NEW EXAMPLES OF BALLISTIC RWRE IN THE LOW DISORDER REGIME

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ABSTRACT. We give a new criterion for ballistic behavior of random walks in random environments which are low disorder perturbations of the simple symmetric random walk on $\mathbb{Z}^d$, for $d \geq 2$. This extends the results from 2003 established by Sznitman in [?] and, in particular, allow us to give new examples of ballistic RWREs in dimension $d = 3$ which do not satisfy Kalikow's condition. Essentially, this new criterion states that ballisticity occurs whenever the average local drift of the walk is not too small when compared to the standard deviation of the environment. Its proof relies on applying coarse-graining methods together with a variation of the Azuma-Hoeffding concentration inequality in order to verify the fulfillment of a ballisticity condition by Berger, Drewitz and Ramírez.

1. INTRODUCTION AND MAIN RESULTS

1.1. **Introduction.** The random walk in a random environment is one of the fundamental models describing the movement of a particle in disordered media (see [?, ?] for a comprehensive overview of the model). For walks on $\mathbb{Z}^d$ with $d \geq 2$, few results exist giving explicit formulas for basic associated quantities such as the velocity, asymptotic direction or variance, or conditions characterizing specific long-term behavior such as transience/recurrence, directional transience and ballistic movement. In particular, it is still a widely open problem to explicitly characterize the (law of the) small disorder necessary to produce ballistic behavior whenever added to the jump probabilities of the simple symmetric random walk, see [?, ?]. In this article we focus on this particular question and generalize previously known conditions by exploring the use of refined concentration inequalities which are variations of the well-known Azuma-Hoeffding inequality (see [?] for an extensive review of general concentration inequalities and their applications).

In the case of an i.i.d. random environment in dimension $d \geq 3$, Sznitman was able to derive in [?] conditions on the small disorder which guarantee that the perturbed walk is ballistic. Essentially, he showed that as long as the average local drift in some direction ℓ of the perturbed random walk is not *too* small with respect to ϵ, the L^∞-norm of the perturbation, one has ballisticity in direction ℓ (see Section 1.3 below for a precise statement). In the present article we improve on this by showing that, in fact, one only needs the average drift to be not too small but compared instead to σ, the standard deviation of the environment, which is always a smaller quantity than ϵ (and, furthermore, could potentially be *much* smaller).

As a consequence of this improvement we are able to obtain new examples of RWREs with ballistic behavior. These are of basically of two types: (i) new examples in all dimensions $d \geq 2$ of (small disorder) RWREs satisfying Kalikow's condition for ballisticity; and, most importantly, (ii) new examples in dimension $d = 3$ of (small disorder) RWREs satisfying the polynomial ballisticity condition from [?] and hence (T') but *not* Kalikow's condition (the first examples of type (ii) appear in [?] for all dimensions $d \geq 3$).

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We prove our results by refining some of the estimates on [?] and ideas developed in [?]. A novel and key ingredient in our approach is the use of fine concentration inequalities, namely a variation of the well-known Azuma-Hoefdding inequality which gives a suitable control on the size of martingale differences in terms of their conditional variance.

Before we state our results more precisely and give further details/discussion, let us formally introduce the model and set up the framework to be used throughout the article.

1.2. **The model.** Fix an integer $d \geq 2$ and for each $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ let $|x| := |x_1| + \cdots + |x_d|$ denote its ℓ^1-norm. Let $V := \{x \in \mathbb{Z}^d : |x| = 1\}$ be the set of canonical vectors in $\mathbb{R}^d$ and denote by $\mathcal{P}$ the set of all probability vectors $\vec{p} = (p(e))_{e \in V}$ on V, i.e. $\vec{p} \in [0,1]^V$ such that $\sum_{e \in V} p(e) = 1$. In addition, consider the product space $\Omega := \mathcal{P}^{\mathbb{Z}^d}$ with its Borel σ-algebra, denoted by $\mathcal{B}(\Omega)$. We will call any element $\omega = (\omega(x))_{x \in \mathbb{Z}^d} \in \Omega$ an *environment* (on $\mathbb{Z}^d$). For each $x \in \mathbb{Z}^d$, $\omega(x)$ is a probability vector on V, whose components we denote by $\omega(x,e)$, i.e. $\omega(x) = (\omega(x,e))_{e \in V}$. The *random walk in the environment* ω starting from $x \in \mathbb{Z}^d$ is then defined as the Markov chain $X = (X_n)_{n \in \mathbb{N}_0}$ on the state space $\mathbb{Z}^d$ which starts from x and is given, for each $y \in \mathbb{Z}^d$ and $e \in V$, by the transition probabilities

