y^2(\tau_{z_n} < \tau_n),$$

where $C = \sup_y (e^{\lambda\eta(y)} - 1)^2$. Our first step will be to show that

$$\sum_{y\in\overline{B}_n} p_y^2(\tau_{z_n} < \tau_n) \leq K_d f_d(n) \tag{15}$$

for some constant K_d and some function $f_d(n)$ depending only on the dimension and such that $f_d(n) := n$ if $d \neq 2$ and $f_2(n) := n^2/\ln n$. The case $d = 1$ is trivial. In fact it is enough to bound the probabilities in the sum of the right-hand side of inequality (15) by 1. In the sequel K_d will denote different constants depending only on the dimension. For the case $d = 2$ we employ the following inequality:

$$\sum_{y\in\overline{B}_n} p_y^2(\tau_{z_n} < \tau_n) \leq \sum_{y\in\overline{B}_{2n}} p_0^2(\tau_y < \tau_{2n}). \tag{16}$$

Now, it can be proved that for every $\beta < 2$ one has that $p_0(\tau_y < \tau_{2n}) = (\ln(2n))^{-1}[\ln(2n) - \ln|y| + o(|x|^{-\alpha}) + O((\ln n)^{-1})]$ (see [12]). This together with the estimate of inequality (16) implies that there is a constant K_2 such that when $d = 2$, one has

$$\sum_{y\in\overline{B}_n} p_y^2(\tau_{z_n} < \tau_n) \leq K_2 \frac{n^2}{\ln n}.$$

In particular this proves inequality (15) in the case $d = 2$. Finally, we consider the case $d \geq 3$. Note that $p_y(\tau_{z_n} < \tau_n) \leq p_y(\tau_{z_n} < \infty) = G(y,z)/G(0,0)$, where $G(y,z)$ is the Green function of the symmetric simple random walk. But $G(x) = a_d|x|^{2-d} + O(|x|^{-d})$ for some constant a_d depending on the dimension (see [12]). We can then conclude that

$$\sum_{y\in\overline{B}_n} p_y^2(\tau_{z_n} < \tau_n) \leq K_d \int_{y:\ |y|\leq n} \frac{1}{|y-z_n|^{2d-4}} \wedge 1\, dy$$

for some constant K_d. We have

$$\begin{aligned}\sum_{y\in\overline{B}_n} p_y^2(\tau_{z_n} < \tau_n) &= K_d \frac{1}{n^{d-4}} \int_{y:\ |y|\leq 1; |y-z_n|\geq 1/n} \frac{1}{|y-z_n/n|^{2d-4}} dy + K_d w_d \\ &\leq K_d \frac{1}{n^{d-4}} \int_{y:\ 1/n\leq|y|\leq 2} \frac{1}{|y|^{2d-4}} dy + K_d w_d \\ &\leq K_d n,\end{aligned}$$

which proves (15) in the case $d \geq 3$.

At this point note that inequality (15) implies that

$$\begin{aligned}&\limsup_{n\to\infty} \frac{1}{n^2} \log E_\eta(e^{\lambda Y_{n,z_n}}) \\ &= \limsup_{n\to\infty} \frac{1}{n^2} \sum_{y\in\overline{B}_n} p_y(\tau_{z_n} < \tau_n)(e^{\lambda\eta(y)} - 1 - a\lambda).\end{aligned}$$

Therefore it is now enough to prove that

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{y\in\overline{B}_n} p_y(\tau_{z_n} < \tau_n)(e^{\lambda\eta(y)} - M_\lambda) = 0 \quad \mu\text{-a.s.} \tag{17}$$

Now if $S_n = (1/n^2)\sum_{y\in\overline{B}_n} p_y(\tau_{z_n} < \tau_n)(e^{\lambda\eta(y)} - M_\lambda)$, we have

$$p_\mu(|S_n| \geq \alpha) \leq \frac{1}{\alpha^2 n^4} N_\lambda \sum_{y\in\overline{B}_n} p_y^2(\tau_{z_n} < \tau_n),$$

where $N_\lambda = \mu(e^{\lambda\eta(y)} - M_\lambda)^2$ and $\alpha > 0$. However, since $\sum_{y\in\overline{B}_n} p_y^2(\tau_{z_n} < \tau_n) \leq K_d n$, it follows that

$$\sum_{n=1}^{\infty} p_\mu(S_n \geq \alpha) < \infty.$$

By Borel–Cantelli, since α is arbitrarily small, (17) follows. □

PROOF OF LEMMA 9. For dimensions $d = 1, 2$ it is trivial that $(1/n^2)\sum_{y\in\overline{B}_n} p_y(\tau_{z_n} < \tau_n)$ is bounded by a constant independent of n. For dimensions $d \geq 3$

note that

$$\frac{1}{n^2}\sum_{y\in\overline{B}_n} p_y(\tau_{z_n} < \tau_n) \le K_d \frac{1}{n^2}\int_{y:\ |y|\le n} \frac{dy}{|y-z_n|^{d-2}}$$

$$= K_d \int_{y:\ |y|\le 1} \frac{dy}{|y-z_n/n|^{d-2}}$$

which is bounded. The lemma now follows. □

3.4. *Proof of the discrete time quenched upper bound.* Here we will prove the following lemma.

LEMMA 10. *Let a and $B_r(\eta)$ be defined as in Theorem* 4. *Then*, μ-a.s. *the following statement is true*: *for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that*

$$P_\eta(D(\eta_{aw_dn^d}) \subset B_{n(1+K\epsilon)} \ \textit{for} \ \ n \ge n_0) \ge 1-\epsilon,$$

where K is a constant.

PROOF.
Step 1. Here we will prove that for every $\epsilon > 0$ there is a n_0 such that

$$P_\eta\left(\sum_{x\in\overline{B}_n^c} (\eta_0(x) - \eta_{aw_dn^d}(x)) \le \epsilon K_0 n^d \text{ for } n \ge n_0\right) \ge 1-\epsilon, \tag{18}$$

where $K_0 = aw_d 2^{d+1}$. First, note that the lower bound of Lemma 6 implies that μ-a.s. the following statement is true: for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that

$$P_\eta\left(\sum_{x\in\overline{B}_{n(1-\epsilon)}} \eta_0(x) \le \sum_{x\in\overline{B}_{n(1-\epsilon)}} (\eta_0(x) - \eta_{aw_dn^d}(x)) \text{ for } n \ge n_0\right) \ge 1-\epsilon.$$

Now,

$$\sum_{x\in\mathbb{Z}^d} (\eta_0(x) - \eta_{aw_dn^d}(x)) = aw_d n^d.$$

Therefore μ-a.s. the following statement is true: for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} P_\eta\Bigg(\sum_{x\in\overline{B}_n^c} (\eta_0(x) - \eta_{aw_dn^d}(x)) &\le aw_d n^d \\ &- \sum_{x\in\overline{B}_{n(1-\epsilon)}} \eta_0(x) \text{ for } n \ge n_0\Bigg) \ge 1-\epsilon. \end{aligned} \tag{19}$$

However, by the strong law of large numbers we know that

$$\lim_{n\to\infty} \frac{1}{n^d(1-\epsilon)^d} \sum_{x\in\overline{B}_{n(1-\epsilon)}} \eta_0(x) = aw_d.$$

Therefore, the following statement is μ-a.s. true: for every $\epsilon > 0$ there is an integer n_2 such that

$$\sum_{x \in \overline{B}_{n(1-\epsilon)}} \eta_0(x) \geq a w_d n^d (1-\epsilon)^d - \epsilon, \tag{20}$$

when $n \geq n_2$. Therefore, combining (19) with (20) we conclude that μ-a.s. the following statement is true: for every $\epsilon > 0$ is an n_0 such that

$$P_\eta\left(\sum_{x \in \overline{B}_n^c} (\eta_0(x) - \eta_{a w_d n^d}(x)) \leq a w_d n^d - a w_d n^d (1-\epsilon)^d + \epsilon \text{ for } n \geq n_0\right)$$
$$\geq 1 - \epsilon.$$

The claim (18) follows now from the inequality $a w_d n^d - a w_d n^d (1-\epsilon)^d + \epsilon \leq a w_d n^d \epsilon 2^{d+1}$.

Step 2. Let us relabel the particles X^{i_j} that exit B_n during the time interval from 0 to $w_d n^d$ as Y^j and consider the embedded growth process $D(i_j)$. Choose $k_0 = [n(1+\epsilon^{1/d})] + 1$ and introduce the quantity

$$Z_k(j) = \sum_{x \in D(i_j) \cap J_{k_0+k}} \eta_0(x) = \sum_{x \in O^c(\eta_{i_j}) \cap J_{k_0+k-1}} \eta_0(x),$$

where $J_k := O(\eta_0) \cap \{x\colon k \leq \|x\| < k+1\}$. This quantity represents the number of dead levels up to the rescaled time j for the obstacles contained in the shell J_{k_0+k}. Now, define the average of $Z_k(j)$ as $\nu_k(j) = E_\eta(Z_k(j))$, $k \geq 1$. Clearly $\nu_1(j) \leq j$ and $\nu_k(0) = 0$ for $k \geq 1$. We shall now prove the following inequality for each j and k:

$$\nu_k(j) \leq n^{d-1}\left[\frac{J}{a}\frac{j}{k}\epsilon^{(1-d)/d} n^{1-d}\right]^k, \tag{21}$$

where $J < \infty$.

First, condition on the time τ at which Y^{l+1} exits $\overline{B}_n$. This gives

$$\nu_k(l+1) - \nu_k(l) = E_\eta(h_{Y^{l+1}(\tau)}(O(\eta_{i_l}), J_{k_0+k})).$$

Now, any walk stopping in J_{k_0+k} must remain within the subset of sites not containing obstacles while it hits the immediately preceding shell J_{k_0+k-1}. Therefore, this is at most

$$E_\eta\big(h_{Y^{l+1}(\tau)}(J_{k_0+k-1}, O^c(\eta_{i_l}))\big) \leq \max_{y \in J_n} E_\eta\big(h_y(J_{k_0+k-1}, O^c(\eta_{i_l}))\big). \tag{22}$$

At this point we make use of the following lemma, proved in [13].

LEMMA 11. *Let $T_k = \min\{t \geq 1\colon X(t) \in J_k\}$ be the hitting time of J_k. There exists a constant $J < \infty$ such that*

$$h_y(J_k, B) \leq J|B|\Delta^{1-d},$$

where $j < k$ and $\Delta = k - j$.

Applying the uniform upper bound of the above lemma to inequality (22) we obtain

$$\nu_k(l+1) - \nu_k(l) \le J \frac{1}{(n\epsilon^{1/d})^{d-1}} E_\eta(|J_{k_0+k-1} \cap O^c(\eta_{i_l})|)$$
$$= \frac{J}{(n\epsilon^{1/d})^{d-1}} \frac{\nu_{k-1}(l)}{a}.$$

Summing this inequality over $l = 0, \dots, j-1$ we get

$$\nu_k(l) \le \frac{J}{a}\left(\frac{1}{n\epsilon^{1/d}}\right)^{d-1} \sum_{l=1}^{j-1} \nu_{k-1}(l)$$

Iteration in k with j fixed yields

$$\nu_k(l) \le \left(\frac{J}{a}\left(\frac{1}{n\epsilon^{1/d}}\right)^{d-1}\right)^{k-1} \frac{j^k}{k!}.$$

Step 3. We shall now establish the upper bound from inequality (21) of Step 2 and inequality (18) of Step 1. By the last inequality we know that μ-a.s. the following it true: for every $\epsilon > 0$ there is an event F and an integer n_0 such that $P_\eta(F) \ge 1 - \epsilon$, where F is the event that for $n \ge n_0$ the number of dead levels outside the ball $\overline{B}_n$ in the set of obstacles $O(\eta_0)$ is smaller than or equal to $\epsilon K_0 n^d$, where $K_0 = a w_d 2^{d+1}$. Therefore, for $K \ge K_0$ we have that the following statement is μ-a.s. true: for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ and an event F such that

$$P_\eta\big(D(a w_d n^d) \not\subseteq B(n(1 + K\epsilon^{1/d}), F\big) \le P_\eta\big(Z_{\eta'}([k_0 \epsilon n^d]) \ge 1\big),$$
$$P_\eta(F) \ge 1 - \epsilon, \tag{23}$$

whenever $n \ge n_0$, where $K_0 = 2^{d+1} a w_d$ and $n' = [n(K-1)\epsilon^{1/d}] - 1$. Now, by the Chebychev inequality, the right-hand side of the inequality (23) is bounded by

$$\nu_{n'}\big([K_0 \epsilon n^d]\big) \le n^{d-1}\left[J_1 \frac{[k_0 \epsilon n^d]}{n(K-1)\epsilon^{1/d} - 1} \epsilon^{(1-d)/d_{n^{1-d}}}\right]^{n'}$$
$$\le n^{d-1}\left(\frac{J_2}{K}\right)^{n'}$$

for a suitable constant J_2. Therefore, we have that the following statement is μ-a.s. true: for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ and an event F such that if $K > J_2$,

$$\sum_{n \ge n_0} P_\eta\big(D(w_d n^d) \not\subseteq B(n(1 + K\epsilon^{1/d}), F\big) \le \sum_{n \ge n_0} e^{-\alpha n} < \infty,$$
$$P_\eta(F) \ge 1 - \epsilon.$$

So by Borel–Cantelli we conclude that μ-a.s. the following statement is true: there exists an event F such that,

$$P_\eta\Big(\limsup_{n\to\infty}\{D(w_d n^d) \not\subseteq B(n(1+K\epsilon^{1/d})\} \cap F\Big) = 0,$$

$$P_\eta(F) \geq 1-\epsilon. \qquad \square$$

3.5. *Proof of the shape theorem.* In this section we will prove Theorem 1 of Section 1. We will follow the notation there introduced. In the sequel, ν will denote an arbitrary probability measure on the space of obstacle configurations $\Gamma = \mathscr{I}^{\mathbb{Z}^d}$.

PROOF. Consider the coordinate process $Z = \{Z_n : n \in \mathbb{N}\}$ on Ω with a law given by $Q_{N,\nu}$. Note that for each $n \in \mathbb{N}$, the process Z_n represents a particle born at time T_n. Following [13], to analyze the set of saturated obstacles S_t [see definition (3)] at time t, it will be more convenient to consider a slightly different particle system but which generates the same probability measure $Q_{N,\nu}$ on the space of trajectories on the obstacle configurations. First, recall that $\zeta(x,t)$ represents the total number of particles at site x at time t [see definition (2) of Section 2]. In this modified model the dynamics of particles is governed by the following rule:

> Consider the depth $\eta(x)$ of an obstacle at site x and the number of particles $\zeta(x,t)$ at the same site and at time t. Let $i_1 < i_2 < \cdots < i_{\zeta(x,t)}$ be the indices of the particles present at site x at time t. Then the first $\eta(x)$ particles (i.e., those with indices $i_1,\dots,i_{\eta(x)}$) remain at x, while the rest (i.e., those with indices $i_{\eta(x)+1},\dots,i_{\zeta(x,t)}$) move like free random walks.

Let $\eta^n(t) := \{\eta^n(x,t) : x \in \mathbb{Z}^d\}$ ($\eta^\infty(t) := \{\eta^\infty(x,t) : x \in \mathbb{Z}^d\}$) be the obstacle configuration filled by the first n particles (all the particles) at time t,

$$\eta^n(x,t) := \Big(\eta(x) - \sum_{k=0}^{n} \delta_x(Z_k(t))\Big)_+,$$

$$\eta^\infty(x,t) := \Big(\eta(x) - \sum_{k=0}^{\infty} \delta_x(Z_k(t))\Big)_+.$$

Note that

$$\eta^n(x,t) \leq \eta^\infty(x,t) \qquad \text{for each } t \geq 0, n \geq 1,$$

where we have defined on Γ the following partial order: if $\eta, \zeta \in \Gamma$ then $\eta \leq \zeta$ if and only if $\eta(x) \leq \zeta(x)$ for every $x \in \mathbb{Z}^d$. On the other hand note that

$$\lim_{t\to\infty} \eta^n(t) = \eta_n, \tag{24}$$

where the convergence is in distribution and η_n is the discrete time stochastic process at time n defined in Section 1 with initial condition $\eta_0 = \eta$. In particular if we define

$$E_{n,t} := O(\eta_0) - O(\eta^n(t)),$$

the limit of equation (24) implies that

$$\lim_{t\to\infty} E_{n,t} = D_n, \tag{25}$$

where the convergence is in distribution and D_n is a set which coincides with the set of saturated obstacles for the discrete time model defined in Theorem 4. Note also that for every $t > 0$ and $n \in \mathbb{N}$,

$$E_{n,t} \subset S_t. \tag{26}$$

We are now ready to prove the upper and lower bounds of Theorem 1. Since the proofs of parts (i) and (ii) of Theorem 1 are very similar, we omit the proof of part (ii).

PROOF OF THE UPPER BOUND OF PART (i). Let $t_0 > 0$. Note that by the limit of equation (25), we have

$$\begin{aligned} &Q_{N,\mu}\Big(S_t \not\subseteq B_{(1+\epsilon)(N(t)/aw_d)^{1/d}} \text{ for } t \geq t_0\Big) \\ &\quad = Q_{N,\mu}\Big(E_{N(t),t} \not\subseteq B_{(1+\epsilon)(N(t)/aw_d)^{1/d}} \text{ for } t \geq t_0\Big) \\ &\quad \leq P_\mu\Big(D_{N(t)} \not\subseteq B_{(1+\epsilon)(N(t)/aw_d)^{1/d}} \text{ for } t \geq t_0\Big). \end{aligned}$$

Now, the upper bound of Theorem 4 implies that this last expression can be made arbitrarily small by choosing t_0 big enough.