$$P_{x,\omega}(X_{n+1} = y + e | X_n = y) = \omega(y,e).$$

We denote its law by $P_{x,\omega}$. We assume throughout that the space of environments Ω is endowed with a probability measure $\mathbb{P}$, called the *environmental law*. We shall call $P_{x,\omega}$ the *quenched law* of the random walk, and also refer to the semi-direct product $P_x := \mathbb{P} \otimes P_{x,\omega}$ on $\Omega \times \mathbb{Z}^{\mathbb{N}}$ given by

$$P_x(A \times B) = \int_A P_{x,\omega}(B) d\mathbb{P}(\omega)$$

as the *averaged* or *annealed law* of the random walk. In general, we will call the sequence $(X_n)_{n \in \mathbb{N}_0}$ under the annealed law a *random walk in a random environment* (RWRE) with environmental law $\mathbb{P}$.

Assumptions 1. *Throughout the sequel we shall make the following assumptions on* $\mathbb{P}$*:*

A1. The probability vectors $\omega(x) := (\omega(x,e))_{e \in V}$ *are i.i.d. in* $x \in \mathbb{Z}^d$ *with some common law* μ*. Equivalently,* $\mathbb{P}$ *is the product measure on* Ω *with marginal distribution* μ*.*

A2. Each $\omega(x)$ *is a small perturbation of the weights of the simple symmetric random walk, i.e.*

$$\epsilon = \epsilon(\mu) := 4d \left\| \omega(0) - \left(\frac{1}{2d}, \dots, \frac{1}{2d} \right) \right\|_{L^\infty(\mu)} \in (0,1), \tag{1}$$

where for any random vector $\vec{p} = (p(e))_{e \in V}$ *we define its* $L^\infty(\mu)$*-norm as*

$$\|\vec{p}\|_{L^\infty(\mu)} := \inf\{M > 0 : |p(e)| \leq M \ \mu\text{-almost surely for all } e \in V\}.$$

Observe that, by (A1), (1) implies that there exists an event Ω_ϵ with $\mathbb{P}(\Omega_\epsilon) = 1$ such that on Ω_ϵ one has that

$$\left| \omega(x,e) - \frac{1}{2d} \right| \leq \frac{\epsilon}{4d} \text{ for all } x \in \mathbb{Z}^d \text{ and } e \in V. \tag{2}$$

In particular, $\mathbb{P}$ is *uniformly elliptic* with ellipticity constant

$$\kappa := \frac{1}{4d}, \tag{3}$$

i.e. $\mathbb{P}$-almost surely one has that $\omega(x,e) \geq \kappa$ for all $x \in \mathbb{Z}^d$ and $e \in V$.

The goal of this article is to study transcience/ballisticity properties of X on fixed directions. Recall that, given $\ell \in S^{d-1}$, one says that the random walk X is *transient in direction* ℓ if

$$\lim_{n \to \infty} X_n \cdot \ell = +\infty \quad P_0 - a.s.,$$

and that it is *ballistic in direction* ℓ if it satisfies the stronger condition

$$\liminf_{n\to\infty} \frac{X_n \cdot \ell}{n} > 0 \qquad P_0 - a.s.$$

Any RWRE which is ballistic in some direction ℓ satisfies also a law of large numbers (see [?]), i.e. there exists a deterministic vector $\vec{v} \in \mathbb{R}^d$ with $\vec{v} \cdot \ell > 0$ such that

$$\lim_{n\to+\infty} \frac{X_n}{n} = \vec{v} \qquad P_0 - a.s..$$

This vector $\vec{v}$ is known as the *velocity* of the random walk.

In the sequel we will fix a certain direction, let us say $e_1 := (1,0,\ldots,0) \in S^{d-1}$ for example, and study transience/ballisticity only in this fixed direction. Thus, whenever we speak of transience or ballisticity of X it will be understood that it is with respect to this given direction e_1. However, we point out that all of our results can be adapted for any other particular direction.