PROOF OF THE LOWER BOUND OF PART (i). By the inclusion (26), we have

$$\Big\{B_{(1-\epsilon)(N(t)/aw_d)^{1/d}} \not\subseteq S_t\Big\} \subset \Big\{B_{(1-\epsilon)(N(t)/aw_d)^{1/d}} \not\subseteq E_{(1-\epsilon/2)N(t),t}\Big\}$$

Now, comparing $E_{(1-\epsilon/2)N(t),u}$ with $D_{(1-\epsilon/2)N(t)}$ we see that the right-hand member of the above inclusion is dominated by

$$\Big\{B_{(1-\epsilon)(N(t)/aw_d)^{1/d}} \not\subseteq D_{(1-\epsilon/2)N(t)}\Big\} \cup \Big\{E_{(1-\epsilon/2)N(t),t} \neq D_{(1-\epsilon/2)N(t)}\Big\}. \tag{27}$$

By the lower bound for the discrete time stochastic process proved in Theorem 4, the first term of the above expression fails eventually in t with $Q_{N,\mu}$ probability 1. For the second term, we define as S the event that one of the $(1-\epsilon/2)N(t)$ particles takes at least a time $(\epsilon/2d)t$ to exit from the set $B_{(1+\epsilon)\left(N(t)/aw_d\right)^{1/d}}$. Then one can dominate the second term of (27) by

$$\{S\} \cup \Big\{S_t \not\subseteq B_{(1+\epsilon)(N(t)/aw_d)^{1/d}}\Big\}. \tag{28}$$

In fact, if $S_t \subset B_{(1+\epsilon)(N(t)/aw_d)^{1/d}}$ and $E_{(1-\epsilon/2)N(t),t} \neq D_{(1-\epsilon/2)N(t)}$, then there must be at least one particle which was born before a time t', defined by the equation $N(t') = (1-\epsilon/2)N(t)$, which is active at time t, and which has not exited the ball $B_{t(1+\epsilon)}$ at time t. But since $N(t) \ll t^{d/2}$, it follows that $t' \leq (\epsilon/2d)t$. Now, the second term of expression (28) fails eventually in t with

$Q_O^{N,\mu}$ probability 1 on account of the previously proved upper bound. Finally, note that the probability of occurrence of the first term is bounded by

$$N(t)P_0\left(T_{B_{(1+\epsilon)(N(t)/aw_d)^{1/d}}} \geq \frac{\epsilon}{2d}t\right) \leq N(t)e^{-C\left(t/N(t)^{2/d}\right)},$$

where P_0 is the probability measure corresponding to a simple random walk starting from the origin, T_D is the exit time of this random walk from a set D and C is a constant depending only on ϵ. By the assumption $N(t) \ll (t^{d/2}/\ln t)$, the proof is now completed via an application of Borel–Cantelli. □

4. Asymptotic behavior of principal eigenvalues. This section is devoted to the proof of a Faber–Krahn-type theorem using a version of the enlargement of obstacle method of Sznitman [17]. More precisely, consider the principal Dirichlet eigenvalue of the discrete Laplacian operator on a subset of a box $[-[t],[t]]^d \cap \mathbb{Z}^d$ consisting of sites $x \in [-[t],[t]]^d \cap \mathbb{Z}^d$ such that $\eta(x) = 0$, where $\{\eta(x): x \in \mathbb{Z}^d\}$ are i.i.d. random variables with a common law μ such that $\mu(\eta(x) = 0) = 1 - p$, $0 < p < 1$. In essence, we will show that the optimal way (in the sense of minimizing the principal Dirichlet eigenvalue) of deleting more than $(\ln t)^{1/d}$ absorbing sites is asymptotically as t diverges, a sphere. In contrast, the deletion of less than $(\ln t)^{1/d}$ absorbing site gives the same asymptotics as no deletion at all. This result will be subsequently applied in the derivation of Theorems 2 and 3 of this paper about the logarithmic asymptotic behavior for large times of the survival probability of particles on the random saturation process.

4.1. *Notation and results.* Let h be a positive real number and consider some subset U of the rescaled lattice $h\mathbb{Z}^d$. Let $V(x)$: $U \to [0,\infty)$ be some positive potential. Define the operator $L_{U,V}$ corresponding to a simple random walk of total jump rate 1 on U with Dirichlet boundary conditions and killed at rate V, by its action on the space $C_0(U)$ of continuous functions from $h\mathbb{Z}^d$ to $\mathbb{R}$ with support on U,

$$L_{U,V}f(x) := \frac{1}{h^2}\frac{1}{2d}\sum_{e\in\mathbb{B}|e|=1}(f(x+he) - f(x)) + \frac{1}{h^2}V(x)f(x). \tag{29}$$

Here $f \in C_0(U)$, $x \in U \cap h\mathbb{Z}^d$ and $\mathbb{B}$ is the set of canonical basis elements on $\mathbb{R}^d$. Note that $L_{U,V}$ is a bounded operator on $C_0(U)$. Let $R_{h,t}^{U,V} := \exp(-tL_{U,V})$ be the corresponding contraction semigroup. Call $\lambda_V^h(U)$ the principal Dirichlet eigenvalue of the operator (29) properly extended to the corresponding Hilbert space. Now if the potential $V(x)$ does not vanish identically for $d = 1, 2$, we can define an equilibrium measure, Green function and capacity as follows. Given a finite set $K \subset U$ and a potential $V(x)$, we will denote by $e_{K,U,V}^h$ the unique equilibrium measure on K with respect to the operator $L_{U,V}$. We will denote by $g_{U,V}^h$ the corresponding Green function and by $\mathrm{cap}_{V,U,h}(K)$ the capacity of K given by $e_{K,U,V}^h(K)$.

For a given configuration of obstacle depth $\eta \in \mathcal{I}^{\mathbb{Z}^d}$, we will denote by $\mathcal{N}_n(\eta)$ the set of configurations obtained from η after deleting n obstacles. Thus, for every $\varsigma \in \mathcal{N}_n(\eta)$ we have $\sum_{x\in\mathbb{Z}^d}(\eta(x) - \varsigma(x)) = n$. Now consider the space $\Upsilon := \{0, 1\}^{\mathbb{Z}^d}$. This represents a space of site configurations on the lattice: sites in state 1 have an obstacle and are absorbing and those in state 0 are empty and nonabsorbing. Next, given $\xi \in \Upsilon$, call the subset of $\varepsilon\mathbb{Z}^d$ without obstacles $\mathcal{E}^\varepsilon(\xi) := \{x \in \varepsilon\mathbb{Z}^d\colon \xi_\varepsilon(x) = 0\}$. Also, for a given subset $A \subset \mathbb{R}^d$ and scale $\varepsilon > 0$ we define $A^\varepsilon := A \cap \varepsilon Z^d$. We can now, given an open subset U of $\mathbb{R}^d$, define the principal Dirichlet eigenvalue on the corresponding random perforated domain as $\lambda^\varepsilon_\xi(U^\varepsilon) := \lambda^\varepsilon_V(U^\varepsilon \cap \mathcal{E}^\varepsilon(\xi))$, where $V = 0$. We also define a mapping $\sigma\colon \mathcal{I}^{\mathbb{Z}^d} \to \Upsilon$ by $\sigma(\eta)(x) = 1$ if $\eta(x) \geq 1$ and $\sigma(\eta)(x) = 0$ if $\eta(x) = 0$. Finally, for given $\eta \in \mathcal{I}^{\mathbb{Z}^d}$, and open set $U \subset \mathbb{R}^d$ we adopt the convention $\lambda_\eta(U) := \lambda^1_{\sigma(\eta)}(U^1)$.

THEOREM 6. *On $\mathcal{I}^{\mathbb{Z}^d}$ consider a product measure μ such that $\mu(\eta(x) \geq 1) = p$, where $\eta \in \mathcal{I}^{\mathbb{Z}d}$ and $0 < p < 1$. Let $f(t)$: $[0 \to \infty) \to [0, \infty)$ be an increasing function such that $f(t) \ll t$, w_d be the volume of a ball on $\mathbb{R}^d$ of unit radius and λ_d the principal Dirichlet eigenvalue of the Laplacian operator on this ball times $1/2d$. Then if $a := \mu(\eta)$, the following statements are true*:

(i) *Suppose that $f(t) \ll (\ln t)^{1/d}$. Then*

$$\lim_{t\to\infty}(\ln t)^{2/d} \inf_{\varsigma\in\mathcal{N}_{aw_d f^d(t)}(\eta)} \lambda_\varsigma((-t, t)^d) = c(d, p) \quad \mu\text{-a.s.},$$

where $c(d, p) := \lambda_d(w_d|\ln(1-p)|)^{2/d}$.

(ii) *Suppose that $f(t) \gg (\ln t)^{1/d}$. Then*

$$\lim_{t\to\infty} f(t)^2 \inf_{\varsigma\in\mathcal{N}_{aw_d f^d(t)}(\eta)} \lambda_\varsigma((-t, t)^d) = \lambda_d \quad \mu\text{-a.s.}$$

(iii) *Suppose that $f(t) \gg (\ln t)^{1/d}$. Then for every function $g(t)$: $[0, \infty) \to [0, \infty)$, such that $(\ln t)^{1/d} \ll g(t) \ll f(t)$ and $\epsilon > 0$, there are constants C_1 and C_2 such that*

$$\mu\left(f(t)^2 \inf_{\varsigma\in\mathcal{N}_{aw_d f^d(t)}(\eta)} \lambda_\varsigma((-t, t)^d) \leq \lambda_d(1-\epsilon)\right) \leq C_1 e^{2d\ln(t/g(t))\vee 1 - C_2 g(t)^d\epsilon^2},$$

whenever $t \geq t_0(\epsilon, g/f)$ where $t_0(\epsilon, g/f)$ depends only on ϵ and the quotient g/f.

4.2. *Enlargement of obstacles on the lattice.* The object of this subsection is to define a version of the enlargement of obstacles technique which will be the main tool in the proof of Theorem 6 for the asymptotic behavior of the principal Dirichlet eigenvalue of the discrete Laplacian on $\mathbb{Z}^d$. Our construction is a translation to the simple random walk of Sznitman's second enlargement of obstacle technique (see [16], [17]) for Brownian motion. The two main results of this construction are two eigenvalue shift estimates (Theorems 7 and 8) and a volume estimate (Theorem 9). Their proof are a straightforward adaptation of Sznitman's version for Brownian motion and are presented in Appendix A.

Let us first introduce some notation and define some concepts that will be needed for the construction of the so-called density set, which is the main object of the enlargement of obstacle method. Given $\varepsilon > 0$, define $\Upsilon_\varepsilon := \{0,1\}^{\varepsilon\mathbb{Z}^d}$. We will call the obstacle configuration attached to $\xi \in \Upsilon$ and $\varepsilon > 0$ the state $\xi_\varepsilon \in \Upsilon_\varepsilon$ given by $\xi_\varepsilon(x) := \xi(x/\varepsilon)$ for $x \in \varepsilon\mathbb{Z}^d$.

The density set $\mathscr{D}_\varepsilon(\xi)$ will be defined in terms of a scale $\varepsilon > 0$ and the obstacle configuration attached to ξ. Furthermore, its definition will depend on two parameters $L \geq 2$ (related to an L-adic decomposition of $\mathbb{R}^d$) and $\delta \geq 0$ (related to a Wiener-type test defining the density set), and three scales defined through the functions $r_\beta, r_\gamma, r_\alpha$: $[0,\infty) \to [0,\infty)$ such that as $\varepsilon \to 0$, $\varepsilon \ll r_\beta(\varepsilon) \ll r_\gamma(\varepsilon) \ll r_\alpha(\varepsilon) \ll 1$.

To this end we define a succession of scales by the cubic lattices $\mathbb{Z}_k^d := (1/L^k)\mathbb{Z}^d$, for $k \geq 0$. They generate a partition of $\mathbb{R}^d$ into boxes $C_z^{(k)} := z + (1/L^k)[0,1)^d$, indexed by $z \in \mathbb{Z}_k^d$. Note that each $z \in \mathbb{Z}_k^d$ has an L-adic expansion as $z = i_0 + i_1/L + \cdots + i_k/L^k$ where $i_0, \dots, i_k \in \{0, \dots, L-1\}^d$. We define the truncation of $z \in \mathbb{Z}_k^d$ to scale $k' < k$ as $[z]_{k'} := i_0 + i_1/L + \cdots + i_{k'}/L^{k'}$. Given $j \in \{0, \dots, L-1\}^d$, we define an extension of $z \in \mathbb{Z}_k^d$ to $\mathbb{Z}_{k+1}^d$ by $z \cdot j := i_0 + i_1/L + \cdots + i_k/L^k + j/L^{k+1}$. Furthermore, we say that $z \succ z'$ whenever $z = i_0 + i_1/L + \cdots + i_k/L^k$ and $z' = i'_0 + i'_1/L + \cdots + i'_{k'}/L^{k'}$ with $k \geq k'$ and $i_j = i'_j$ for $1 \leq j \leq k'$.

Next, let us introduce the notation $n_\beta(\varepsilon) := [\ln(1/r_\beta(\varepsilon))/\ln(L)]$, $n_\gamma(\varepsilon) := [\ln(1/r_\gamma(\varepsilon))/\ln(L)]$ and $n_\alpha(\varepsilon) := [\ln(1/r_\alpha(\varepsilon))/\ln(L)]$. Note that $r_\beta(\varepsilon) \approx L^{-n_\beta(\varepsilon)}$, $r_\gamma(\varepsilon) \approx L^{-n_\gamma(\varepsilon)}$ and $r_\alpha(\varepsilon) \approx L^{-n_\alpha(\varepsilon)}$. Now fix $L \geq 2$ and $k \geq 0$. We then define for $z \in \mathbb{Z}_k^d$ the subset of sites of $(C_z^{(k)})^\varepsilon$ having obstacles and rescaled by L^k as

$$K_z^{(k)} := L^k \cdot \Big\{x \in (C_z^{(k)})^\varepsilon \colon \xi_\varepsilon(x) = 1\Big\}.$$

Note that $K_z^{(k)}$ is a subset of the rescaled cubic lattice $\varepsilon L^k \mathbb{Z}^d$.

We are now in a position to define the quantitative Wiener criterion. For a given obstacle configuration $\xi \in \Upsilon$, scale $\varepsilon > 0$, functions r_γ and r_α (defining n_γ and n_α, respectively) and parameters $L \geq 2$, $\delta > 0$ we will say that $z \in \mathbb{Z}_{n_\gamma(\varepsilon)}^d$ is a density index if

$$\sum_{n_\alpha(\varepsilon) \leq k \leq n_\gamma(\varepsilon)} \mathrm{cap}_{\varepsilon,[z]_k} \geq \delta(n_\gamma(\varepsilon) - n_\alpha(\varepsilon)). \tag{30}$$

Here $\mathrm{cap}_{\varepsilon,[z]_k} := \mathrm{cap}_{V,\varepsilon L^k\mathbb{Z}^d,\varepsilon L^k}(K_{[z]_k}^{(k)})$ and $V(x) = 0$ for $d \geq 3$, $V(x) = 1$ for $d = 1, 2$. If the criterion (30) does not hold, but the box $K_z^{(k)}$ is not empty, we will say that $z \in \mathbb{Z}_{n_\gamma(\varepsilon)}^d$ is a rarefaction index. At the smaller scale defined by $n_\beta(\varepsilon)$, we say that $z \in \mathbb{Z}_{n_\beta(\varepsilon)}^d$ is a bad index if the box $C_z^{(n_\beta)}$ contains at least one obstacle, and if $z \succ z'$, where z' is a rarefaction index. Furthermore, we define the density set $\mathscr{D}_\varepsilon(\xi)$ and the bad set $\mathscr{B}_\varepsilon(\xi)$, subsets of the rescaled lattice $\varepsilon\mathbb{Z}^d$, as

$$\mathscr{D}_\varepsilon(\xi) := \bigcup_{\substack{z\in\mathbb{Z}^d_{n_\gamma(\varepsilon)} \\ z \text{ density index}}} \left(C_z^{(n_\gamma)}\right)^\varepsilon,$$

$$\mathscr{B}_\varepsilon(\xi) := \bigcup_{\substack{z\in\mathbb{Z}^d_{n_\beta(\varepsilon)} \\ z \text{ bad index}}} \left(C_z^{(n_\beta)}\right)^\varepsilon.$$

We proceed now to state the three theorems that will be proved in this section. They are the cubic lattice translation of corresponding results by Sznitman on $\mathbb{R}^d$. For a given subset U of $\varepsilon\mathbb{Z}^d$ we denote by $|U|_\varepsilon := \sum_{x\in U}$ the number of sites within U. The first one says that for small $\varepsilon > 0$ the eigenvalue $\lambda_\xi^\varepsilon(U^\varepsilon)$ does not change too much if the density set $\mathscr{D}_\varepsilon(\xi)$ is erased from U^ε.

THEOREM 7. *There exists a constant $c(d) > 0$ such that for every function $h_\rho(\varepsilon)\colon [0,\infty) \to [0,\infty)$ that satisfies $h_\rho(\varepsilon) \gg ((d+2)/(c(d)\delta \ln L))(r_\gamma(\varepsilon)/r_\alpha(\varepsilon))$ and every $M > 0$ one has,*

$$\lim_{\varepsilon\to 0} \frac{1}{h_\rho(\varepsilon)} \sup_{\xi,\,U}\big(\lambda_\xi^\varepsilon(U^\varepsilon)\wedge M - \lambda_\xi^\varepsilon(U^\varepsilon\backslash\mathscr{D}_\varepsilon(\xi))\wedge M\big) = 0,$$

where the supremum is taken over $\xi \in \Upsilon$ and open subsets U of $\mathbb{R}^d$.

The next theorem also provides an eigenvalue control but at a larger scale. To state it we need to introduce the concepts of clearing and forest boxes. For $\varepsilon \in (0,1)$, $\xi \in \Upsilon$ and $r \in (0, 1/4)$ we define the clearing boxes as the boxes $C_z^{(0)}$, $z \in \mathbb{Z}^d$ such that

$$\varepsilon^d \left|\left(C_z^{(0)}\right)^\varepsilon \backslash \mathscr{D}_\varepsilon(\xi)\right|_\varepsilon \geq r^d.$$

Otherwise $C_z^{(0)}$ will be called a forest box. Now define the clearing set as

$$\mathscr{A}_\varepsilon(\xi) := \bigcup_{\substack{z\colon C_z^{(0)} \\ \text{clearing box}}} \left(C_z^{(0)}\right)^\varepsilon$$

and for a given function $R(\varepsilon)\colon (0,1) \to \mathbb{R}^d$, its neighborhood,

$$\mathscr{O}_{R,\,r}(\xi) := \{z \in \varepsilon\mathbb{Z}^d\colon \operatorname{dist}(z, \mathscr{A}_\varepsilon(\xi)) \leq R(\varepsilon)\},$$

where for $x \in \varepsilon\mathbb{Z}^d$ and $A \subset \varepsilon\mathbb{Z}^d$ we have defined $\operatorname{dist}(x, \mathscr{A}) := \inf_{y\in A}\{\sup_{1\leq i\leq d} |x_i - y_i|\}$.

THEOREM 8. *There exist constants* $c_3 \in (0, \infty)$, $c_4 \in (1, \infty)$, $r_0 \in (0, 1/4)$ *such that whenever* $R/4r > c_4$, $L^{-n_\alpha(\varepsilon)} < r < r_0$ *and* $M > 0$, *one has*

$$\lim_{\varepsilon \to 0} \frac{1}{h_2(\epsilon)} \sup_{\xi, U} \big(\lambda_\xi^\varepsilon(U^\varepsilon) \wedge M - \lambda_\xi^\varepsilon(U^\varepsilon \cap \mathscr{O}_{R,r}) \wedge M\big) \le 1,$$

where $h_2(\varepsilon) := e^{-c_3[R/4r]}$ *and the supremum is taken over* $\xi \in \Upsilon$ *and open subsets* $U \subset \mathbb{R}^d$.

The last theorem gives a control on the volume of the bad set $\mathscr{B}_\varepsilon(\xi)$. We first need to define the following constants:

$$c_8(d, L) = \begin{cases} \frac{L^2}{3^d+1}, & \text{when } d \ge 3, \\ \frac{L^2}{10c \ln L}, & \text{when } d = 2, \end{cases}$$

$$\delta_0 = \begin{cases} \frac{3}{8L^d G(1/2L)}, & \text{when } d \ge 2, \\ \frac{1}{2}\mathrm{Cap}\{0\}, & \text{when } d = 1, \end{cases}$$

where c is some constant, $G(|x - y|) := g(x, y)$, $g(x, y)$ is the Green function of the continuous Laplacian divided by $2d$ on $\mathbb{R}^d$ and with potential $V = 1$ for $d = 2$ and $V = 0$ for $d \ge 3$ and $\mathrm{Cap}\{0\}$ is the capacity with respect to one-half the continuous Laplacian on $\mathbb{R}$ of the point 0.

THEOREM 9. *Assume that* L *is large enough so that* $c_8(d, L) > 1$ *and* $\delta < \delta_0(d, L)$. *Then the following statements are true*:

(i) *If* $d = 1$, *then for any* $\xi \in \Upsilon$, *the set* $\mathscr{B}_\varepsilon(\xi)$ *is empty*.

(ii) *If* $d \ge 2$, *then*

$$\limsup_{\varepsilon \to 0} \frac{1}{h_\kappa(\epsilon)} \sup_{z \in \mathbb{Z}^d,\, \xi \in \Upsilon} \varepsilon^d \Big| \Big(C_z^{(0)}\Big)^\varepsilon \cap \mathscr{B}_\varepsilon(\xi) \Big|_\varepsilon < \infty,$$

where

$$h_\kappa(\epsilon) := \left(\frac{r_\gamma}{r_\alpha}\right)^{(2 - \ln(3^d+1)/\ln L)(1 - \delta/\delta_0) 1/\ln L} \left(\frac{r_\beta}{\varepsilon}\right)^{d-2}.$$

For the applications that will follow, we introduce the term an *admissible collection of parameters* to denote a collection of functions $r_\alpha, r_\beta, r_\gamma, R, r, h_\rho \colon [0, \infty) \to [0, \infty)$ and parameters L, δ such that

$$\varepsilon \ll r_\beta \ll r_\gamma \ll r_\alpha \ll 1,$$

$$L \ge 2 \text{ with } L \text{ such that } c_8(d, L) > 1, \text{ when } d \ge 2,$$

$$\delta > 0, \text{ with } \delta < \delta_0(d, L),$$

$$h_\rho(\varepsilon) \gg \frac{d+2}{c(d)\delta \ln L} \frac{r_\gamma(\varepsilon)}{r_\alpha(\varepsilon)},$$

$$\left(\frac{r_\gamma}{r_\alpha}\right)^{(2-\ln(3^d+1)/\ln L)(1-\delta/\delta_0)(1/\ln L)}\left(\frac{r_\beta}{\varepsilon}\right)^{d-2} \ll 1,$$

$$r_\alpha < r < r_0 \text{ and } r \ll R.$$

Finally we would like to point out that the fact that the scale of the bad set, given by r_β, can be chosen much smaller than the scale of the density set, given by r_γ, will turn out to be essential to state Theorem 6 with sharp hypothesis ($f(t) \ll \ln t$ or $\ln t \ll f(t)$). This is a particular feature of Sznitman's second construction of the enlargement of obstacle technique, in contrast to the first one [15].

4.3. *Proof of asymptotic estimates.* We now proceed to prove Theorem 6. First, we need to introduce some more notation. For $\varepsilon > 0$ and $\eta \in \Gamma$ (recall that $\Gamma := \mathscr{I}^{\mathbb{Z}^d}$), we define $\eta_\varepsilon \in \mathscr{I}^{\varepsilon\mathbb{Z}^d}$ by $\eta_\varepsilon(x) := \eta(x/\varepsilon)$, where $x \in \varepsilon\mathbb{Z}^d$. For η_1 we drop the subscript and write η and with a slight abuse of notation we will write $\mathscr{D}_\varepsilon(\eta) := \mathscr{D}_\varepsilon(\sigma(\eta))$ and $\mathscr{B}_\varepsilon(\eta) := \mathscr{B}_\varepsilon(\sigma(\eta))$. Also, if $U \in \varepsilon\mathbb{Z}^d$ we define $\|U\|_\eta$, the total obstacle depth in U of the configuration η_ε, by

$$\|U\|_\eta = \sum_{x\in U} \eta_\varepsilon(x).$$

Now, for a given subset $K \subset \varepsilon\mathbb{Z}^d$ we define its boundary $\delta K := \{z \in K^c\colon |z-y| = 1 \text{ for some } y \in K\}$ and its closure $\overline{K} := K \cup \delta K$. Finally, we define for $r > 0$, the ball $B_r := \{x \in \mathbb{R}^d\colon |x| < r\}$. Now, for the proof of Theorem 6 the following three lemmas will be needed. The first one is an a adaptation to the cubic lattice of Lemma 4.4.4 of Sznitman [17] (see also Lemma 2.1 of [18]).

LEMMA 12. *Let $\varepsilon := (\ln t)^{-1/d}$ and $\eta \in \Gamma$. Consider an admissible collection of parameters $r_\gamma, r_\alpha, r_\beta, R, r, h_\rho, \delta, L$, defining the density set $\mathscr{D}_\varepsilon(\eta)$, the bad set $\mathscr{B}_\varepsilon(\eta)$ and clearing set $\mathscr{A}_{R,r}(\eta)$ of U^ε, where $U := [-t\varepsilon, t\varepsilon]^d$. Assume that $1 \ll R(\varepsilon) \ll r_\beta(\varepsilon)(\ln t)^{1/d}$, and that $\eta \in \Gamma$ is distributed according to some probability measure μ such that $\mu(\eta(x) \geq 1) = p$, with $p \in (0,1)$. Let $\mathscr{C}_t$ be the collection of blocks $z + (0, [R(\varepsilon)])^d$, $z \in \mathbb{Z}^d$, which intersect the set U^ε. Let $G := \{\sup_{B\in\mathscr{C}_t} \varepsilon^d |B\backslash(\mathscr{D}_\varepsilon(\eta)\cup\mathscr{B}_\varepsilon(\eta))|_\varepsilon \leq \frac{d}{|\ln(1-p)|} + \frac{\varepsilon^d}{g(\varepsilon)^d}\}$ where $\varepsilon \ll g(\varepsilon) \ll \frac{r_\beta(\varepsilon)}{R(\varepsilon)}$. Then for ε small enough*,

$$\mu(G^c) \leq e^{-(|\ln(1-p)|/2)(1/g(\varepsilon)^d)}$$

The next lemma is along the lines of Lemma 2.6 of [1] and is a Faber–Krahn-type inequality for the discrete Laplacian on the cubic lattice.

LEMMA 13. *Let $K \subset \varepsilon\mathbb{Z}^d$. Then*

$$\lambda^\varepsilon(K) \geq \lambda_d\left(\frac{w_d}{\varepsilon^d|\overline{K}|_\varepsilon}\right)^{2/d}\frac{1}{1 + C_d\varepsilon^2\lambda^\varepsilon(K)}, \tag{31}$$

where $C_d := 3d^2 2^{d-1}$.

The final lemma, which will be applied only for parts (ii) and (iii) of Theorem 6, gives an estimate of the obstacle density in the cubic lattice at scales larger than $\ln t$ and distributed according to some product measure.

LEMMA 14. *On Γ consider a product measure μ such that $\mu(\eta(x)) = a$, with $0 < a < \infty$. Let $g(t)$: $[0, \infty) \to [0, \infty)$ be an increasing function such that $g(t) \gg (\ln t)^{1/d}$. Let $\{D_z := z + [0, g(t))^d,\ z \in ([-1, 1]^d)^{(g/t)}\}$ denote the partition of disjoint semiopen boxes of the interval $[-t, t]^d$. Then there are constants C_1 and C_2 such that for every $\epsilon > 0$ one has*

$$\mu\left(\bigcup_{z\in([-1,1]^d)^{(g/t)}}\left\{\frac{\|D_z \cap \mathbb{Z}^d\|_\eta}{g(t)^d} \le a(1-\epsilon)\right\}\right) \le C_1 e^{2d\ln(t/g(t))\vee 1 - C_2 g(t)^d \epsilon^2}$$

PROOF OF PART (i) OF THEOREM 6. The upper bound,

$$\limsup_{t\to\infty}(\ln t)^{2/d} \inf_{\varsigma\in\mathcal{N}_{aw_d f^d(t)}(\eta)} \lambda_\varsigma((-t, t)^d) \ge c(d, p) \quad \mu\text{-a.s.}$$

follows from the inequality $\inf_{\varsigma\in\mathcal{N}_{aw_d f^d(t)}(\eta)} \lambda_\varsigma((-t, t)^d) \le \lambda_\eta((-t, t)^d)$ and [2].

We next prove the lower bound,

$$\liminf_{t\to\infty}(\ln t)^{2/d} \inf_{\varsigma\in\mathcal{N}_{aw_d f^d(t)}(\eta)} \lambda_\varsigma((-t, t)^d) \ge c(d, p) \quad \mu\text{-a.s.}$$

For this, we set $\varepsilon := (\ln t)^{-1/d}$ and consider an admissible collection of parameters $r_\gamma, r_\alpha, r_\beta, R, r, h_\rho, \delta, L$, defining the density set $\mathscr{D}_\varepsilon$, the bad set $\mathscr{B}_\varepsilon$ and clearing set $\mathscr{A}_{R,r}$ of U^ε, where $U := [-t\varepsilon, t\varepsilon]^d$. We choose r_γ in such a way that

$$r_\gamma(\varepsilon) \ll \frac{1}{f(t)} \tag{32}$$

and $R(\varepsilon)$ so that

$$1 \ll R(\varepsilon) \ll \min\left\{\frac{1}{h_\kappa^{1/d}(\varepsilon)}, r_\beta(\varepsilon)(\ln t)^{1/d}\right\}. \tag{33}$$

We also let r constant. Now let $\epsilon > 0$ and consider the event,

$$\mathscr{V}_{t,\epsilon} := \left\{\eta \in \Upsilon\colon (\ln t)^{2/d} \inf_{\varsigma\in\mathcal{N}_{aw_d f^d(t)}(\eta)} \lambda_\varsigma((-t, t)^d) \le \frac{c(d, p)}{(1+\epsilon)^{2/d}}\right\}.$$

By the eigenvalue estimates of Theorems 7 and 8 of the enlargement of obstacle subsection, for t big enough, $V_{t,\epsilon}$ is contained in

$$\mathscr{W}^1_{t,\varepsilon} := \left\{\eta \in \Upsilon\colon \inf_{\varsigma\in\mathcal{N}_{aw_d f^d(t)}(\eta)} \lambda^\varepsilon(\mathscr{O}_{R(\varepsilon),r} \backslash \mathscr{D}_\varepsilon(\varsigma)) \le \frac{c(d, p)}{(1+\epsilon)^{2/d}} + h_\rho(\varepsilon) + h_2(\varepsilon)\right\},$$

where $h_2(\varepsilon) := e^{-c_3[R/r]}$. At this point let $\mathscr{G}_t(\varsigma)$ be the collection of connected components of the set $\mathscr{O}_{R(\varepsilon),r} \backslash \mathscr{D}_\varepsilon(\varsigma)$. Note that

$$\mathscr{W}^1_{t,\epsilon} = \left\{ \eta \in \Upsilon : \inf_{\varsigma \in \mathscr{N}_{aw_d f^d(t)}(\eta)} \inf_{B \in \mathscr{G}_t(\varsigma)} \lambda^\varepsilon(B) \le \frac{c(d,p)}{(1+\epsilon)^{2/d}} + h_\rho(\varepsilon) + h_2(\varepsilon) \right\}.$$

Therefore, by Lemma 13 we can conclude that for t big enough, the set $\mathscr{W}^1_{t,\varepsilon}$ is contained in

$$\mathscr{W}^2_{t,\epsilon} := \left\{ \eta \in \Upsilon : \sup_{\varsigma \in \mathscr{N}_{aw_d f^d(t)}(\eta)} \sup_{B \in \mathscr{G}_t(\varsigma)} \varepsilon^d |\overline{B}|_\varepsilon \ge \frac{d}{|\ln(1-p)|} \frac{1+\epsilon}{(1+o(\varepsilon))} \right\}.$$

Now, each set $B \in \mathscr{G}_t(\varsigma)$ is the union of boxes of side $r_\gamma(\varepsilon)$ and hence $|\delta B|_\varepsilon \le d!(\varepsilon/r_\gamma(\varepsilon))|B|_\varepsilon$. Therefore the set $\mathscr{W}^2_{t,\varepsilon}$ is contained in

$$\mathscr{W}^3_{t,\epsilon} := \left\{ \eta \in \Upsilon : \sup_{\varsigma \in \mathscr{N}_{aw_d f^d(t)}(\eta)} \sup_{B \in \mathscr{G}_t(\varsigma)} \varepsilon^d |B|_\varepsilon \ge \frac{d}{|\ln(1-p)|} \frac{1+\epsilon}{(1+o(\varepsilon))} \right\}.$$

Consider now the collection $\mathscr{C}_{c,t}$ of blocks $z + ([0, cR(\varepsilon))^d)^\varepsilon$, $z \in \mathbb{Z}^d$, which intersect $([-t/(\ln t)^{1/d}, t/(\ln t)^{1/d}]^d)^\varepsilon$. Note that for $c \ge a/r^d$ it is true that the set $\mathscr{W}^3_{t,\epsilon}$ is contained in

$$\mathscr{W}^4_{t,\epsilon} := \left\{ \eta \in \Upsilon : \sup_{\varsigma \in \mathscr{N}_{aw_d f^d(t)}(\eta)} \sup_{B \in \mathscr{C}_{c,t}} \varepsilon^d |C \backslash \mathscr{D}_\varepsilon(\varsigma)|_\varepsilon \ge \frac{d}{|\ln(1-p)|} \frac{1+\epsilon}{(1+o(\varepsilon))} \right\}.$$

However, by the volume estimate, Theorem 9 of the previous section, it is true that

$$|C \backslash \mathscr{D}_\varepsilon|_\varepsilon \le |C \backslash (\mathscr{D}_\varepsilon \cup \mathscr{B}_\varepsilon)|_\varepsilon + R(\varepsilon)^d \varepsilon^{-d} h_\kappa(\varepsilon),$$

whenever t is big enough. Now, by our choice (33) of $R(\varepsilon)$ we have $\lim_{t\to\infty} R(\varepsilon)^d h_\kappa(\varepsilon) = 0$. Therefore, the set $\mathscr{W}^4_{t,\epsilon}$ is contained in

$$\mathscr{W}^5_{t,\epsilon} := \left\{ \eta \in \Upsilon : \sup_{\varsigma \in \mathscr{N}_{aw_d f^d(t)}(\eta)} \sup_{B \in \mathscr{C}_{c,t}} \varepsilon^d |C \backslash (\mathscr{D}_\varepsilon \cup \mathscr{B}_\varepsilon)(\varsigma)|_\varepsilon \right.$$
$$\left. \ge \frac{d}{|\ln(1-p)|} \frac{1+\epsilon}{(1+o(\varepsilon))} \right\}.$$

We can now conclude that for t big enough one has that $\mathscr{W}^5_{t,\epsilon}$ is contained in

$$\mathscr{W}^6_{t,\epsilon} := \left\{ \eta \in \Upsilon : \sup_{\varsigma \in \mathscr{N}_{aw_d f^d(t)}(\eta)} \sup_{B \in \mathscr{C}_{c,t}} \varepsilon^d |C \backslash (\mathscr{D}_\varepsilon \cup \mathscr{B}_\varepsilon)(\varsigma)|_\varepsilon \right.$$
$$\left. \ge \frac{d}{|\ln(1-p)|} (1+\epsilon/2) \right\}.$$

Each density and bad box has at least one obstacle. Therefore for every $\varsigma \in \mathscr{N}_{aw_d f^d(t)}(\eta)$ and $C \in \mathscr{C}_{c,t}$ one has

$$\varepsilon^d(|C\backslash(\mathscr{D}_\varepsilon \cup \mathscr{B}_\varepsilon)(\varsigma)|_\varepsilon - |C\backslash(\mathscr{D}_\varepsilon \cup \mathscr{B}_\varepsilon)(\eta)|_\varepsilon)\frac{1}{r_\beta(\varepsilon)^d}\bar{a} \leq aw_d f^d(t)$$

Now, by the choice (32), we have $\lim_{t\to\infty} f(t)r_\beta(\varepsilon) = 0$. We can finally conclude that for t big enough the event $\mathscr{V}_{t,\epsilon}$ is contained in

$$\mathscr{W}^7_{t,\epsilon} := \left\{\eta \in \Upsilon : \sup_{B\in\mathscr{C}_{c,t}} \varepsilon^d |C\backslash(\mathscr{D}_\varepsilon \cup \mathscr{B}_\varepsilon)(\eta)|_\varepsilon \geq \frac{d}{|\ln(1-p)|}(1+\epsilon/4)\right\}.$$

By Lemma 12 this implies that there is a function $g(\varepsilon)$: $[0,\infty) \to [0,\infty)$ such that $R(\varepsilon)(1/r_\beta(\varepsilon)) \ll \frac{1}{g(\varepsilon)} \ll (\ln t)^{1/d}$ and such that for t big enough,

$$\mu(\mathscr{V}_{t,\epsilon}) \leq e^{-(|\ln(1-p)|/2)(1/g(\varepsilon)^d)}.$$

Choosing $g(\varepsilon)$ close enough to $(\ln t)^{-1/d}$, setting $t = 2^n$, $n \geq 1$, in the above inequality, using Borel–Cantelli and using the fact that $\inf_{\varsigma\in\mathscr{N}_{aw_d f^d(t)}(\eta)} \lambda_\varsigma((-t,t)^d)$ is decreasing in t, the proof is complete. □

PROOF OF PART (ii) OF THEOREM 6. In the sequel of this proof we set $\varepsilon := 1/f(t)$. We begin by showing the following upper bound:

$$\limsup_{t\to\infty} f(t)^2 \inf_{\varsigma\in\mathscr{N}_{aw_d f^d(t)}(\eta)} \lambda_\varsigma((-t,t)^d) \leq \lambda_d \quad \mu\text{-a.s.} \tag{34}$$

Note that for this it is enough to prove that for every $\epsilon > 0$, μ-a.s. there is a sequence of configurations $\{\varsigma_t \in \mathscr{N}_{aw_d f^d(t)}(\eta)\}_{t\geq 0}$ such that

$$\limsup_{t\to\infty} f(t)^2 \lambda_{\varsigma_t}((-t,t)^d) \leq \frac{\lambda_d}{(1-\epsilon)^2}. \tag{35}$$

Now, to find such a sequence, since $f(t) \gg (\ln t)^{1/d}$, note that for every $\epsilon > 0$ there exist constants C_1 and C_2 such that

$$\mu\left(\frac{\|(B_{f(t)(1-\epsilon)})^1\|_\eta}{w_d f(t)^d(1-\epsilon)^d} \geq \frac{a}{(1-\epsilon)^d}\right) \leq C_1 e^{-C_2 f(t)^d \epsilon^2}.$$

It follows that for every $\epsilon > 0$, μ-a.s. there is a $t_0 > 0$ such that

$$\|(B_{f(t)(1-\varepsilon)})^1\|_\eta \leq aw_d f(t)^d \quad \text{for } t \geq t_0.$$

This implies that for every $\epsilon > 0$, μ-a.s. there is a $t_0 > 0$ and a sequence of configurations $\{\varsigma_t \in \mathscr{N}_{aw_d f(t)^d}\}_{t\geq 0}$ such that when $t > t_0$ there are no obstacles inside a ball of radius $f(t)(1-\epsilon)$ for the configuration ς_t. Hence,

$$\limsup_{t\to\infty} f(t)^2 \lambda_{\varsigma_t}((-t,t)^d) \leq \limsup_{\varepsilon\to 0} \lambda^\varepsilon((B_{(1-\epsilon)})^\varepsilon).$$

By Lemma 2.4 of [1], inequality (35) is proved.

We now proceed to prove the following lower bound:

$$\liminf_{t\to\infty} f(t)^2 \inf_{\varsigma\in\mathcal{N}_{aw_d f^d(t)}(\eta)} \lambda_\varsigma((-t,t)^d) \geq \lambda_d \quad \mu\text{-a.s.} \tag{36}$$

Call $\mathscr{F}$ the set of configurations where inequality (36) is not satisfied: if $\eta \in \mathscr{F}$ then $\liminf_{t\to\infty} f(t)^2 \inf_{\varsigma\in\mathcal{N}_{aw_d f(t)^d}(\eta)} \lambda_\varsigma((-t,t)^d) < \lambda_d$. We will show that $\mu(\mathscr{F}) = 0$.