1.3. **Main results.** For $x \in \mathbb{Z}^d$ define the *local drift of the RWRE at site* x as the random vector

$$\vec{d}(x) := \sum_{e\in V} \omega(x,e)e.$$

and let λ denote the *average local drift in direction* e_1 by the formula

$$\lambda = \lambda(\mu) := \mathbb{E}(\vec{d}(0)\cdot e_1) = \mathbb{E}(\omega(0,e_1) - \omega(0,-e_1)).$$

A natural question to ask is whether, under Assumptions 1, X is ballistic as soon as $\lambda > 0$ holds. Unfortunately, this is not the case, as one can see from [?] where examples of random walks with diffusive behavior and non-vanishing λ are constructed. However, in [?] it is shown that if $d \geq 3$ and λ is not *too* small with respect to ϵ then the random walk is indeed ballistic and, in fact, satisfies the so-called (T′) condition for ballisticity (see Section below 2 for further details). The validity of (T′) not only implies ballisticity of X, but also a CLT and large deviation controls for the sequence $(\frac{X_n}{n})_{n\in\mathbb{N}}$, see [?]. More precisely, in [?] it is shown that, under Assumptions 1, given any $\eta \in (0,1)$ there exists some $\epsilon_0 = \epsilon_0(\eta,d) \in (0,1)$ such that if $\epsilon \leq \epsilon_0$ and

$$\lambda \geq \begin{cases} \epsilon^{2.5-\eta} & \text{if } d = 3 \\ \epsilon^{3-\eta} & \text{if } d \geq 4 \end{cases} \tag{4}$$

then X satisfies condition (T′) in direction e_1 and is, therefore, ballistic in this direction.

Our goal with this article is to extend the results from [?], and thus to provide new instances of ballistic behavior, by considering the *standard deviation of the environment* ω, defined as

$$\sigma = \sigma(\mu) := \sqrt{\sum_{e\in V} \mathrm{Var}_\mu(\omega(0,e))} = \|\tilde{\delta}(0)\|_{L^2(\mu)},$$

where, for each $x \in \mathbb{Z}^d$, $\tilde{\delta}(x) = (\tilde{\delta}(x,e))_{e\in V}$ denotes the centered vector given by

$$\tilde{\delta}(x,e) := \omega(x,e) - \mathbb{E}(\omega(x,e)). \tag{5}$$

Our first result is an extension of the main result from [?] in the case of dimension $d = 3$.

Theorem 2. *Suppose that $d = 3$ and Assumptions 1 hold. Then, for any given $\eta \in (0,1)$ there exists $\epsilon_0 = \epsilon_0(\eta) \in (0,1)$ such that if $\epsilon \leq \epsilon_0$ and*

$$\lambda \geq \sigma\epsilon^{1.5-\eta} \tag{6}$$

then condition (T′) is satisfied in direction e_1. In particular, X is ballistic in direction e_1.

Note that, since $\sigma \leq \epsilon$ automatically holds by properties of the L^p norms, (6) is indeed weaker than the $\lambda \geq \epsilon^{2.5-\eta}$ condition stated in [?] for $d = 3$. Furthermore, as opposed to Sznitman's original result, (6) shows that it is possible to have ballistic behavior for environments with drift λ as small as one wants with respect to ϵ, as long as their standard deviation σ is accordingly small. Indeed, Theorem 2 suggests that, in order to see ballistic behavior, what is important is that λ is not too small but with respect to σ rather than ϵ.

Next, we investigate the validity of Kalikow's condition for ballisticity. We refer to Section 2 for a precise definition of this condition, but for now recall the reader that it is in general strictly stronger than (T'), and that it implies slightly stronger results on the sequence $(\frac{X_n}{n})_{n\in\mathbb{N}_0}$, see [?]. We have the following result, which is valid for all dimensions $d \geq 2$.

Theorem 3. *Suppose that $d \geq 2$ and Assumptions 1 hold. Then, if*

$$\lambda > 4d\sigma^2(1 + 9\epsilon)$$

Kalikow's condition is satisfied in direction e_1. In particular, X is ballistic in direction e_1.

Note that, whenever $\sigma \ll \epsilon$, Theorem 3 refines [?, Theorem 2], where it was shown that Kalikow's condition holds if $\lambda > \frac{1}{d}\epsilon^2$.

Combining both theorems together with Sznitman's result, we obtain the following corollary which yields the full map of scenarios of ballisticity known so far in this context for all $d \geq 2$.

Corollary 4. *If Assumptions 1 hold then for any $\eta \in (0, 1)$ there exists $\epsilon_0 = \epsilon_0(\eta, d) \in (0, 1)$ such that if $\epsilon \leq \epsilon_0$ and*

$$\lambda \geq \begin{cases} (4d+\eta)\sigma^2 & \text{if } d = 2 \\ \min\{\sigma\epsilon^{1.5-\eta}, (4d+\eta)\sigma^2\} & \text{if } d = 3 \\ \min\{\epsilon^{3-\eta}, (4d+\eta)\sigma^2\} & \text{if } d \geq 4 \end{cases}$$

then X satisfies condition (T') in direction e_1. In particular, X is ballistic in direction e_1. Furthermore, if

$$\lambda \geq (4d + \eta)\sigma^2$$

then in fact X satisfies Kalikow's stronger ballisticity condition.

Our third and last result is a variation of [?, Theorem 5.1] which gives scenarios under which Kalikow's condition fails to hold.

Theorem 5. *Suppose that $d \geq 2$. Then, for any given $\rho \in (0, 1]$ there exists $\epsilon_0 = \epsilon_0(d, \rho) > 0$ such that if the environment site distribution μ satisfies the following conditions:*

- *K1. $\epsilon(\mu) \leq \epsilon_0$,*
- *K2. If $d \geq 3$ (respectively, if $d = 2$) then μ is invariant under all rotations (respectively, isometries) which preserve $\mathbb{Z}^d$ and e_1,*
- *K3. $Var_\mu(\omega(0, e_1)) = Var_\mu(\omega(0, -e_1))$,*
- *K4. $Var_\mu(\omega(0, e_1)) - Cov_\mu(\omega(0, e_1), \omega(0, -e_1)) \geq \rho\sigma^2 > 0$,*
- *K5. $\rho\sigma^2 > 32d^2\lambda \geq 0$,*

then Kalikow's condition fails to hold in all directions $\ell \in S^{d-1}$.

Theorem 5 is important mainly for two reasons. First, it shows that the condition $\lambda \geq c_1(d)\sigma^2$ from Theorem 3 is essentially optimal for the validity of Kalikow's condition, in the sense that there exists $c_2(d) > 0$ such that if $\lambda \leq c_2(d)\sigma^2$ then one can already find examples of RWREs which do not satisfy Kalikow's condition. But, most importantly, in combination with Theorem 2 it also shows new examples (apart from those already given in [?]) of ballistic walks for $d = 3$ which satisfy (T') but not Kalikow's condition. Indeed, as a direct consequence of Theorems 2-5, we obtain the following result.

Corollary 6. *Given any $\epsilon_0 > 0$ one can construct a RWRE in dimension $d = 3$ verifying Assumptions 1 and such that:*

i. $\epsilon \leq \epsilon_0$ and $\lambda \leq \epsilon^{2.5}$ (so that the hypothesis (4) *from* [**?**] *is not satisfied)*
ii. (T′) is verified in direction e_1 but Kalikow's condition fails in all directions.

One such example can be constructed as in [**?**] by first fixing $\rho \in (0,1]$ and by setting μ to be the law of the random probability weight $\omega(0)$ on $\mathcal{P}$ given, for each $e \in V$, by

$$\omega(0,e) = p(e) + \frac{\lambda}{2} e \cdot e_1,$$

where $\lambda > 0$ is a constant to be specified later and $\vec{p} = (p(e))_{e\in V}$ is a random probability vector with distribution $\hat{\mu}$ on $\mathcal{P}$ satisfying:

- $\hat{\mu}$ is isotropic, i.e. invariant under rotations of $\mathbb{R}^3$ which preserve $\mathbb{Z}^3$,
- $\mathrm{Var}_{\hat{\mu}}(p(e_1)) - \mathrm{Cov}_{\hat{\mu}}(p(e_1), p(-e_1)) \geq \rho\sigma(\hat{\mu}) > 0$.

It is simple to check that, for μ defined in this way, given any $\eta \in (0, \frac{1}{2})$ one can choose constants $k_1, k_2, k_3 > 0$ (depending only on η, ϵ_0 and ρ) in such a way if $\hat{\mu}$ is taken to also satisfy

$$\epsilon(\hat{\mu}) \leq k_1\epsilon_0 \qquad \text{and} \qquad k_2(\epsilon(\hat{\mu}))^{1.5-\eta} \leq \sigma(\hat{\mu}) \leq k_3(\epsilon(\hat{\mu}))^{1+\eta}$$

then there exists a nonempty interval of values of λ for which the associated walk satisfies (C1) and the hypotheses of both Theorems 2 and 5, so that (C2) also holds. We omit the details.