Note that if $\eta \in \mathscr{F}$ there are sequences $\{t_n > 0\}_{n\geq 1}$ and $\{\varsigma_{t_n} \in \mathcal{N}_{aw_d f(t_n)^d}(\eta)\}_{n\geq 1}$ such that

$$\limsup_{n\to\infty} f(t_n)^2 \lambda_{\varsigma_{t_n}}\big((-t_n,t_n)^d\big) < \lambda_d. \tag{37}$$

At this point we consider an admissible collection of parameters $r_\alpha, r_\gamma, r_\beta$, R, r, δ, L and h_ρ, h_κ defining the density set $\mathscr{D}_\varepsilon$, bad set $\mathscr{B}_\varepsilon$ and clearing set $\mathscr{A}_{R,r}$ of U^ε, where $U := [-t\varepsilon, t\varepsilon]^d$. Since we are assuming that $f(t) \gg (\ln t)^{1/d}$, note that we can always choose the scale of the density set in such a way that $f(t) r_\gamma(\varepsilon) \gg (\ln t)^{1/d}$. We choose $R(\varepsilon)$, r and $h_\kappa(\varepsilon)$ so that $R(\varepsilon) \gg 1$, $R^d(\varepsilon) h_\kappa(\varepsilon) \ll 1$ and r is constant. We also define $\varepsilon_n := 1/f(t_n)$. Now, by the eigenvalue estimates (Theorems 7 and 8) of the enlargement of obstacle method of Section 4.2, it follows that

$$\limsup_{n\to\infty} \lambda^{\varepsilon_n}(\mathscr{O}_{R(\varepsilon_n),r} \backslash \mathscr{D}_{\varepsilon_n}(\varsigma_{t_n})) < \lambda_d. \tag{38}$$

Now note that the set $(\mathscr{O}_{R(\varepsilon_n),r} \backslash \mathscr{D}\varepsilon_n(\varsigma_{t_n}))$ is a union of boxes of side $r_\gamma(\varepsilon)$. Each one of them has $d! r_\gamma(\varepsilon)^{d-1} \varepsilon^{-(d-1)}$ boundary points. Thus, $|\delta(\mathscr{O}_{R(\varepsilon_n),r} \backslash \mathscr{D}\varepsilon_n (\varsigma_{t_n}))|_\varepsilon \leq d!(\varepsilon/r_\gamma(\varepsilon))|(\mathscr{O}_{R(\varepsilon_n),r} \backslash \mathscr{D}_{\varepsilon_n}(\varsigma_{t_n}))_\varepsilon$. Then, an application of Lemma 13 and of inequality (38) implies that if $\eta \in \mathscr{F}$ there are sequences $\{t_n > 0\}_{n\geq 1}$ and $\{\varsigma_{t_n} \in \mathcal{N}_{aw_d f(t_n)^d}(\eta)\}_{n\geq 1}$ such that

$$\liminf_{n\to\infty} \varepsilon_n^d |\mathscr{O}_{R,r} \backslash D_{\varepsilon_n}(\varsigma_{t_n})|_{\varepsilon_n} > w_d. \tag{39}$$

Now, since we have made a choice of r_γ such that $f(t) r_\gamma(\varepsilon) \gg (\ln t)^{1/d}$, there exists a function $g(t)\colon [0,\infty) \to [0,\infty)$ such that $(\ln t)^{1/d} \ll g(t) \ll f(t) r_\gamma(\varepsilon)$. By Lemma 14, the condition $g(t) \gg (\ln t)^{1/d}$ implies that for every $\epsilon > 0$ one has that

$$\liminf_{t\to\infty} \inf_{z\in([-1,1])^{(g/t)}} \frac{\|D_z \cap \mathbb{Z}^d\|_\eta}{g(t)^d} \geq (1-\epsilon) \quad \mu\text{-a.s.}, \tag{40}$$

where $D_z := z + [0, g(t))^d$. Now, the condition $g(t) \ll f(t) r_\gamma(\varepsilon)$, ensures that each density box of the set $\mathscr{O}_{R,r}$, having a size of order $r_\gamma(\varepsilon)$, is equal to the union of boxes of size g/f [scale at which we measure the obstacle density of $\varepsilon\mathbb{Z}^d$ in (40)]. Therefore, for each obstacle depth configuration η, the set $|\mathscr{O}_{R,r} \backslash \mathscr{D}_\varepsilon|$ has $(f/g)^d (1/f^d) |\mathscr{O}_{R,r} \backslash \mathscr{D}_\varepsilon|_\varepsilon$ boxes of side g/f. Thus, for every $\epsilon > 0$ we get that

$$\liminf_{t\to\infty} \inf_{\varsigma\in\mathcal{N}_{aw_d f(t)^d}(\eta)} \frac{\|\mathscr{O}_{R,r} \backslash \mathscr{D}_\varepsilon(\varsigma)\|_n}{|\mathscr{O}_{R,r} \backslash \mathscr{D}_\varepsilon(\varsigma)|_\varepsilon} \geq a(1-\epsilon) \quad \mu\text{-a.s.} \tag{41}$$

Call $\mathscr{G}_\epsilon$ the set where the above inequality is satisfied. Since $\mu(\mathscr{F}\cap\mathscr{G}_\epsilon^c)=0$, it is enough to show that for some $\epsilon>0$ one has $\mu(\mathscr{F}\cap\mathscr{G}_\epsilon)=0$. So let $\eta\in\mathscr{F}\cap\mathscr{G}_\epsilon$, where ϵ will be chosen later. Then,

$$
\begin{aligned}
&\limsup_{t\to\infty}\sup_{\varsigma\in\mathscr{N}_{aw_d f(t)^d}(\eta)}\frac{\|\mathscr{O}_{R,r}\backslash D_\varepsilon(\varsigma)\|_\varsigma}{|\mathscr{O}_{R,r}\backslash\mathscr{D}_\varepsilon(\varsigma)|_\varepsilon}\\
&\quad=\lim_{t\to\infty}\sup_{\varsigma\in\mathscr{N}_{aw_d f(t)^d}(\eta)}\frac{\sum_{z\in\mathbb{Z}^d:\ (C_z^{(0)})^\varepsilon\subset\mathscr{O}_{R,r}}\|(C_z^{(0)})^\varepsilon\backslash\mathscr{D}_\varepsilon(\varsigma)\|_\varsigma}{|\mathscr{O}_{R,r}\backslash\mathscr{D}_\varepsilon(\varsigma)|_\varepsilon}\\
&\quad\le\lim_{t\to\infty}\sup_{\varsigma\in\mathscr{N}_{aw_d f(t)^d}(\eta)}\frac{\varepsilon^d|\mathscr{A}_{R,r}|_\varepsilon w_d R(\varepsilon)^d\sup_{z\in\mathbb{Z}^d}\|(C_z^{(0)})^\varepsilon\backslash\mathscr{D}_\varepsilon(\varsigma)\|_\varsigma}{|\mathscr{O}_{R,r}\backslash\mathscr{D}_\varepsilon(\varsigma)|_\varepsilon} \qquad (42)\\
&\quad\le\lim_{t\to\infty}\sup_{\varsigma\in\mathscr{N}_{aw_d f(t)^d}(\eta)}\varepsilon^d\frac{w_d}{r^d}\frac{|\mathscr{O}_{R,r}\backslash\mathscr{D}_\varepsilon(\varsigma)|_\varepsilon R(\varepsilon)^d\sup_{z\in\mathbb{Z}^d}\|(C_z^{(0)})^\varepsilon\backslash\mathscr{D}_\varepsilon(\varsigma)\|_\varsigma}{|\mathscr{O}_{R,r}\backslash\mathscr{D}_\varepsilon(\varsigma)|_\varepsilon}\\
&\quad\le\lim_{t\to\infty}\frac{w_d}{r^d}\varepsilon^d R(\varepsilon)^d\sup_{\eta\in\mathscr{I}^{\mathbb{Z}^d}\,z\in\mathbb{Z}^d}\left\|(C_z^{(0)})^\varepsilon\backslash\mathscr{D}_\varepsilon(\varsigma)\right\|_\varsigma\\
&\quad=0,
\end{aligned}
$$

where in the last inequality we have made use of Theorem 9 which enables us to control the number of obstacles in the rarefaction set. Combining inequality (41) with the limit in (42) it follows that whenever $\eta\in\mathscr{F}\cap\mathscr{G}_\epsilon$, one has

$$
\liminf_{t\to\infty}\inf_{\varsigma\in\mathscr{N}_{aw_d f(t)^d}(\eta)}\frac{\|\mathscr{O}_{R,r}\backslash\mathscr{D}_\varepsilon(\varsigma)\|_\eta-\|\mathscr{O}_{r,r}\backslash\mathscr{D}_\varepsilon(\varsigma)\|_\varsigma}{|\mathscr{O}_{R,r}\backslash\mathscr{D}_\varepsilon(\varsigma)|_\varepsilon}\ge a(1-\epsilon).
$$

By the volume estimate (39), it now follows that for each $\eta\in\mathscr{F}\cap\mathscr{G}_\epsilon$ there are sequences $\{t_n>0\}_{n\ge1}$ and $\{\varsigma_{t_n}\in\mathscr{N}_{aw_d f(t_n)^d}(\eta)\}_{n\ge1}$ such that

$$
\liminf_{n\to\infty}\varepsilon_n^d\big(\|\mathscr{O}_{R(\varepsilon_n),r}\backslash\mathscr{D}_{\varepsilon_n}(\varsigma_{t_n})\|_\eta-\|\mathscr{O}_{R(\varepsilon_n),r}\backslash\mathscr{D}_{\varepsilon_n}(\varsigma_{t_n})\|_{\varsigma_{t_n}}\big)\ge av(1-\epsilon), \qquad (43)
$$

where $v:=\limsup_{n\to\infty}\varepsilon_n^d|\mathscr{O}_{R(\varepsilon_n),r}\backslash\mathscr{D}_{\varepsilon_n}(\varsigma_{t_n})|_{\varepsilon_n}>w_d$. But by definition it is true that

$$
\limsup_{n\to\infty}\varepsilon_n^d\big(\|\mathscr{O}_{R(\varepsilon_n),r}\backslash\mathscr{D}_{\varepsilon_n}(\varsigma_{t_n})\|_\eta-\|\mathscr{O}_{R(\varepsilon_n),r}\backslash\mathscr{D}_{\varepsilon_n}(\varsigma_{t_n})\|_{\varsigma_{t_n}}\big)\le aw_d. \qquad (44)
$$

Inequalities (43) and (44) are incompatible whenever ϵ is chosen in such a way that $v(1-\epsilon)>w_d$. Hence, whenever $0<\epsilon<(v-w_d)/v$, the set $\mathscr{F}\cap\mathscr{G}_\epsilon$ is empty. This concludes the proof of the lower bound (36) and of part (ii) of Theorem 6. □

PROOF OF PART (iii) OF THEOREM 6. Let $\varepsilon:=1/f(t)$. We choose an admissible collection of parameters $r_\gamma, r_\alpha, r_\beta, r, R, \delta, L$ and h_p, h_k defining the density set $\mathscr{D}_\varepsilon$, the bad set B_ε and clearing set $\mathscr{A}_{R,r}$ of U^ε, where $U:=[-t\varepsilon, t\varepsilon]^d$.

Note that we can always choose r_γ in such a way that $r_\gamma(\varepsilon) \gg g(t)/f(t)$. Let $\epsilon > 0$ and $t > 0$ and call $\mathscr{H}_\epsilon$ the event that

$$f(t)^2 \inf_{\varsigma \in \mathscr{N}_{aw_d f(t)^d}(\eta)} \lambda_\varsigma((-t,t)^d) \le \lambda_d(1-\epsilon).$$

We will show that under our assumption on f, there is a $t_0 > 0$ such that $\mathscr{H}_\epsilon \subset \mathscr{E}_\epsilon$ whenever $t \ge t_0$, where $\mathscr{E}_\epsilon$ is the event that

$$\bigcup_{z \in ([-1,1]^d)^{(g(t)/t)}} \left\{ \frac{\|D_z \cap \mathbb{Z}^d\|_\eta}{g(t)} < a(1-\epsilon/3) \right\}.$$

Lemma 14 enables us to estimate the probability of $\mathscr{E}_\epsilon$, so that we can conclude that there are constants C_1 and C_2 such that

$$\mu(\mathscr{E}_\epsilon) \le C_1 e^{2d(t/g(t))\vee 1 - C_2 g(t)^d \epsilon^2}$$

whenever $t \ge t_0$. This proves part (iii) of Theorem 6.

We proceed to prove our claim that there is a $t_0 > 0$ such that $\mathscr{H}_\epsilon \subset \mathscr{E}_\epsilon$ for $t \ge t_0$. So fix $\epsilon > 0$, and let $\eta \in \mathscr{H}_\epsilon$. First, note that by the eigenvalue estimates, Theorems 7 and 8 Section 4.2, there is a $t_1 > 0$ such that

$$\inf_{\varsigma \in \mathscr{N}_{aw_d f^d(t)}(\eta)} \lambda^\varepsilon(\mathscr{O}_{R,r} \backslash \mathscr{D}_\varepsilon(\varsigma)) \le \lambda_d(1-\epsilon) + h_p(\varepsilon),$$

whenever $t \ge t_1$. Therefore, by the volume estimate of Lemma 13, there is a $t_2 \ge t_1$ such that

$$\sup_{\varsigma \in \mathscr{N}_{aw_d f(t)^d}(\eta)} \varepsilon^d |\overline{\mathscr{O}_{R,r} \backslash \mathscr{D}_\epsilon(\varsigma)}|_\varepsilon \ge \frac{w_d(1-C\varepsilon^2)}{((1-\epsilon)+h_p(\varepsilon))^{d/2}},$$

whenever $t \ge t_2$, where $C = 2C_d\lambda_d$. However, $|\delta(\mathscr{O}_{R,r} \backslash \mathscr{D}_\varepsilon(\varsigma)|_\varepsilon \le d!(\varepsilon/r_\gamma(\varepsilon))$ $|\mathscr{O}_{R,r} \backslash \mathscr{D}_\varepsilon(\varsigma)|_\varepsilon$. It follows that there is a $t_3 \ge t_2$ such that

$$\sup_{\varsigma \in \mathscr{N}_{aw_d f(t)^d}(\eta)} \varepsilon^d |\mathscr{O}_{R,r} \backslash \mathscr{D}_\varepsilon(\varsigma)|_\varepsilon \ge \frac{w_d}{(1-\epsilon/2)}, \tag{45}$$

whenever $t \ge t_3$.

Now assume that $\eta \notin \mathscr{E}_\epsilon$. Then, using the fact that boxes of side $g(t)/f(t)$ are much smaller than density boxes [having side of length $r_\gamma(\varepsilon)$], we see that for every $\varsigma \in N_{aw_d f(t)^d}(\eta)$,

$$\|\mathscr{O}_{R,r} \backslash \mathscr{D}_\varepsilon(\varsigma)\|_\eta \ge \left(\frac{f(t)}{g(t)}\right)^d \left(\frac{1}{f(t)}\right)^d |\mathscr{O}_{R,r} \backslash \mathscr{D}_\varepsilon(\varsigma)| g(t)^d a(1-\epsilon/3).$$

This together with the obstacle density estimate of Theorem 9 of Section 4.2 implies that there is a $\varsigma \in \mathscr{N}_{aw_d f(t)^d}(\eta)$, and a $t_4 \ge t_3$ such that

$$\begin{aligned} &\|\mathscr{O}_{R,r} \backslash \mathscr{D}_\varepsilon(\varsigma)\|_\eta - \|\mathscr{O}_{R,r} \backslash \mathscr{D}_\varepsilon(\varsigma)\|_\varsigma \\ &\qquad \ge |\mathscr{O}_{R,r} \backslash \mathscr{D}_\varepsilon(\varsigma)|_\varepsilon a(1-\epsilon/3) - h_k(\varepsilon) f(t)^d \end{aligned} \tag{46}$$

whenever $t \geq t_4$. Therefore, from inequalities (45) and (46) we conclude that if $\eta \in \mathscr{H}_\epsilon$ and $\eta \notin \mathscr{E}_\epsilon$, there is a $t_0 > 0$ and a $\varsigma \in \mathscr{N}_{aw_d f(t)^d}(\eta)$ such that

$$\|\mathscr{O}_{R,r}\backslash\mathscr{D}_\varepsilon(\varsigma)\|_\eta - \|\mathscr{O}_{R,r}\backslash\mathscr{D}_\varepsilon(\varsigma)\|_\varsigma > f(t)^d a w_d,$$

whenever $t \geq t_0$. This contradicts the fact that $\|\mathscr{O}_{R,r}\backslash\mathscr{D}_\varepsilon(\varsigma)\|_\eta - \|\mathscr{O}_{R,r}\backslash\mathscr{D}_\varepsilon(\varsigma)\|_\varsigma \leq f(t)^d a w_d$. □

PROOF OF LEMMA 12. First note that for every $B \in \mathscr{C}_t$,

$$\mu\left(\varepsilon^d|B\backslash(\mathscr{D}_\varepsilon \cup \mathscr{B}_\varepsilon)|_\varepsilon \geq \frac{d}{|\ln(1-p)|} + \varepsilon^{d(1-\beta)}\right) \leq 2^{2R(\varepsilon)^d r_\gamma(\varepsilon)^{-d}} e^{-d\varepsilon^{-d} - g(\varepsilon)^{-d}|\ln(1-p)|}$$

Therefore,

$$\begin{aligned}\mu(G^c) &\leq (2t\varepsilon)^d \mu\left(\varepsilon^d|B\backslash(\mathscr{D}_\varepsilon \cup \mathscr{B}_\varepsilon)|_\varepsilon \geq \frac{d}{|\ln(1-p)|} + \varepsilon^{d(1-\beta)}\right)\\ &\leq 2^d \varepsilon^d 2^{2R(\varepsilon)^d r_\gamma(\varepsilon)^{-d}} e^{-g(\varepsilon)^{-d}|\ln(1-p)|}\\ &\leq e^{-\frac{|\ln(1-p)|}{2}\frac{1}{g(\varepsilon)^d}},\end{aligned}$$

where the last inequality is true for t big enough. □

PROOF OF LEMMA 13. Consider the space $L^2(\varepsilon\mathbb{Z}^d, \varepsilon^d|\cdot|_\varepsilon)$. Given two functions $f_1, f_2 \in L^2(\varepsilon\mathbb{Z}^d, \varepsilon^d|\cdot|_\varepsilon)$, we will denote by $(f_1, f_2)_\varepsilon := \varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d} f_1(x) f_2(x)$ their inner product. Note that $(f_1, L_K f_2)_\varepsilon = \varepsilon^{d-2}\frac{1}{2}\sum_{x\in\varepsilon\mathbb{Z}^d}\sum_{e\in\mathbb{B}:|e|=1} (f(x+\varepsilon e) - f(x))^2$, whenever $f_1, f_2 \in C_0(K)$. First note that there is a function $f \in C_0(K)$ such that $\lambda^\varepsilon(K) = (f, L_K f)_\varepsilon/(f,f)_\varepsilon$. To prove inequality (31) we will show that there exists a function $g \in H_0^1(\mathbb{R}^d)$ such that:

1. $(f, L_K f)_\varepsilon = \int_{\mathbb{R}^d} |\nabla_g|^2\,dx$.
2. $|(f,f)_\varepsilon - \int_{\mathbb{R}^d} g^2\,dx| \leq \varepsilon^2 C(f, L_K f)_\varepsilon$.
3. $m(\{x\colon g(x) > 0\}) \leq \varepsilon^d|\overline{K}|_\varepsilon$, where m is Lebesgue measure.