The proof of Theorem 2 is an adaptation of the approach developed in [**?**], which consists of verifying the effective criterion given in [**?**] for the validity of (T′). Instead of this criterion, we will verify that the polynomial condition from [**?**] holds, which is equivalent to (T′) and more convenient for our purposes. Crucial to this verification are estimates on the average value and size of fluctuations of the Green's operator of the RWRE killed upon exiting a slab of diameter proportional to ϵ^{-1}. In [**?**] it is shown that these quantities can be suitably controlled by ϵ, the L^∞-norm of the perturbation. However, to obtain our results we will need to show that these can still be controlled by σ, the L^2-norm of the perturbed environment, which is a smaller quantity. This task requires redoing some of the estimates from [**?**] but now in L^2 (and thus refining them from their original L^∞-form), and introducing new tools to the analysis, like the generalized version of Azuma-Hoeffding's inequality used in Lemma 9. The proof of Theorem 3 is inspired by that of [**?**, Theorem 2], and is based on the version of Kalikow's formula proved in [**?**] and a careful application of Kalikow's criteria for ballisticity. Finally, Theorem 5 follows from adapting the argument in [**?**, Theorem 5.1] to the L^2-setting.

Finally, we mention that the ideas here can also be used to extend the bounds on the velocity developed in [**?**]. Indeed, the same approach used here goes through in [**?**] to show that:

B1. In dimension $d = 3$, given two quantities $\delta < \eta \in (0,1)$ there exists $\epsilon_0 \in (0,1)$ and $c_0 > 0$ depending on δ and η such that if $\epsilon \leq \epsilon_0$ and $\lambda \geq \sigma\epsilon^{1.5-\eta}$ then X is ballistic in direction e_1 with velocity $\vec{v}$ satisfying

$$0 < \vec{v} \cdot e_1 \leq \lambda + c_0\sigma\epsilon^{1.5-\delta}.$$

B2. For all dimensions $d \geq 2$, given any $\eta \in (0,1)$ there exists $\epsilon_0 = \epsilon_0(d,\eta) \in (0,1)$ such that if $\epsilon \leq \epsilon_0$ and $\lambda > (4d+\eta)\sigma^2$ then X is ballistic in direction e_1 with velocity $\vec{v}$ satisfying

$$|\vec{v} - \lambda| \leq (4d+\eta)\sigma^2.$$

However, since this extension of the results [**?**] is completely analogous to the one of [**?**] we shall do here, we will omit the details of how to prove (B1-B2). The reader interested in a proof will know how to proceed from [**?**] after reading Sections 3 and 4 below.

The remainder of the article is organized as follows. In Section 2 we introduce general notation and establish a few preliminary facts about the RWRE model, including the precise definitions of

Kalikow's and (T′) conditions for ballisticity. Afterwards, Sections 3,4 and 5 are each devoted to the proofs of Theorems 2, 3 and 5, respectively.

2. Preliminaries

In this section we introduce the general notation to be used throughout the article, as well as review some preliminary notions about RWREs that we shall require for the proofs.

2.1. **General notation.** Given any subset $B \subset \mathbb{Z}^d$, we define its (outer) boundary as

$$\partial B := \{x \in \mathbb{Z}^d - B : |x - y| = 1 \text{ for some } y \in B\}.$$

and the first exit time of the random walk from B as

$$T_B := \inf\{n \geq 0 : X_n \notin B\}.$$

Furthermore, for $L \in \mathbb{N}$ we define the L-slab in direction e_1

$$U := \left\{ y \in \mathbb{Z}^d : -L \leq y \cdot e_1 < L \right\}, \tag{7}$$

where we are consciously omitting the dependence on L from the notation U for convenience. Finally, for each $M \in \mathbb{N}$ we also define the box

$$B_M := \left\{ y \in \mathbb{Z}^d : -\frac{M}{2} < y \cdot e_1 < M \text{ and } |y \cdot e_i| < 25M^3 \text{ for } 2 \leq i \leq d \right\} \tag{8}$$

together with its *frontal side*

$$\partial_+ B_M := \{y \in \partial B_M : y \cdot e_1 \geq M\},$$

and its *middle-frontal part*

$$B^*_M := \left\{ y \in B_M : \frac{M}{2} \leq y \cdot e_1 < M, \ |y \cdot e_i| < M^3 \text{ for } 2 \leq i \leq d \right\}.$$

2.2. **Green's functions and operators.** Let us now introduce some notation we shall use related to the Green's functions of the RWRE and of the simple symmetric random walk (SSRW).

Given a subset $B \subseteq \mathbb{Z}^d$, the Green's functions of the RWRE and SSRW killed upon exiting B are respectively defined for $x, y \in B \cup \partial B$ as

$$g_B(x, y, \omega) := E_{x,\omega}\left(\sum_{n=0}^{T_B} \mathbb{1}_{\{X_n = y\}} \right) \qquad \text{and} \qquad g_{0,B}(x, y) := g_B(x, y, \omega_0),$$

where ω_0 is the corresponding weight of the SSRW, given for all $x \in \mathbb{Z}^d$ and $e \in V$ by

$$\omega_0(x, e) = \frac{1}{2d}.$$

Furthermore, if $\omega \in \Omega$ is such that $E_{x,\omega}(T_B) < +\infty$ for all $x \in B$, we can define the corresponding Green's operator on $L^\infty(B)$ by the formula

$$G_B[f](x, \omega) := \sum_{y \in B} g_B(x, y, \omega) f(y) = E_{x,\omega}\left(\sum_{n=0}^{T_B - 1} f(X_n) \right).$$

Notice that g_B, and therefore also G_B, depends on ω only though its restriction $\omega|_B$ to B. Finally, it is straightforward to check that if U is the slab defined in (7) then both g_U and G_U are well-defined for all environments $\omega \in \Omega_\epsilon$.

2.3. **Ballisticity conditions.** We now recall the two conditions for ballisticity we shall work with: condition (T′), originally introduced by Sznitman in [**?**], and Kalikow's condition from [**?**]. For simplicity, instead of giving the original formulation of (T′) by Sznitman, we will state here the polynomial condition (P) for ballisticity, which was shown in [**?**] to be equivalent to (T′). We warn the reader that there exist also other ballisticity conditions, such as condition (T), but we shall not use them in the sequel and hence they will not be discussed here. The interested reader is invited to consult a more detailed exposition about such conditions in [**?**]. For simplicity, we consider only ballisticity in direction e_1.

Condition (P). We will say that condition (P) is satisfied (in direction e_1) if there exists $M \geq M_0$ such that

$$\sup_{x \in B_M^*} P_x\left(X_{T_{B_M}} \notin \partial_+ B_M\right) \leq \frac{1}{M^{15d+5}}, \tag{9}$$

where

$$M_0 := \exp\{100 + 4d(\log\kappa)^2\}$$

and κ is the uniform ellipticity constant of the RWRE, which in our case can be taken to be $\kappa = \frac{1}{4d}$. We mention once again that in [**?**] it was shown that (P) is equivalent to (T′).

Introducing Kalikow's walk. Given a nonempty connected strict subset $B \subsetneq \mathbb{Z}^d$, for each $x \in B$ we define *Kalikow's walk* on B (starting from x) as the random walk starting from x which is killed upon exiting B and has transition probabilities determined by the environment $\omega_B \in \mathcal{P}^B$ given by

$$\omega_B^x(y,e) := \frac{\mathbb{E}(g_B(x,y,\omega)\omega(y,e))}{\mathbb{E}(g_B(x,y,\omega))}. \tag{10}$$

It is straightforward to check that by the uniform ellipticity of $\mathbb{P}$ we have $0 < \mathbb{E}(g_B(x,y,\omega)) < +\infty$ for all $y \in B$, so that the environment ω_B^x is well-defined. In accordance with our present notation, we will denote the law of Kalikow's walk on B by P_{x,ω_B^x} and its Green's function by $g_B(x,\cdot,\omega_B^x)$. Given a direction $\ell \in S^{d-1}$, we will say that Kalikow's condition holds (in direction ℓ) if

$$\varepsilon_K(\ell) := \inf\{\vec{d}_{B,0}(y)\cdot\ell : B \subsetneq \mathbb{Z}^d \text{ connected with } 0, y \in B\} > 0, \tag{11}$$

where $\vec{d}_{B,0}$ denotes the drift of Kalikow's walk in B at 0 defined as

$$\vec{d}_{B,0}(y) := \sum_{e \in V} \omega_B^x(y,e)e.$$

If Kalikow's condition holds in some direction $\ell \in S^{d-1}$ then the walk X is ballistic in direction ℓ, see [**?**] for further details.