This will be enough. In fact,

$$\begin{aligned}\frac{(f, L_K f)_\varepsilon}{(f,f)_\varepsilon} &\geq \frac{\int |\nabla_g|^2\,dx}{\int g^2\,dx + \varepsilon^2 C(f, L_K f)_\varepsilon}\\ &\geq \frac{\int |\nabla_g|^2\,dx}{\int g^2\,dx}\frac{1}{1+\varepsilon^2 C\lambda^\varepsilon(K)}\\ &\geq \lambda_d\left(\frac{w_d}{\varepsilon^d|\overline{K}|_\varepsilon}\right)^{2/d}\frac{1}{1+\varepsilon^2 C\lambda^\varepsilon(K)},\end{aligned}$$

where in the last inequality we have used Faber–Krahn for the continuous Laplacian and property (3) of g.

We now proceed to construct the function g. We will denote by $x_i,\ 1 \le i \le d$, the canonical coordinates of a point $x \in \mathbb{R}^d$. Consider the following hyperplanes in $\mathbb{R}^d$:

$$x_i = \varepsilon n, \tag{47}$$

$$x_i = x_j + \varepsilon m, \tag{48}$$

where n and m integers and $1 \le i \ne j \le d$. We will denote by $\mathbb{H}_i$ the hyperplane defined by (47) and by $\mathbb{H}_{i,j}$ the one defined by (48). They define a partition of the space $\mathbb{R}^d$. Also, note that the points of intersection of the hyperplanes defined by (47) are the cubic lattice $\varepsilon\mathbb{Z}^d$. The fundamental region of this lattice (which is bounded by the hyperplanes $x_i = 0$ and $x_i = \varepsilon$, for $1 \le i \le d$), is partitioned in $d!$ close subregions having an intersection of zero Lebesgue measure by the hyperplanes $x_i = x_j,\ i \ne j$. The boundary of each one of them is formed by two hyperplanes H_i and H_j of type (47), with $i \ne j$, and $d-1$ hyperplanes $H_{i,i_1}, H_{i_1,i_2}, \dots, H_{i_{d-2},i_{d-1}}$ of type (48), with $i_k \ne i_{k'}$ and $i_k, i_{k'} \ne i, j$. We will call $R_k,\ 1 \le k \le d!$, the subregions thereby formed. Also, denote by $v_1^k, \dots, v_{d+1}^k$ the $d+1$ points of the cubic lattice $\varepsilon\mathbb{Z}^d$ contained in the boundary of R_k. We now define g on the fundamental region as the unique continuous function which is linear on each subregion R_k, and such that $g(v_i^k) = f(v_i^k),\ 1 \le k \le d!$ and $1 \le i \le d+1$. Similarly we define g on $\mathbb{R}^d$.

Henceforth, we proceed to check the three properties of g stated above. Properties 1 and 3, stating that $(f, L_K f)_\varepsilon = \int |\nabla g|^2\, dx$ and that the Lebesgue measure of the support of g is smaller than or equal to $\varepsilon^d |\overline{K}|_\varepsilon$, respectively, follow directly from the definition. To verify property 2 note that

$$\int_{R_k} g^2\, dx = \varepsilon^d \frac{2}{(d+2)!}\left(\sum_{i=1}^{d+1} f^2(v_i^k) + \sum_{i \ne j} f(v_i^k) f(v_j^k)\right).$$

Now fix $i' \in (1, 2, \dots, d+1)$. A straightforward calculation shows that

$$\begin{aligned}
&\left|\frac{2}{(d+2)!}\left(\sum_{i=1}^{d+1} f^2(v_i^k) + \sum_{i\ne j} f(v_i^k) f(v_j^k)\right) - \frac{1}{(d+1)!}\sum_{i=1}^{d+1} f^2(v_i^k)\right| \\
&\quad = \left|\frac{1}{(d+1)!}\left(\sum_{j\ne i'}\sum_{l \ne i', j} (f(v_{i'}^k) - f(v_j^k))(f(v_{i'}^k) - f(v_l^k)) \right.\right.\\
&\qquad \left.\left. - d\sum_{j \ne i'} (f(v_{i'}^k) - f(v_j^k))^2\right)\right| \\
&\quad \le \frac{1}{(d+1)!}(2d-1)\sum_{j\ne i'} (f(v_{i'}^k) - f(v_j^k))^2 \\
&\quad \le \frac{3d^2}{(d+1)!}\sum_{e \in \varepsilon\mathbb{E}_0:\ e=(x,y)} (f(x) - f(y))^2,
\end{aligned}$$

where $\varepsilon\mathbb{E}_0$ is the set of edges of the fundamental region. But since there are $d!$ subregions for each integer translate of the fundamental cube and each edge is contained in 2^{d-1} cubes, it is not difficult to deduce that

$$\left| \int g^2\, dx - (f, f)_\varepsilon \right| \le \varepsilon^d \frac{3d^2 2^{d-1}}{d+1} \sum_{e \in \varepsilon\mathbb{E}:\ e=(x,y)} (f(x) - f(y))^2,$$

where $\varepsilon\mathbb{E}$ is the set of edges of the cubic lattice $\varepsilon\mathbb{Z}^d$. This finishes the proof of the lemma. □

PROOF OF LEMMA 14. It is elementary to check that there are constants B_1 and B_2 such that $\mu(\|D_z \cap \mathbb{Z}^d\|_\eta / g(t)^d \le a(1-\epsilon)) \le B_1 e^{-B_2 \epsilon^2 g^d(t)}$. Therefore,

$$\mu\left(\bigcup_{z \in ([-1,1]^d)^{g/t}} \left\{ \frac{\|D_z \cap \mathbb{Z}^d\|_\eta}{g(t)^d} > a(1-\epsilon) \right\} \right) \ge \left(1 - B_1 e^{-B_2 g(t)^d \epsilon^2}\right)^{([t/g(t)])^d}$$

$$\ge 1 - \sum_{k=1}^{(1/2)([t/g(t)])^d+1} \left(\left[\frac{t}{g(t)} \right] \right)^{(2k-1)d} \left(B_1 e^{-B_2 g(t)^d \epsilon^2} \right)^{(2k-1)}$$

$$\ge 1 - \left(\left[\frac{t}{g(t)} \right] \right)^{2d} B_1 e^{-B_2 g(t)^d \epsilon^2}.$$

The lemma now follows easily. □

5. Asymptotic behavior of the survival probability in the random saturation process. In this section we will prove Theorems 2 and 3 describing the asymptotic behavior of the survival probability of the $k(t)$th born particle, where $k(t)\colon [0,\infty) \to \mathbb{N}$ is an increasing function, in the random saturation process.

First, we introduce some notation that will be used. Let us recall that, according to the notation defined in Section 2, given a random saturation process on an obstacle configuration η and driven by an injection N, we denote by τ_k, the time at which the random walk Z_k is frozen. Also, recall that T_k denotes the birth time of Z_k, $g(t) := T_{k(t)}$, $\zeta(x,t)$ the total number of random walks at site x and time t [see definition (2) of Section 2] and S_t the set of saturated obstacles at time t [definition (3)]. We now define by S_t^k the set of sites having "saturated obstacles" produced by the set of random walks $\{Z_j\colon j \ne k\}$ at time t,

$$S_t^k := \left\{ x \in \mathbb{Z}^d : \sum_{n \in \mathbb{N};\, h \ne k} 1_{Z_n(t)}(x) \ge \eta(x) > 0 \right\}.$$

Next, given $\varepsilon > 0$ and $u > 0$, we define,

$$F_u := \{ Z \in \Omega\colon B_{(1-\epsilon)f(t)} \subset S_t \text{ for } t \ge u \},$$

$$F_u^k := \{ Z \in \Omega\colon B_{(1-\epsilon)f(t)} \subset S_t^k \text{ for } t \ge u \}.$$

Now define

$$N'(t) = \begin{cases} N(t), & \text{if } t < T_{k(t)}, \\ N(t) - 1, & \text{if } t \ge T_{k(t)}. \end{cases}$$

This represents an injection obtained from N after "erasing" the random walk $Z_{k(t)}$. The corresponding probability measure will be denoted $Q_{N',\nu}$.

Also, denote by $P_{x,t}$ the probability distribution on of a simple random walk X, of total jump rate 1, born at site x at time t. Now, the killing time $\tau_{k(t)}$ of the $k(t)$th tagged particle has a distribution under $Q_{N,\nu}$ which is equal to the distribution of the first exit time, of a random walk distributed according to $P_{0,g(t)}$, from the set $\{x \in \mathbb{Z}^d\colon \zeta(x,t) > 0\}$ distributed according to $Q_{N',\nu}$. We will call this exit time τ_ζ.

5.1. *Proof of Theorem* 2. We will prove (i) and (ii) separately.

PROOF OF PART (i). In the sequel we assume that $1 \ll N(t) \ll \min\{(t-g(t))^{d/2}, t^{(d/2)-\epsilon}\}$ for some $\epsilon > 0$.

Part 1. Here we will prove the lower bound of the limit (4) under the additional assumption that either $\ln(t-g(t)) \ll N(t)$ or $p > 1-p_c(d)$. Let $u > 0$. For the lower bound, first note that

$$\begin{aligned} Q_{N,\eta}(\tau_{k(t)} > t) &\geq E_{Q_{N,\eta}}(I_{\tau\zeta > t} F_u^k) \\ &= E_{Q_{N',\eta}}(P_{0,g(t)}(\tau_\zeta > t) F_u) \\ &= E_{Q_{N',\eta}}(P_{0,g(t)}(\tau_\zeta > t) | F_u) Q_{N',n}(F_u). \end{aligned}$$

Now, by part (ii) of Theorem 1, we can conclude that μ-a.s. there is a u_0 such that

$$Q_{N,\eta}(\tau_{k(t)} > t) \geq \frac{1}{2} E_{Q_{N',n}}(P_{0,g(t)}(\tau_\zeta > t) | F_{u_0}), \tag{49}$$

where for $A \subset \Omega$, I_A is defined as the indicator function of A. Now let σ be the first time that the random walk leaves the origin. Then, by the strong Markov property,

$$\begin{aligned} &E_{Q_{N',n}}(P_{0,g(t)}(\tau_\zeta > t) | F_{u_0}) \\ &\quad\geq E_{0,g(t)}\left(\sigma \geq u_0, P_{0,\sigma}\left(\sup_{\sigma \leq s \leq t-g(t)} \frac{|X_s|}{(1-\epsilon)f(s)} \leq 1\right)\right) \\ &\quad\geq P_{0,0}(\sigma \geq (u_0 - g(t))_+) P_{0,u_0}\left(\sup_{u_0 \leq s \leq t-g(t)} \frac{|X_s|}{(1-\epsilon)f(s)} \leq 1\right), \end{aligned} \tag{50}$$

whenever $t \geq u_0$. Clearly, the first term of the right-hand side of the last inequality in (50) can be bounded from below by some strictly positive constant. The second one represents the probability that the exit time from a time-dependent ball of radius $(1-\epsilon)f(s)$, of a random walk born at time u_0, is smaller than or equal to $t-g(t)$. An application of Lemma 20 of Appendix B together with inequality (49) enables us to conclude that,

$$\liminf_{t\to\infty} \frac{1}{\lambda_d \int_{g(t)}^t (aw_d/N(s))^{2/d}\, ds} \ln Q_{N,\eta}(\tau_{k(t)} > t) \geq -1 \quad \mu\text{-a.s.}$$

Part 2. Here we prove the upper bound of the limit in (4) under the assumption that $\ln(t-g(t)) \ll N(t)$. First we define the following auxiliary function:

$$h(t) := \frac{t-g(t)}{\ln\ln(t-g(t)/N(t)^{2/d})}.$$

Now, let $T := \inf\{s \geq 0\colon |X_s| \geq h(t)\}$ be the first exit time of a random walk X_s from the interval $(-h(t), h(t))^d$. Then,

$$E_{Q_{N',\eta}}(P_{0,g(t)}(\tau_\zeta > t)) = E_{Q_{N',\eta}}(P_{0,g(t)}(T_\zeta > t)) + (P_{0,g(t)}(T \leq t)), \tag{51}$$

where $T_\zeta := T \wedge \tau_\zeta$. Note that the second term of equality (51) satisfies $\limsup_{t\to\infty}(t-g(t)/h(t)^2)\ln P_{0,g(t)}(T \leq t) < 0$ (see, e.g., Theorem 3.7.1 in [4], describing the moderate deviations of sums of i.i.d. random variables). Now, the assumption $N(t) \gg \ln(t-g(t))$ implies that $\int_{g(t)}^t (ds/N(s)^{2/d}) \ll (t-g(t)/\ln(t-g(t))^{2/d})$. Therefore, since $(h(t)^2/t-g(t)) \gg (t-g(t)/\ln(t-g(t))^{2/d})$ it is enough to prove that

$$\limsup_{t\to\infty} \frac{1}{\lambda_d \int_{g(t)}^t (aw_d/N(s))^{2/d}\,ds} \ln E_{Q_{N',\eta}}\big(P_{0,g(t)}(T_\zeta > t)\big) \leq -1. \tag{52}$$

Let $n := [(t-g(t) \vee h(t))/h(t)]$. Also define $E_{x,s}$, for $x \in \mathbb{Z}^d$, $s \geq 0$, as the expectation with respect to the random walk probability measure $P_{x,s}$, $T_{\zeta_s} := \inf\{u \geq 0\colon X_u \geq h(t) \text{ or } \eta(X_u) > \zeta_s(X_u)\}$ the exit time from the set $\{x \in \mathbb{Z}^d \cap (-h(t), h(t))^d\colon \eta(x) \leq \zeta_s(x)\}$ representing sites without active obstacles at time s inside the interval $(-h(t), h(t))^d$, and $Y_k := X_{kh(t)}$. Then by the Markov property we have

$$\begin{aligned} P_{0,g(t)}(T_\zeta > t) &\leq P_{0,g(t)\vee h(t)}(T_\zeta > t) \\ &\leq E_{0,0}(T_{\zeta_{g\vee h+h}} > h, E_{Y_1,0}(T_{\zeta_t} > h)) \\ &\leq E(T_{\zeta_{g\vee h+h}} > h, E(T_{\zeta_{g\vee h+2h}} > h, \dots, \\ &\qquad E(T_{\zeta_{g\vee h+nh}} > h, E(T_{\zeta_t} > h)\cdots), \end{aligned} \tag{53}$$

where in the last inequality we have used induction and have dropped the subindices of the expectations. Then, adopting the notation of Section 4 and using Lemma 15 of Appendix A, we see that the expectations inside the right-hand side of inequality (53) can be bounded as follows:

$$\begin{aligned} &E_{Y_{k-1},0}(T_{\zeta_{g\vee h+kh}} > h) \\ &\quad \leq c(d)((\lambda_{\varsigma_{g\vee h+kh}}((-h,h)^d)h)^{d/2}+1)e^{-h\lambda_{\varsigma_{g\vee h+kh}}((-h,h)^d)}. \end{aligned}$$

Now, define $I_k := (-kh(t), kh(t))^d$. Note that for $k \geq 1$, $\lambda_{\varsigma_{g\vee h+kh}}((-h(t), h(t))^d) \geq \lambda_{\varsigma_{g\vee h+kh}}(I_k) \geq \inf_{\varsigma\in\mathcal{N}_{aw_d f(g\vee h+kh)^d}(\eta)} \lambda_\varsigma(I_k)$. On the other hand, since we are assuming that $N(t) \gg (\ln(t-g(t))^{d+1}$ and that $g(t)$ is increasing, it follows that $f(g\vee h+kh) \gg (\ln(kh+g-g(kh+g)))^{(d+1)/d} \geq (\ln(kh))^{(d+1)/d}$. We can now apply part (ii) of Theorem 6 (Section 4), describing the almost sure asymptotics of the principal Dirichlet eigenvalue after a large number of obstacles is deleted, to conclude that μ-a.s. for every $\epsilon > 0$ there is a $t_0 > 0$ such that

$$\sup_{1\le k\le n-1}\left|\lambda_d - f(g(t)\vee h(t)+kh(t))^2 \inf_{\varsigma\in\mathcal{N}_{aw_d f(g\vee h+kh)^{d}(\eta)}} \lambda_\varsigma(I_k)\right| \le \epsilon \quad \text{for } t\ge t_0. \tag{54}$$

It follows that μ-a.s. for every $\epsilon\in(0,1]$ and $1\le k\le n$, there is a $t_0>0$ such that

$$\lambda_{\varsigma_{g\vee h+kh}}((-h(t),h(t))^d) \ge \frac{\lambda_d-\epsilon}{f(t_k)^2} \quad \text{for } t\ge t_0, \tag{55}$$

where we have defined $t_k := g\vee h + kh$ for $1\le k\le n-1$ and $f_n := f(t)$. But the assumption $N(t)\ll (t-g(t))^{d/2}$, ensures that $h(t)\gg f(t)^2\ge f(t_k)^2$. This, inequality (55) and the fact that the function $(x^{d/2}+1)e^{-x}$ is decreasing for $x\ge d/2$ now implies that μ-a.s., for every $\epsilon>0$ and $1\le k\le n$ there is a $t_0\ge 0$ such that

$$E_{Y_{k-1},0}(T_{\zeta_{g+kh}} > h) \le c(d)\lambda_d \frac{h(t)}{f(t_k)^2} e^{-(h/f(t_k)^2)(\lambda_d-\epsilon)}, \tag{56}$$

whenever $t\ge t_0$. Thus, a recursive substitution of this estimate on inequality (53) enables us to conclude that μ-a.s. for every $\epsilon>0$ there is a $t_0>0$ such that

$$E_{Q_{N',\eta}}(P_{0,g(t)}(T_\zeta > t)) \le c_2(d)^n \prod_{k=1}^n \frac{h(t)}{f(t_k)^2} e^{-(h/f(t_k)^2)(\lambda_d-\epsilon)}, \tag{57}$$

whenever $t\ge t_0$, where $c_2(d) := c(d)\lambda_d$. Therefore, μ-a.s. for every $\epsilon>0$ there is a $t_0>0$ such that

$$\begin{aligned}
&\frac{1}{\int_g^t (ds/f(s)^2)} \ln E_{Q_{N',\eta}}(P_{0,g(t)}(T_\zeta > t))\\
&\quad\le \frac{1}{\int_g^t (ds/f(s)^2)}\Biggl(\frac{(t-g(t)\vee h(t))}{h(t)}\ln(c_2(d))\\
&\qquad + \sum_{k=1}^n \left(\ln\left(\frac{h(t)}{f(t_k)^2}\right) - \frac{h(t)}{f(t_k)^2}(\lambda_d-\epsilon)\right)\Biggr)\\
&\quad\le \frac{1}{\int_g^t (ds/f(s)^2)} \ln\ln\left(\frac{t-g(t)}{N(t)^{2/d}}\right)\ln(c_2(d))\\
&\qquad + \frac{1}{\int_g^t (ds/f(s)^2)}\sum_{k=1}^n\left(\ln\left(\frac{h(t)}{f(t_k)^2}\right) - \frac{h}{f(t_k)^2}(\lambda_d-\epsilon)\right),
\end{aligned} \tag{58}$$

whenever $t\ge t_0$. Now, since $\int_g^t (ds/N(s)^{2/d}) \ge ((t-g(t))/N(t)^{2/d})$ and $N(t)^{2/d}\ll t-g(t)$ the first term of the left-hand side of the last inequality vanishes as $t \to \infty$. Thus,

$$\begin{aligned}(59)\quad &\limsup_{t\to\infty} \frac{1}{\lambda_d \int_{g(t)}^t (ds/f(s)^2)} \ln E_{Q_{N',\eta}}\big(P_{0,\,g(t)}(T_\zeta > t)\big)\\ &\leq \limsup_{t\to\infty} \frac{1}{\lambda_d \int_{g(t)}^t (ds/f(s)^2)} \sum_{k=1}^n \left(\ln\left(\frac{h(t)}{f(t_k)^2}\right) - \frac{h}{f(t_k)^2}(\lambda_d - \epsilon)\right).\end{aligned}$$

However, because $h(t)/f(t_k)^2 \gg 1$, the first term inside the summand of the left-hand side of inequality (59) is negligible with respect to the second one, so that we can drop it to conclude that

$$\begin{aligned}(60)\quad &\limsup_{t\to\infty} \frac{1}{\lambda_d \int_{g(t)}^t (ds/f(s)^2)} \ln E_{Q_{N',\eta}}\big(P_{0,\,g(t)}(T_\zeta > t)\big)\\ &\leq -\frac{\lambda_d - \varepsilon}{\lambda_d} \limsup_{t\to\infty} \frac{1}{\int_{g(t)}^t (ds/f(s)^2)} \sum_{k=1}^n \frac{h}{f(t_k)^2}.\end{aligned}$$

By Lemma 22 of Appendix B the lim sup of the right-hand side is 1. Letting epsilon to 0 gives (52).

Part 3. Here we will prove the upper bound of the limit of (4) under the assumption that $p > 1 - p_c(d)$. By Part 2, without loss of generality we can also suppose that $N(t) \ll (\ln(t - g(t)))^2$.

First, let

$$h(t) := \frac{t - g(t)}{\ln\ln(t - g(t))}.$$

Then define $T := \inf\{s \geq h(t)\colon |X_s| > \exp(\sqrt{N(s)})\}$ as the first exit time bigger than $h(t)$ of the random walk X_s from the time-dependent set $[\exp(\sqrt{N(s)}), \exp(\sqrt{N(s)})]^d$. We now claim that by Lemma 23 of Appendix B, and the assumption $p > 1 - p_c(d)$ we know that μ-a.s. there is a t_0 such that

$$E_{Q_O^{N',\eta}}\big(P_{0,\,g(t)}(\tau_\zeta > t)\big) = E_{Q_O^{N',\eta}}\big(P_{0,\,g(t)}(T_\zeta > t)\big),$$

whenever $t \geq t_0$, where $T_\zeta := T \wedge \tau_\zeta$. In fact, by Lemma 23, μ-a.s. there is an n_0 such that whenever $n \geq n_0$, the largest connected component of the set free of obstacles within a box $[-n, n]^d$, has $\ln n$ sites. Since $1 \ll N(t)$, this means that μ-a.s. there is a t_0 such that whenever $t \geq t_0$, every connected component of the set free of obstacles within the box $\big[-\exp\big(\sqrt{N(t)}\big), \exp\big(\sqrt{N(t)}\big)\big]^d$ has a diameter smaller than $\sqrt{N(t)}$. Thus, if the random walk X_s exits such a box at time t, at least $\exp\big(\sqrt{N(t)}\big)/\sqrt{N(t)}$ random walks must be absorbed. Since the number of particles born at time t is $N(t)$, this is a contradiction when t is large enough.

We can now, by analogy to the steps of Part 2 leading to inequality (57), conclude that μ-a.s. for every $\epsilon > 0$ there is a $t_0 > 0$ such that

$$E_{Q_{N',\,\eta}}(P_{0,\,g(t)}(T_\zeta > t)) \leq c_2(d)^n \prod_{k=1}^n \frac{h(t)}{f(t_k)^2} e^{-(h/f(t_k)^2)(\lambda_d - \epsilon)}$$

whenever $t \geq t_0$, where $c_2(d) := c(d)\lambda_d$. Since the rest of the reasoning is the same as the one leading to the conclusion of Part 2 beginning from inequality (57), we omit it.

PROOF OF PART (ii). First note that without loss of generality we can assume that the origin belongs to the unique infinite trap free cluster which occurs with probability 1 because $p < 1 - p_c(d)$. In fact, by part (i) of Theorem 1, μ-a.s. with $Q_O^{N,\eta}$ probability greater than 1/2 there is a time t_0 such that for $t \geq t_0$ the origin is contained in the infinite trap free cluster. Define $M(t) := 1$. Now consider the probability measure $Q_{M,\eta}$. This represents the law of a single random walk on a time-independent random environment of obstacles given by η. Note that $Q_{N,\eta}(\tau_k(t) > t) \geq Q_{M,\eta}(\tau_0 > t - g(t))$. For $p < 1 - p_c$ call $\mathscr{C}^\infty$ the μ-a.s. a unique infinite trap free cluster. At this point we can apply a result of [2] to obtain the following lower bound: if $d \geq 2$, $p < 1 - p_c$ and $\varepsilon > 0$ we have μ-a.s. on the set $\{0 \in \mathscr{C}^\infty\}$ that

$$\liminf_{t\to\infty} \frac{\ln(t - g(t))^{2/d}}{c_q(p,d)(t - g(t))} \ln Q_{M,\eta}(\tau_0 > t - g(t)) \geq -1,$$

where $c_q(p,d) = \lambda_d\big((w_d|\ln(1-p))|/(2/d)\big)^d$. This proves the lower bound of part (ii).

For the upper bound, by analogy to the proof of part (i), we obtain the following inequality:

$$Q_{N,\eta}\big(\tau_{k(t)} > t\big) \leq E_{Q_{N',\eta}}\big(P_{0,g(t)}(T_\zeta > t)\big) + P_{0,g(t)}(T \leq t),$$

where this time $T := \inf\{s \geq 0 : |X_s| \geq t - g(t)\}$ and $T_\zeta := T \wedge \tau_\zeta$. Now, the second term of the right-hand side of the above inequality satisfies

$$\limsup_{t\to\infty} \frac{1}{t - g(t)} \ln P_{0,g(t)}(T \leq t) < 0.$$

Therefore it is enough to show that μ-a.s.,

$$\limsup_{t\to\infty} \frac{\ln(t - g(t))^{2/d}}{c_q(p,d)(t - g(t))} \ln E_{Q_{N',\eta}}\big(P_{0,g(t)}(T_\zeta > t)\big) \leq -1. \tag{61}$$

Now,

$$\begin{aligned} P_{0,g(t)}(T_\zeta > t) &\leq P_{0,g(t)}(T_{\zeta_t} > t) \\ &\leq c(d)\big((\lambda_{\varsigma t}(I_t)(t-g))^{d/2} + 1\big)e^{-(t-g)\lambda_{\varsigma_t}(I_t)}, \end{aligned} \tag{62}$$

where $I_t := (-(t - g(t)), t - g(t))^d$ and where in the last inequality we have made use of Lemma 15 of Appendix A. An application of part (i) of Theorem 6, describing the almost sure asymptotics of the principal Dirichlet eigenvalue for a small number of obstacles deleted, to inequality (62) enables us to conclude our claim (61).

5.2. *Proof of Theorem* 3. We will prove (i) and (ii) separately.

PROOF OF PART (i). Note that the case $p = 1$ is included in the statement of part (i) of the quenched Theorem 2. Therefore, in the sequel, we consider the case $(t - g(t))^{d/(d+2)} \ll N(t) \ll (t - g(t))^{d/2}$.

By analogy to the argument used to prove the lower bound of part (i) of Theorem 2, using part (i) of the annealed shape Theorem 1 we know that there is a $u_0 > 0$ such that

$$E_{Q_{N',u}}(P_{0,\,g(t)}(\tau_\zeta > t)|F_{u_0}) \ge P_{0,0}(\sigma \ge (u_0 - g(t))_+)P_{0,\,u_0}\left(\sup_{u_0 \le s \le t-g(t)} \frac{|X_s|}{(1-\epsilon)f(s)} \le 1\right),$$

whenever $t \ge u_0$. Here σ is the first time that the random walk leaves the origin. An application of Lemma 20 of Appendix B then shows that

$$\liminf_{t\to\infty} \frac{1}{\lambda_d \int_{g(t)}^t (aw_d/N(s))^{2/d}\,ds} \ln Q_{N,\mu}(\tau_{k(t)} > t) \ge -1.$$

We now proceed to prove the upper bound. Let $T := \inf\{s \ge 0\colon |X_s| \ge t - g(t)\}$ and $T_\zeta := T \wedge \tau_\zeta$. Note that

$$Q_{N,\mu}(\tau_{k(t)} > t) \le \int_{\mathscr{I}^{\mathbb{Z}^d}} E_{Q_{N',\eta}}(P_{0,\,g(t)}(T_\zeta > t))d\mu + P_{0,\,g(t)}(T \le t). \tag{63}$$

Now, the second term of the right-hand side of inequality (63) decreases like $e^{-C(t-g(t))+o((t-g(t)))}$ for some constant C. On the other hand, the condition $N(t) \gg (t - g(t))^{d/(d+2)}$ implies that $\int_{g(t)}^t (ds/N(s)^{2/d}) \ll (t - g(t))^{d/(d+2)}$. Thus, it is enough to show that

$$\limsup_{t\to\infty} \frac{1}{\lambda_d \int_{g(t)}^t (aw_d/N(s))^{2/d}\,ds} \ln \int_{\mathscr{I}^{\mathbb{Z}^d}} E_{Q_{N',\eta}}(P_{0,\,g(t)}(T_\zeta > t))d\mu \le -1 \tag{64}$$

The first step in order to prove this inequality is the control of the probability that the principal Dirichlet eigenvalue of the Lapalacian on a subset $[-t, t]^d \cap \mathbb{Z}^d$, chosen by first deleting each site independently with a positive probability and then replacing $f(t)^d$ deleted sites, deviates from the value $\lambda_d/f(t)^2$. So for $\varepsilon > 0$ and $u \ge 0$ define

$$G_u := \left\{\eta \in \mathscr{I}^{\mathbb{Z}^d}\colon f(u)^2 \inf_{\varsigma \in \mathscr{N}_{aw_d f^d(u)}(\eta)} \lambda_\varsigma\big((-(t - g(t)), t - g(t))^d\big) \ge \lambda_d(1-\epsilon)\right\},$$

where we have adopted the notation of Section 4. Let $I(t) := \int_{g(t)}^t (ds/N(s)^{2/d})$ and let $h(t) := \frac{t-g(t)}{\ln\ln(t-g(t)/(N(t)^{2/d} \vee I(t)^{(d+2)/d}))}$ and $n := (t - g(t) \vee h(t))/h(t)$. We next define the event that G_u occurs at times $t_k := g \vee h + (t - g \vee h)(k/n)$, for $0 \le k \le n$ by $\mathscr{G}_t := \cap_{k=0}^n G_{t_k}$. Then,

$$\int_{\mathscr{I}^{\mathbb{Z}^d}} E_{Q_{N',\eta}}\big(P_{0,\,g(t)}(T_\zeta > t)\big)\,d\mu \le \int_{\mathscr{G}_t} E_{Q_{N',\eta}}\big(P_{0,\,g(t)}(T_\zeta > t)\big)\,d\mu + \mu(\mathscr{G}_t^c) \tag{65}$$

Now, there is a $t_0 > 0$ such that

$$\begin{aligned}
\mu(\mathscr{G}_t^c) &\le \sum_{k=0}^{n} \mu(G_{t_k}) \\
&\le \sum_{k=0}^{n} C_1 \exp\left(-C_2\left(\frac{f(t_k)}{\ln\ln\left((t_k - g(t_k))/I(t_k)^{(d+2)/2}\right)}\right)^d \epsilon^2\right) \\
&\le C_1(t-g(t)) \exp\left(-\frac{C_2}{aw_d}\frac{N_{h(t)}}{\ln\ln\left((t_k - g(t_k))/I(t_k)^{(d+2)/2}\right)}\right) \\
&\le C_1(t-g(t)) \exp\left(-\frac{C_2}{aw_d}\frac{(t-g(t))^{d/(d+2)}}{\ln\ln\left((t - g(t))/I(t)^{(d+2)/2}\right)}\right),
\end{aligned} \tag{66}$$

whenever $t \ge t_0$. The second inequality is a consequence of part (iii) of Theorem 6 describing the eigenvalue asymptotics deviation and the assumption $N(t) \gg (t-g(t))^{d/(d+2)}$. This last assumption also implies that $I(t) = \int_g^t (ds/N(s)^{2/d}) \ll (t-g(t))^{d/(d+2)}$, so it follows that

$$\limsup_{t\to\infty} \frac{1}{\int_{g(t)}^t ds/f(s)^2} \ln \mu(\mathscr{G}_t^c) = -\infty.$$

Thus, we only have to examine the first term of the estimate (65). Now, using the strong Markov property, Lemma 15 and the fact that the function $x^{d/2}e^{-x}$ is increasing for $x \ge d/2$, as in the proof of part (i) of Theorem 2, we obtain that for every $\epsilon > 0$,

$$\int_{\mathscr{G}_t} E_{Q_{N',\eta}}\left(P_{0,\,g(t)}(T_\zeta > t)\right) d\mu \le c_2(d)^n \prod_{k=1}^{n} \frac{h(t)}{f(t_k)^2} e^{(-h(t)/f(t_k)^2)(1-\epsilon)\lambda_d} \tag{67}$$

The passage from inequality (67) to

$$\limsup_{t\to\infty} \frac{1}{\lambda_d \int_{g(t)}^t (ds/f(s)^2)} \ln \int_{\mathscr{G}_t} E_{Q_{N',\eta}}\left(P_{0,\,g(t)}(T_\zeta > t)\right) d\mu \le -1 + \epsilon \tag{68}$$

follows an analogous reasoning to the passage from estimate (57) and (52) of the proof of part (i) of Theorem 2. Therefore we omit the details. We end the proof by taking the limit $\epsilon \to 0$ in inequality (68).

PROOF OF PART (ii). Let $M(t) := 1$. By analogy to the proof of the lower bound of part (ii) of Theorem 2, consider the probability measure $Q_{M,\mu}$ representing the law of a single random walk on an obstacle environment distributed according to μ. Note that $Q_{N,\mu}(\tau_{k(t)} > t) > Q_{M,\mu}(\tau_0 > t - g(t))$. Now, by a result of [7] (see also [1,2]) we know that, $\liminf_{t\to\infty}(1/(c_a(p,d)(t - g(t))^{d/(d+2)})) \ln Q_{M,\mu}(\tau_0 > t-g(t)) \ge -1$. Hence,

$$\liminf_{t\to\infty} \frac{1}{c_a(p,d)(t-g(t))^{d/(d+2)}} \ln Q_{N,\mu}(\tau_{k(t)} > t) \ge -1.$$

So let ξ_t be the number of different sites visited at time t by the canonical process X_t representing a simple random walk. Then we have the estimate

$$Q_{N,\mu}\big(\tau_{k(t)} > t\big) \le E_{0,\,g(t)}\big(q^{\xi_t - N_t/a}\big) = q^{-N_t/a} E_{0,\,0}\big(q^{\xi_t - g(t)}\big).$$

Here $q := 1-p$ is the probability that a given site of the lattice has no obstacles and $\bar{a}$ is the maximum number of particles that an obstacle can absorb. Now, using the above relation and the fact that $N(t) \ll (t - g(t))^{d/(d+2)}$, it is clear that

$$\lim_{t\to\infty} \frac{1}{(t-g(t))^{d/(d+2)}} \ln Q_{N,\mu}(\tau_{k(t)} > t)$$
$$\le \lim_{t\to\infty} \frac{1}{(t-g(t))^{d/(d+2)}} \ln E_{0,\,0}(q^{\xi_{t-g(t)}}).$$

Now the left-hand side expression can be bounded above by $c_a(p, d)$ using, for example, the large deviation techniques of [7] (see also [1]).

APPENDIX A

This Appendix contains the proofs of Theorems 7, 8 and 9 of the version of the enlargement of obstacle method used in the proof of Theorem 6. All of the results that will be stated have their counterpart in [16, 17]. Most of the notation is the same as that of Section 4. Also, for $\varepsilon > 0$, consider the Skorokhod space $\Gamma_\varepsilon := D([0,\infty), \varepsilon\mathbb{Z}^d)$ endowed with its Borel σ-field $\mathscr{F}$ and let P_z^ε be the probability measure on $(\Gamma_\varepsilon, \mathscr{F})$ under which the canonical coordinate process $(Z_t^\varepsilon)_{t\ge 0}$ is a simple random walk starting at $z \in \varepsilon\mathbb{Z}^d$ and having jump intensity $1/\varepsilon^2$. Furthermore, for a given subset U of $\varepsilon\mathbb{Z}^d$, we denote by

$$T_U := \inf\{t \ge 0 \colon Z_t^\varepsilon \notin U\}$$

the first exit time from U and

$$H_U := \inf\{s \ge 0 \colon Z_s^\varepsilon \in U\}$$

the first hitting time to U. On the other hand, given $x, y \in \varepsilon\mathbb{Z}^d$, we define the distance $d(x, y) := \sup_{i=1,\dots,d} |x_i - y_i|$.

A.1. *Preliminary lemmas.* The first two lemmas are a straightforward adaptation of proposition A.1 of [16] to the random walk situation.

LEMMA 15. *Let $\varepsilon \in (0,1)$, V a potential on the cubic lattice and $U \subset \varepsilon\mathbb{Z}^d$. Then for every bounded function $f\colon \varepsilon\mathbb{Z}^d \to \mathbb{R}$ we have*

$$\sup_{x\in\varepsilon\mathbb{Z}^d} E_x^\varepsilon\Big(f(X_t)1_{T_U>t}\Big) \le c(d)\Big(\big(\lambda_V^\varepsilon(U)t\big)^{d/2} + 1\Big) e^{-\lambda_V^\varepsilon(U)t} \sup_{x\in\varepsilon\mathbb{Z}^d} f(x).$$

Here E_x^ε is the expectation corresponding to the probability measure P_x^ε, $\mathbf{1}_A$ is the indicator function of $A \in \mathscr{B}$ and $c(d)$ is a constant depending on the dimension.

LEMMA 16. *There exists a constant $K(d) \in (1, \infty)$ such that for $\rho \in (0, 1)$, $U \subset \varepsilon\mathbb{Z}^d$ and V a potential on the cubic lattice,*

$$\sup_{\varepsilon\in(0,1)} \sup_{z\in\mathbb{Z}^d} \left(1 + \int_0^\infty (1-\rho)\lambda_V^\varepsilon(U) e^{(1-\rho)\lambda_V^\varepsilon(U)s} R_{\varepsilon,s}^{U,V}\mathbf{1}(x)\,ds\right) \le \frac{K(d)}{\rho^{d/2+1}}.$$

Now consider a stopping time S_1 and introduce the sequence of iterates of S_1,

$$S_{k+1} := S_1 \circ \theta_{S_k} + S_k \le \infty, \tag{69}$$

where $S_0 := 0$. Then we have the following proposition.

PROPOSITION 1. *Let $\varepsilon > 0, U$ be a bounded subset of $\varepsilon\mathbb{Z}^d$ and consider a potential V on $\mathbb{Z}^d$. Let $\lambda > 0$ and S_1 be a stopping time. Assume that*:

(i) *For all* $x \in \mathbb{Z}^d$, $\lim_{k\to\infty} S_k \ge T_U$, P_x^ε-a.s.
(ii) $\alpha := \sup_x E_x^\varepsilon\left(S_1 < T_U, e^{\lambda S_1 - \int_0^{S_1} \varepsilon^{-2}V(Z_s^\varepsilon)\,ds}\right) < 1$.
(iii) $\beta := \sup_x \int_0^\infty \lambda e^{\lambda u} E_x^\varepsilon\left(S_1 \wedge T_U > u, e^{-\int_0^u \varepsilon^{-2}V(Z_s^\varepsilon)\,ds}\right) du < \infty$.

Then $\lambda \le \lambda_V^\varepsilon(U)$ *and*

$$\sup_x \int_0^\infty \lambda e^{\lambda u} R_{\varepsilon,u}^{U,V}\mathbf{1}(x)\,du \le \frac{\beta}{1-\alpha}.$$

As in [17], the following proposition is an application of the previous one which enables a comparison between $\lambda_V^\varepsilon(O_1)$ and $\lambda_V^\varepsilon(O_2)$ for suitable subsets O_1 and O_2 of $\varepsilon\mathbb{Z}^d$. For a given stopping time τ we will pick up S_1 in (69) as

$$S_1 = \tau \circ \theta_{T_{O_1}} + T_{O_1}. \tag{70}$$

PROPOSITION 2. *Let $\varepsilon > 0$, O_1 and O_2 be bounded subsets of $\varepsilon\mathbb{Z}^d$, and consider a potential V on $\mathbb{Z}^d$. Let $\lambda > 0$ and define S_1 as in* (70) *for some stopping time τ. Assume that*:

(i) *For all* $x \in \mathbb{Z}^d$, $\lim_{k\to\infty} S_k \ge T_{O_2}$, P_x^ε-a.s.
(ii) $A := \sup_{x\in\mathbb{Z}^d}\left(1 + \int_0^\infty \lambda e^{\lambda u} R_{\varepsilon,u}^{O_1,V}\mathbf{1}(x)\,du\right) < \infty$.
(iii) $B := \sup_{x\notin O_1} \int_0^\infty \lambda e^{\lambda u} E_x^\varepsilon\left(\tau \wedge T_{O_2} > u, e^{-\int_0^u \varepsilon^{-2}V(Z_s^\varepsilon)\,ds}\right) du < \infty$.
(iv) $C := \sup_{x\notin O_1} E_x^\varepsilon\left(\tau < T_{O_2}, e^{\lambda\tau - \int_0^\tau \varepsilon^{-2}V(Z_s^\varepsilon)\,ds}\right)$.

Then $\lambda \le \lambda_V^\varepsilon(O_2)$.

A.2. *Proof of Theorem* 7. For $L \ge 2$ and $k \ge 0$ we define the stopping time H_k by

$$H_k := \inf\{s \ge 0, d(Z_s^\varepsilon, Z_0^\varepsilon) \ge L^{-k}\}.$$

The main step leading to the proof of Theorem 7 is the following lemma.

LEMMA 17. *There exists a constant* $c_1(d) > 0$ *such that for each* $\xi \in \Upsilon$, $z \in \mathbb{Z}^d_{n_\gamma(\epsilon)}$ *and* $x \in C_z^{(n_\gamma(\epsilon))}$ *one has*

$$P_x^\varepsilon(H_{n_\alpha(\varepsilon)} < T) \le e^{-c_1(d)\sum_{n\alpha(\varepsilon)<k\le n_\gamma(\varepsilon)} \mathrm{cap}_{\varepsilon,[z]_k}}, \tag{71}$$

where T *is the entrance time in the obstacle set* $\{x \in \varepsilon\mathbb{Z}^d : \xi_\varepsilon(x) = 1\}$.

PROOF. Let $x \in \varepsilon\mathbb{Z}^d$, $k \ge 0$ and define $D_k := [x - L^{-k+1}, x + L^{-k+1}]$. Also, let $E_k := L^{-k}K^{(k)}_{[m]k}$. Then, by the strong Markov property,

$$P_x^\varepsilon(H_{n_\alpha(\varepsilon)} < T) = P_x^\varepsilon(H_{n_\alpha(\varepsilon)+1} < T), P_{Z^\varepsilon_{H_{n\alpha(\varepsilon)+1}}}(T_{D^\varepsilon_{n\alpha(\varepsilon)+1}} < T), \tag{72}$$

where T_U denotes the first exit time from the set U. We will show that there is a constant $c_1(d)$ such that for each $n_\alpha(\varepsilon) < k \le n_\gamma(\varepsilon)$ one has

$$\inf_{y\in E_k} P_y^\varepsilon(T_{D_k^\varepsilon} < T) \le 1 - c_1(d)\mathrm{cap}_{\varepsilon,[z]_k}. \tag{73}$$

Combining the capacity estimate (73) with equation (72), we obtain that

$$P_x^\varepsilon(H_{n_\alpha(\varepsilon)} < T) \le \Pi_{n_\alpha(\varepsilon)<k\le n_{\gamma(\varepsilon)}}(1 - c_1(d)\mathrm{cap}_{\varepsilon,[z]_k}).$$

This together with the fact that $1 - x \le e^{-x}$ for $0 \le x \le 1$ implies our claim (71). So it remains to prove the capacity estimate (73). Then

$$P_y^\varepsilon\Big(T_{D_k^\varepsilon} < T\Big) \le P\Big(T_{D_k^\varepsilon \le H_{E_k}}\Big). \tag{74}$$

However, $P_y^\varepsilon(T_{D_k^\varepsilon} > H_{E_k}) = P_{L^k y - L^k x}(H_{K^{(k)}_{[z]_k} - L^k x} < T_{F^{L^{k_\varepsilon}}})$, where $F^{L^{k_\varepsilon}} = ([-L, L]^d)^{L^{k_\varepsilon}}$. It follows that

$$P_y^\varepsilon(T_{D_k^\varepsilon} > H_{E_k}) \ge \int_{K^{(k)}_{[z]_k} - L^k x} g^{L^{k_\varepsilon}}_{F^{L^{k_\varepsilon}}, V}(L^k y - L^k x, y') e^{L^{k_\varepsilon}}_{K^{(k)}_{[z]_k}, F^{L^{k_\varepsilon}}, V}(dy'),$$

where $V = 1$ for $d = 1, 2$ and $V = 0$ otherwise. Now, clearly, we have $\int e^{L^{k_\varepsilon}}_{K^{(k)}_{[z]_k}, F^{L^{k_\varepsilon}}, V}(dy') \ge \mathrm{cap}_{\varepsilon,[z]_k}$. Thus

$$P_y^\varepsilon(H_{E_k} < T_{D_k^\varepsilon}) \ge \mathrm{cap}_{\varepsilon,[z]_k}\left(\inf_{[-1,1]^{\varepsilon L^k} \times [-1.25, 1.25]^{\varepsilon L^k}} g^{\varepsilon L^k}_{F^{\varepsilon L^k}, V}(\cdot,\cdot)\right).$$

Standard estimates on the Green function of a simple random walk on the cubic lattice (see, e.g., [12]) enable us to conclude that there is a constant $c_1(d)$ independent of ε such that,

$$\inf_{[-1,1]^{\varepsilon L^k} \times [-1.25, 1.25]^{\varepsilon L^k}} g^{\varepsilon L^k}_{F^{\varepsilon L^k}, V}(\cdot,\cdot) \ge c_1(d).$$

Substituting this back in inequality (74) we obtain the capacity inequality (73). This finishes the proof of the lemma. □

We are now ready to prove Theorem 7. Let $g(\varepsilon)$: $[0,\infty) \to [0,\infty)$ be a function such that $g(\varepsilon) \geq 0$ and $(r_\gamma(\varepsilon)/r_\alpha(\varepsilon))^{(d+2)/c_1\delta \ln L} \ll g(\varepsilon)$. Pick $M > 0$, $\xi \in \Upsilon$ and U an open subset of $\mathbb{R}^d$ and define

$$\lambda := (\lambda_\xi^\varepsilon(U_1^\varepsilon) \wedge M - g(\varepsilon))_+,$$

where $U_1^\varepsilon = U^\varepsilon \backslash \mathscr{D}_\varepsilon(\xi)$. Without loss of generality we can assume that $\lambda > 0$, and then $M > g(\varepsilon)$, so that

$$\lambda \leq \lambda_\xi^\varepsilon(U_1^\varepsilon)\left(1 - \frac{g(\varepsilon)}{M}\right), \tag{75}$$

Now let $\tau := H_{n_\alpha}(\varepsilon)$ and $S_1 := \tau \circ \theta_{T_{U_1^\varepsilon}} + T_{U_1^\varepsilon}$. We recall [see definition (69)] that we define the iterates of S_1 by $S_k := S_1 \circ \theta_{S_{k-1}} + S_{k-1}$. We will apply Proposition 2, choosing $O_1 = U_1^\varepsilon$ as above and $O_2 := U^\epsilon$. It is enough to show that the constants A, B and C satisfy $A < \infty$, $B < \infty$ and $AC < 1$.

Then $A < \infty$ follows from inequality (75) and Lemma 16 of Appendix A. In fact,

$$A \leq K(d)\left(\frac{M}{g(\varepsilon)}\right)^{d/2+1},$$

where $K(d)$ is a constant depending only on the dimension. Next, choose ε small enough so that $E_0^\varepsilon(e^{2M\tau}) \leq K'(d) < \infty$, for some dimension dependent constant $K'(d)$. Since $\lambda \leq M$ we get

$$B \leq K'(d) < \infty.$$

Finally, note that

$$C^2 \leq \sup_{x\in\mathbb{R}^d} E_x^\varepsilon(e^{2M\tau}) \sup_{c\notin U_1^\varepsilon} E_x^\varepsilon(\tau < T_{U^\varepsilon},\ \tau < T).$$

From the capacity estimate of Lemma 71 and our estimate on A we get that

$$AC \leq K(d, M)e^{(d/2+1)\ln(1/g(\varepsilon)) - (c_1/2)\delta(n_\gamma(\varepsilon) - n_\alpha(\varepsilon))}.$$

Our hypothesis on $g(\varepsilon)$ ensures that $AC < 1$, which proves Theorem 7. □

A.3. *Proof of Theorem* 8. The main ingredient of the proof is the following analog of Proposition 2.4 of [17].

LEMMA 18. *There exists a constant* $c_2(d) \in (0,\infty)$ *such that when* $\varepsilon \in (0,1)$ *and* $r \in (0, 1/4)$ *satisfy*

$$\begin{gathered} L^{-n_\gamma(\varepsilon)} < L^{-n_\alpha(\varepsilon)} < r, \\ \delta c_1(d)(n_\gamma(\varepsilon) - n_\alpha(\varepsilon)) > \ln 2. \end{gathered} \tag{76}$$

Then for any $\xi \in \Upsilon$ *and open set* $U \in \mathbb{R}^d$ *such that* $\sup_{z\in\mathbb{Z}^d} \varepsilon^d |(U^\varepsilon \backslash (\mathscr{D}_\varepsilon \cap C_z)|_\varepsilon < r^d$, *one has*

$$\lambda_\xi^\varepsilon(U^\varepsilon) > \frac{c^2(d)}{r^2}.$$

PROOF. Let $S_1 := \inf\{s \geq 0, d(Z_s^\varepsilon, Z_0^\varepsilon) \geq 4r\}$ and for $z \in \varepsilon\mathbb{Z}^d$ and $a \in (0,\infty)$ let $B_d(z,a) := \{x \in \varepsilon\mathbb{Z}^d : d(x,z) \leq a\}$. First note that

$$\begin{aligned}
P_z^\varepsilon(S_1 < T_{U^\varepsilon} \wedge T) &\leq P_z^\varepsilon\big(T_{B_d^\varepsilon(z,3r)} \leq H_{\mathscr{D}_\varepsilon \cup (U^\varepsilon)^c}\big) \\
&\qquad + P_z^\varepsilon\big(H_{\mathscr{D}_\varepsilon \cup (U^\varepsilon)^c} < T_{B_d^\varepsilon(z,3r)} < S_1 < T_{U^\varepsilon} \wedge T\big) \\
&\leq 1 - P_z^\varepsilon\big(H_{\mathscr{D}_\varepsilon \cup (U^\varepsilon)^c} < T_{B_d^\varepsilon(z,3r)}\big) + P_z^\varepsilon\big(H_{\mathscr{D}_\varepsilon \cup (U^\varepsilon)^c} \\
&\qquad < T_{B_d^\varepsilon(z,3r)} \wedge T_{U^\varepsilon}, P^\varepsilon_{Z^\varepsilon_{H_{\mathscr{D}_\varepsilon \cup (U^\varepsilon)^c}}}(H_{n_\gamma(\varepsilon)} < T)\big) \\
&\leq 1 - P_z^\varepsilon\big(H_{\mathscr{D}_\varepsilon \cup (U^\varepsilon)^c} < T_{B_d^\varepsilon(z,3r)}\big) \\
&\qquad + e^{-c_1(d)\delta(n_\gamma(\varepsilon) - n_\alpha(\varepsilon))} P_z^\varepsilon\big(H_{\mathscr{D}_\varepsilon \cup (U^\varepsilon)^c} < T_{B_d^\varepsilon(z,3r)}\big) \\
&\leq 1 - \tfrac{1}{2} P_z^\varepsilon\big(H_{\mathscr{D}_\varepsilon \cup (U^\varepsilon)^c} < T_{B_d^\varepsilon(z,3r)}\big),
\end{aligned}$$

where in the second to last inequality we have used Lemma 17 and in the last one assumption (76). Now note that

$$\begin{aligned}
|B_d^\varepsilon(x,2r) \cap (\mathscr{D}_\varepsilon(U^\varepsilon)^c) \cap C_z^{(0)}|_\varepsilon &\geq |B_d^\varepsilon(x,2r) \cap C_z^{(0)}|_\varepsilon - |(U^\varepsilon \backslash \mathscr{D}_\varepsilon) \cap C_z^{(0)}|_\varepsilon \\
&\geq \varepsilon^{-d} r^d.
\end{aligned}$$

An application of the Dirichlet principle implies that

$$\begin{aligned}
&P_z^\varepsilon\big(H_{\mathscr{D}_\varepsilon \cup (U^\varepsilon)^c} < T_{B_d^\varepsilon(z,3r)}\big) \\
&\quad \geq \frac{r^{2d}}{\varepsilon^{2d} \sum_{x,\, y \in B_d^\varepsilon(0,2r)} g^\varepsilon_{B_d^\varepsilon(0,3r), V^{(x,y)}}} \inf_{y \in B_d^\varepsilon(0,2r)} g^\varepsilon_{B_d^\varepsilon(0,3r), V^{(0,y)}},
\end{aligned}$$

where $V := 1$ for $d = 1, 2$ and $V := 0$ for $d \geq 3$. As in Lemma 15, it is possible to find a constant $c_1(d)$ which bounds the left-hand side of the above equation from below. Therefore,

$$\sup_{z \in \varepsilon\mathbb{Z}^d} P_z^\varepsilon(S_1 < T_{U^\varepsilon} \wedge T) \leq \gamma(d) < 1.$$

We can now choose $c_2(d)$ small enough so that $E_0^\varepsilon(e^{2c_2 T_{B_d^\varepsilon(0,4)}}) < 1/\gamma(d)$. Choosing $\lambda = c_2/r^2$ it is now easy to see that the conditions $\alpha < 1$ and $\beta < \infty$ of Proposition 1 of Appendix A are satisfied. This proves the lemma. □

As in [7, 16], the proof of Theorem 8 is now a straightforward application of the above lemma and of Proposition 1 of Appendix A. We therefore omit it.

A.4. *Proof of Theorem* 9. Theorem 9 is elementary in dimension $d = 1$. In fact, if we denote by Cap$\{0\}$ the capacity with respect to one-half of the continuous Laplacian on $\mathbb{R}$ of the point 0, it is easy to verify that by definition the bad set $\mathscr{B}_\varepsilon$ is empty whenever $\delta \leq \frac{1}{2}\text{Cap}\{0\}$. The following theorem is the key step to prove the volume estimate of part (ii) of Theorem 9 in dimensions $d \geq 2$.

THEOREM 10. *Assume that $d \geq 2$ and that L is large enough so that $c_8(d, L) > 1$, where $c_8(d, L) := L^2/(3^d + 1)$ if $d \geq 3$ and $c_8(2, L) := L^2/(c \ln L)$ for some numerical constant c. Let $\delta_0 := 3/(8L^d G(1/2L))$. Then there exist constants $c_4(d, L)$ and $c_5(d, L)$ such that for any $z_0 \in \mathbb{Z}^d_{n_\alpha(\varepsilon)}$ we have*

$$\frac{1}{L^{d(n_\gamma - n_\alpha)}} \sum_{\substack{z \succ z_0 \\ z \in \mathbb{Z}^d_{n_\gamma},\, C_z^{(n_\gamma)} \text{ rarefaction box}}} \mathrm{cap}_z \leq c_4 e^{-c_5(1-\delta/\delta_0)(n_\gamma - n_\alpha)}.$$

PROOF. In the sequel we will denote by V the potential on the lattice $h\mathbb{Z}^d$, for $h > 0$, which equals 0 for $d \geq 3$ and 1 otherwise. With no loss of generality we assume that $n_\alpha = 0$, $z_0 = 0$ and let $l := n_\gamma$. Also the following relations will be used in the sequel:

$$K_z^{(k)} \subset L^k (C_z^{(k)})^\varepsilon, \tag{77}$$

$$K_z^{(k)} = \bigcup_{y \in \mathbb{Z}^d_{k+1} \cap C_z^{(k)}} \frac{1}{L} K_y^{(k+1)}. \tag{78}$$

Step 1. Here we will prove three inequalities involving the capacities cap_z. First note that there is a constant $c_6(d)$ such that

$$\mathrm{cap}_z \leq c_6(d), \tag{79}$$

Whenever $z \in \mathbb{Z}^d_k$ for $k \geq 1$. In fact this is a consequence of inequality (77), the monotonicity of capacity and the fact that the capacity of the set $([0, 1]^d)^{L^k \varepsilon}$ converges to the capacity with respect to the continuous Laplacian on $\mathbb{R}^d$ of the square $[0, 1]^d$.

Now note that for $d \geq 3$ and $h \in (0, \infty)$, the following scaling relation is satisfied by the discrete Laplacian Green function:

$$g^{h/L}_{(h/L)Z^d, 0}\left(\frac{x}{L}, \frac{y}{L}\right) = L^{(d-2)} g^h_{hZ^d, 0}(x, y) \tag{80}$$

for $x, y \in h\mathbb{Z}^d$. Now for $d = 2$ it is true that

$$g^{h/L}_{(h/L)Z^2, 1}\left(\frac{x}{L}, \frac{y}{L}\right) \leq c(\ln L) g^h_{hZ^2, 1} \tag{81}$$

for some constant c. This is an elementary computation (it is also a corollary of Lemma 1.9 of [14]). Now, from the fact that $L^{-1} K^{(k+1)}_{z \cdot j} \subset K_z^{(k)}$, from Dirichlet principle and inequalities (80) and (81), we obtain our second relation,

$$\mathrm{cap}_{\varepsilon, z \cdot j} \leq c_7 \mathrm{cap}_{V, \varepsilon L^k, \varepsilon L^k Z^d}\left(\frac{1}{L} K^{(k)}_{z \cdot j}\right) \leq c_7 \mathrm{cap}_{\varepsilon, z} \tag{82}$$

for $z \in \mathbb{Z}^d_k$, $0 < k \leq l$, where $c_7(d, L) := L^{d-2}$ when $d \geq 3$ and $c_7(d, L) := c \ln L$ for $d = 2$.

Now we define the constants $\delta_1(d, L), \bar{\delta}_1(d, L)$ and $c_8(d, L)$ via the equations,

$$\frac{4}{3}\bar{\delta}_1 L^d G\left(\frac{1}{2L}\right) = 1,$$

$$\delta_1 = c_7 \bar{\delta}_1,$$

$$c_8 = \frac{L^d}{(3^d + 1)c_7},$$

where $G(|x - y|) := g(x, y)$ and g is the Green function of the continuous Laplacian divided by $2d$ on $\mathbb{R}^d$ and with potential V.

Our third relation is the content of the following lemma.

LEMMA 19. *For* $z \in \mathbb{Z}_k^d$, $0 < k < l$,

$$\mathrm{cap}_z \geq c_8 \frac{1}{L^d} \sum_{j \in \{0, \ldots, L-1\}^d} \mathrm{cap}_{z \cdot j} \wedge \delta_1. \tag{83}$$

PROOF. With no loss of generality we can assume that $\mathrm{cap}_{z \cdot j} = 0$ for at least one $j \in \{0, \ldots, L-1\}^d$. Now choose subsets $\overline{K}_j$ of $(1/L)K_{z \cdot j}$ and constants γ_j for $j \in \{0, \ldots, L-1\}^d$ such that

$$\begin{gathered}
\overline{K}_j = \frac{1}{L} K_{z \cdot j} \quad \text{if } \mathrm{cap}_{V, \varepsilon L^k Z^d, \varepsilon L^k}\left(\frac{1}{L} K_{z \cdot j}\right) \leq \gamma_j, \\
\mathrm{cap}_{V, \varepsilon L^k Z^d, \varepsilon L^k}(\overline{K}_j) \quad \text{if } \mathrm{cap}_{V, \varepsilon L^k Z^d, \varepsilon L^k}\left(\frac{1}{L} K_{z \cdot j}\right) > \gamma_j, \\
|\gamma_j - \bar{\delta}_1| \leq \mathrm{cap}_{V, \varepsilon L^k Z^d, \varepsilon L^k}(\{0\}).
\end{gathered} \tag{84}$$

That this is possible follows from the subadditivity of capacity, which implies that when one site is deleted from a finite set its capacity decreases at most by $\mathrm{cap}_{V, \varepsilon L^k Z^d, \varepsilon L^k}(\{0\})$.

So denote by ν_j the equilibrium measure of $\overline{K}_j$ for $j \in \{0, \ldots, L-1\}^d$ so that

$$\nu_j(\overline{K}_j) = \mathrm{cap}_{V, \varepsilon L^k Z^d, \varepsilon L^k}\left(\frac{1}{L} K_{z \cdot j}\right) \wedge \gamma_j.$$

Consider now the probability measure $\nu := 1/(\sum_j \mathrm{cap}_{V, \varepsilon L^k Z^d, \varepsilon L^k}(\overline{K}_j)) \sum_j \nu_j$ on K_z. Defining the inner product $\langle \mu, \nu \rangle_h := \int g^h_{\varepsilon L^k Z^d, V}(x, y) \mu(dx) \nu(dx)$ between two measures μ and ν in $h\mathbb{Z}^d$, by the Dirichlet principle we have that

$$\begin{aligned}
\frac{1}{\mathrm{cap}_z} \leq \langle \nu, \nu \rangle_{\varepsilon L^k} = {} & \frac{1}{\left(\sum_j \mathrm{cap}_{V, \varepsilon L^k Z^d, \varepsilon L^k}(\overline{K}_j)\right)^2} \\
& \times \left(1 + \sum_j \sum_{j' : j' \neq j} \left\langle \frac{\nu_j}{\sum_{j''} \mathrm{cap}_{V, \varepsilon L^k Z^d, \varepsilon L^k}(K_{j''})}, \nu_{j'} \right\rangle\right).
\end{aligned} \tag{85}$$

However,

$$\sum_{j'\neq j}\int g^{\varepsilon L^k}_{\varepsilon L^k Z^d,\,V}(x,y)\nu_{j'}(dy)\le 3^d-1+\sum_{j'}{}'\int g^{\varepsilon L^k}_{\varepsilon L^k Z^d,\,V}(x,y)\nu_{j'}(dy),$$

where the term 3^d-1 is obtained from those indices j' such that $C^{(k+1)}_{z\cdot j'}$ is a nearest or diagonal neighbor of $C^{(k)}_{z\cdot j}$, and the sum $\sum'_{j'}$ goes over the other indices j'. Now, from the fact that $K_z\subset L^kC^{(k)}_z$ we see that $|x-y|\ge 1/(2L)$ whenever $x\in\operatorname{supp}(\nu_j)$ and $y\in\operatorname{supp}(\nu_{j'})$, with the corresponding boxes not nearest nor diagonal neighbors. This, the fact that $G(r)$ is decreasing for r large enough, and a standard approximation of the Green function of the discrete Laplacian to the Green function of the continuous Laplacian permits us to conclude that $g^{\varepsilon L^k}_{\varepsilon L^k Z^d,\,V}(x,y)\le (4/3)G(1/2L)$, whenever ε is small enough, $x\in\operatorname{supp}(\nu_j)$ and $y\in\operatorname{supp}(\nu_{j'})$. We can now conclude that for $x\in\operatorname{supp}(\nu_j)$,

$$\sum_{j'\neq j}\int g^{\varepsilon L^k}_{\varepsilon L^k Z^d,\,V}(x,y)\nu_{j'}(dy)\le 3^d-1+L^d\frac{4}{3}G\left(\frac{1}{2L}\right)\bar\delta_1=3^d.$$

Thus, for ε small enough,

$$\begin{aligned}\operatorname{cap}_z&\ge\frac{c_8}{L^d}\sum\nolimits_j\operatorname{cap}_{z\cdot j}\wedge(c_7\gamma_j)\\&\ge\frac{c_8}{L^d}\sum\nolimits_j\operatorname{cap}_{z\cdot j}\wedge(\delta_j),\end{aligned}$$

where we have used inequality (84) and the fact that in dimensions $d\ge 2$, $\lim_{\varepsilon\to 0}\operatorname{cap}_{V,\,\varepsilon L^kZ^d,\varepsilon L^k}(\{0\})=0$. $\square$

Now we will use the three basic relations (79), (82) and (83) that have been proved. Introduce the probability space $\Sigma:=(\{0,\dots,L-1\}^d)^l$ endowed with the uniform probability Q. Denote by $X_1,\dots,X_l$ the canonical $\{0,\dots,L-1\}^d)\}^d$ valued coordinates on this space and by $\mathscr{G}_k$, $k\ge 0$, the filtration on Σ generated by $X_1,\dots,X_{k\wedge l}$. Viewing $(0,X_1,\dots,X_k)$, $1\le k\le l$ as a random index, we can now define the stochastic process

$$Y_k:=\operatorname{cap}_{1+X_1/L+\cdots X_k/L^k},\qquad 1\le k\le l.$$

We can now reexpress our relations (79), (82) and (83) as

$$\begin{aligned}0&\le Y_k\le c_6,\\Y_{k+1}&\le c_7Y_k,\\Y_k&\ge c_8E(Y_{k+1}\wedge\delta_1|\mathscr{G}_k)\end{aligned}$$

for $1 \le k \le l$. Now let $\delta_0 := \frac{1}{2}\delta_1 c_7^{-1}$ and define the stopping times,

$$\begin{aligned}
\tau_1 &:= \inf\{k \ge 1, Y_k \le \delta_0\} \wedge l,\\
\sigma_1 &:= \inf\{k \ge \tau_1, Y_k \ge 2\delta_0\} \wedge l,\\
\tau_i &:= \inf\{k \ge \sigma_{i-1}, Y_k \le \delta_0\} \wedge l, \qquad i \ge 2,\\
\sigma_i &:= \inf\{k \ge \tau_i, Y_k \ge 2\delta_0\} \wedge l, \qquad i \ge 2.
\end{aligned}$$

Now, as in [17], by a standard supermartingale argument it is possible to prove that

$$U_n^i := c_8^{n\wedge(\sigma_i-\tau_i)} Y_{(\tau_i+n)\wedge\sigma_i}, \qquad n \ge 0$$

is a $\mathscr{G}_{\tau_i+n}, n \ge 0$ supermartingale. Using this fact and relation (79), we conclude as in Sznitman that

$$E\Big(c_8^{\sum_i(\sigma_i-\tau_i)} Y_l\Big) \le \frac{1}{c_9} E(Y_{\tau_1}) \le \frac{c_6}{c_9}, \tag{86}$$

where $c_9 := (\delta_1/c_7c_6) \wedge 1$. However, on rarefaction boxes, when $\delta < \delta_0$, the following lower bound is satisfied:

$$1 + \sum_{i\ge1}(\sigma_i - \tau_i) \ge \left(1 - \frac{\delta}{\delta_0}\right) l. \tag{87}$$

Combining (86) with (87) finishes the proof of the lemma. □

The next theorem is the final step before the proof of the volume estimate Theorem 9. It is the discrete Laplacian version on the cubic lattice of Theorem 4.3.5 of [17]. Since the reasoning is the same, we omit the proof.

THEOREM 11. *Assume that $d \ge 2$. Then,*

$$\limsup_{\varepsilon\to0}\left(\frac{\varepsilon}{r_\beta(\varepsilon)}\right)^{d-2} \sup_{\xi\in\Upsilon,\, z\in Z^d_{n_\gamma(\varepsilon)}} \frac{\operatorname{cap}_{V,\,\varepsilon L^{n_\gamma} Z^d,\,\varepsilon L^{n_\gamma}}(L^{n_\gamma}\mathscr{B}_\varepsilon(\xi) \cap C_z^{(n_\gamma)})}{\operatorname{cap}_{\varepsilon,\,z}} < \infty.$$

We are now ready to prove the volume estimate of Theorem 9. First note that by Dirichlet principle, for any subset K of a box $(C_z^{(0)})^{\varepsilon L^k}$, for $z \in \mathbb{Z}^d$ and $k \ge 0$, we have

$$\operatorname{cap}_{V,\,\in L^k Z^d,\,\varepsilon L^k}(K) \ge \frac{(\varepsilon L^k)^d |K|_{\varepsilon L^k}}{(\varepsilon L^k)^d \sum_{x\in([-1,1])^{\varepsilon L^k}} g^{\varepsilon L^k}_{V,\,\varepsilon L^k Z^d}(0,x)}.$$

Now since for any compact set C one has that

$$\lim_{h\to0} h^d \sum_{x\in C^h} g^h_{V,\,hZ^d}(0,x) = \int_C g_{V,\,\mathbb{R}^d(0,x)}\,dx,$$

where $g_{V,\mathbb{R}^d}$ is the Green function of $1/(2d)$ times the Laplacian operator on $\mathbb{R}^d$ plus a potential V. It follows that for ε small enough one has

$$\mathrm{cap}_{V,\,\varepsilon L^k Z^d,\,\varepsilon L^k}(K) \geq \frac{(\varepsilon L^k)^d |K|_{\varepsilon L^k}}{c_{10}(d)}, \tag{88}$$

where $c_{10}(d) := \int_{[-1,1]^d} g_{V,\mathbb{R}^d}(0,x)\,dx$. Now, for $z \in \mathbb{Z}^d$,

$$\begin{aligned}
\varepsilon^d \left|\mathscr{B}_\varepsilon(\xi) \cap C_z^{(0)}\right|_\varepsilon &= \frac{1}{L^{dn_\gamma(\varepsilon)}} \sum_{\substack{z' \succ z,\, z' \in Z^d_{n_\gamma} \\ z' \text{a rarefaction index}}} (\varepsilon L^{n_\gamma})^d \left|L^{n_\gamma}(\mathscr{B}_\varepsilon(\xi) \cap C_{z'}^{(n_\gamma)}\right|_\varepsilon \\
&\leq c_{10}(d)\left(\sup_{\xi \in \Upsilon,\, z' \in Z^d_{n_\gamma}} \frac{\mathrm{cap}_{V,\,\varepsilon L^{n_\gamma} Z^d_{n_\gamma},\,\varepsilon L^{n_\gamma}}(L^{n_\gamma}(\mathscr{B}_\varepsilon(\xi) \cap C_{z'}^{(n_\gamma)}))}{\mathrm{cap}_{\varepsilon,\,z'}}\right) \\
&\quad \times \frac{1}{L^{dn_\gamma}} \sum_{\substack{z' \succ z,\, z' \in Z^d_{n_\gamma} \\ z' \text{a rarefaction index}}} \mathrm{cap}_{\varepsilon,\,z'} \\
&\leq C\left(\frac{r_\beta(\varepsilon)}{\varepsilon}\right)^{d-2} e^{-c_5(1-\delta/\delta_0)(n_\gamma - n_\alpha)}.
\end{aligned}$$

where the first inequality is a consequence of the estimate (88), the second of Lemma 10 and C is a constant. This proves Theorem 9. □

APPENDIX B

In this Appendix we prove several results that are needed for the proofs of Theorems 3 and 2 concerning the asymptotic behavior in time of the survival probability of particles in the random saturation process. In what follows we let Z_t be a symmetric nearest neighbor random walk on $\mathbb{Z}^d$ with total jump rate equal to 1 and when $Z_0 = x \in \mathbb{Z}^d$ we denote by P_x the corresponding probability measure defined on $D([0,\infty); \mathbb{Z}^d)$.

LEMMA 20. *Let $f(t)\colon [0,\infty) \to (0,\infty)$ be an increasing function such that $1 \ll f(t) \ll t^{1/2-\epsilon}$ for some $\epsilon > 0$. Then*

$$\lim_{t\to\infty} \frac{1}{\lambda_d \int_0^t (ds/f(s)^2)} \ln P_0\left(\sup_{0\leq s\leq t} \frac{|Z_s|}{f(s)} \leq 1\right) \geq -1,$$

where λ_d is principal Dirichlet eigenvalue of the continuous Laplacian divided by 2d on a ball of unit radius in $\mathbb{R}^d$.

PROOF. Let $h(t)$ be an increasing function such that

$$f(t)^2 \ll h(t) \ll t.$$

Such a choice is possible because of the hypothesis $f(t) \ll t^{1/2-\epsilon}$ for some $\epsilon > 0$. let $n := [t/h(t)]$. Now we denote by A_k the event that between time

$kh(t)$ and $(k+1)h(t)$, the random walk does not exit a ball of radius $f(kh(t))$ with center at the origin,

$$A_k := \left\{ \sup_{kh(t)\le s\le (k+1)h(t)} |Z_s| \le f(kh(t)) \right\}$$

and we denote by B_k the event that at time $(k+1)f(h(t))$ it is inside the ball of unit radius centered at the origin

$$B_k := \{|Z_{(k+1)h(t)}| \le 1\}.$$

Then if we define $C_k := A_k \cap B_k$ for $0 \le k \le n-1$, we have by the Markov property that

$$P_0\left(\sup_{0\le s\le t} \frac{|Z_s|}{f(s)} \le 1\right) \ge \Pi_{k=0}^n P_0(C_k). \tag{89}$$

However, note that for $x \in \mathbb{Z}^d$ such that $|x| \le f(kh(t))$ one has that

$$\begin{aligned} P_x(C_k) &= \sum_{n=1}^{\infty} e^{-h(t)\lambda_n(f(kh(t)))}(\phi_{n,\, f(kh(t))}(x))^2 \\ &\ge e^{-h(t)\lambda_1(f(kh))}\phi^2_{1,\, f(kh)}(x), \end{aligned} \tag{90}$$

where $\lambda_n(r)$, $n \in \mathbb{N}$, are the set of eigenvalues in increasing order of the discrete Laplacian operator on the set $\{y \in \mathbb{Z}^d : |y| \le r\}$ with Dirichlet boundary conditions, and $\phi_{n,r}$ are the corresponding eigenfunctions. It now follows from inequalities (89) and (90) that

$$P_0\left(\sup_{0\le s\le t} \frac{|Z_s|}{f(s)} \le 1\right) \ge \prod_{k=0}^{n} e^{-h(t)\lambda_1(f(kh))}\phi^2_{1,\, f(kh)}(0).$$

Now, it is a standard fact that $\lim_{r\to 0} r^d \phi_{1,r}(0) = \phi_1(0)$, where $\phi_1(x)$ is the principal Dirichlet eigenfunction of the Laplacian operator on the unit ball of $\mathbb{R}^d$ divide by $2d$. Thus,

$$P_0\left(\sup_{0\le s\le t} \frac{|Z_s|}{f(s)} \le 1\right) \ge \prod_{k=0}^{n} \frac{1}{f(kh)^d} e^{-h(t)\lambda_1(f(kh))}. \tag{91}$$

Now the following lemma proved by [2] will be used. Here keep track of the rate of convergence.

LEMMA 21. *There is a constant C such that,*

$$\lambda^1(t) \le \frac{1}{t^2}\lambda(t) + \frac{C}{t^3},$$

where $\lambda(t)$ is the principal Dirichlet eigenvalue of the continuous Laplacian on a ball of radius t, B_t.

Combining this lemma with our previous estimate (91), we now obtain that

$$P_0\left(\sup_{0\le s\le t}\frac{|Z_s|}{f(s)}\le 1\right)\ge \prod_{i=0}^{n} e^{-\lambda_d(h(t)/f(kh)^2)-C(h(t)/f(kh)^3)-d\ln f(kh)}.$$

Our assumption on $h(t)$ and the hypothesis that $1\ll f(t)\ll t^{1/2-\epsilon}$ for some $\epsilon>0$ now proves the lemma. □

LEMMA 22. *Let $f(t), u(t)\colon[0,\infty)\to[0,\infty)$ be functions such that $f(t)\ll t^\beta$ for some $\beta\in(0,\infty)$, and $u(t)\ll t$. Assume that $f(t)/t^\beta\ge\sup_{s\ge t}(f(s)/s^\beta)$. Then*,

$$\frac{u(t)}{f(u(t))^{1/\beta}}\ll\int_{u(t)}^{t}\frac{ds}{f(s)^{1/\beta}}.$$

PROOF. Note that $\int_{u(t)}^t(ds/f(s)^{1/\beta})\ge\left(\inf_{s\ge u(t)}\frac{s^\beta}{f(s)}\right)^{1/\alpha}\ln\frac{t}{u(t)}$. Therefore,

$$\frac{u(t)/f(u(t))^{1/\beta}}{\int_{u(t)}^t(ds/f(s)^{1/\beta})}\le\frac{1}{\ln(t/u(t))}\left(\frac{\sup_{s\ge u(t)}(f(s)/s^\beta)}{f(u(t))/u(t)^\beta}\right)^{1/\beta}$$

But the left-hand side of this last inequality converges to 0 as $t\to\infty$. □

The last lemma is a well-known fact for specialists. It is a direct consequence of the exponential tail of the vacant cluster distribution for supercritical Bernoulli percolation on $\mathbb{Z}^d$ (see, e.g., Grimmett [10]).

LEMMA 23. *Consider the cubic lattice $\mathbb{Z}^d$ where all sites are independently occupied with probability p and denote by μ the corresponding probability measure. Let $p_c(d)$ be the critical probability. Then if $p>1-p_c(d)$,* μ-a.s. *there is an n_0 such that whenever $n\ge n_0$, the largest vacant cluster within a box $[-n,n]^d$ has $\ln n$ sites.*

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