3. PROOF OF THEOREM 2

We will divide the proof of Theorem 2 into three parts, each carried out in a separate subsection. Throughout this section we shall assume that the environmental law $\mathbb{P}$ satisfies Assumptions 1.

3.1. **Lower bound on $\mathbb{E}(G_U[\vec{d}\cdot e_1](0))$.** The first step is to give a lower bound on $\mathbb{E}(G_U[\vec{d}\cdot e_1](0))$, the expectation of Green's operator on the local drift at 0. The precise bound we need is contained in the following proposition, which is a generalization of [**?**, Proposition 3.1].

Proposition 7. *Suppose that $d \geq 3$ and Assumptions 1 hold. Then, there exist constants $c_1, c_2 > 0$ depending only on d such that if $\epsilon \leq \frac{1}{8d}$ then for any integer $L \geq 2$ satisfying $\epsilon L < c_1$ one has that*

$$\mathbb{E}(G_U[\vec{d}\cdot e_1](0)) \geq \frac{2}{5}d\lambda L^2 \tag{12}$$

provided that

$$\lambda \geq c_2\sigma^2 \left(\epsilon \log L + \frac{1}{L} \right). \tag{13}$$

Remark 8. *Since $\sigma \leq \epsilon$ by standard properties of the L^p norms, Proposition 7 is indeed a generalization of* [**?**, Proposition 3.1] *because the required lower bound on the drift λ in* (13) *is now smaller.*

To begin the proof of Proposition 7, for $x \in \mathbb{Z}^d$, $e \in V$ and $\omega \in \Omega_\epsilon$ let us set

$$\tilde{\delta}(x,e) := \omega(x,e) - \mathbb{E}(\omega(x,e)) \qquad \text{and} \qquad \tilde{d}(x,\omega) := \vec{d}(x,\omega) - \mathbb{E}(\vec{d}(x,\cdot)).$$

Following the proof of [**?**, Proposition 3.1], one can check that (12) will follow if we show that whenever (13) holds one has

$$|\mathbb{E}(G_U[\tilde{d} \cdot e_1](0))| \leq \frac{2}{5} d\lambda L^2.$$

Now, by the proof of [**?**, Proposition 3.1], we can write

$$G_U[\tilde{d} \cdot e_1](0) = A + B + C$$

with $\mathbb{E}(A) = 0$,

$$B = -\sum_{x \in U} \frac{P_{0,\omega}(H_x < T_U)}{P_{x,\bar{\omega}}(\tilde{H}_x > T_U)} (\tilde{d}(x) \cdot e_1) \sum_{e \in V} \tilde{\delta}(x,e) \frac{P_{x+e,\omega}(H_x > T_U)}{P_{x,\bar{\omega}}(\tilde{H}_x > T_U)}$$

and

$$|C| \leq 2\frac{\epsilon}{d} \sum_{x \in U} \frac{P_{0,\omega}(H_x < T_U)}{P_{x,\bar{\omega}}(\tilde{H}_x > T_U)} \left(\sum_{e \in V} |\tilde{\delta}(x,e)| \frac{P_{x+e,\omega}(H_x > T_U)}{P_{x,\bar{\omega}}(\tilde{H}_x > T_U)} \right)^2,$$

where $\bar{\omega} \in \Omega_\epsilon$ is the environment given by the formula

$$\bar{\omega}(y,e) := \begin{cases} \omega(y,e) & \text{if } y \neq x \\ \mathbb{E}(\omega(x,e)) & \text{if } y = x. \end{cases}$$

Then, let us show first that $\mathbb{E}(|C|) \leq \frac{1}{5} d\lambda L^2$ holds if c_2 is chosen appropriately large. Indeed, from the inequality

$$P_{x+e,\omega}(H_x > T_U) \leq 4d P_{x,\bar{\omega}}(\tilde{H}_x > T_U) \quad \forall\, e \in V \tag{14}$$

we obtain

$$\left(\sum_{e \in V} |\tilde{\delta}(x,e)| \frac{P_{x+e,\omega}(H_x > T_U)}{P_{x,\bar{\omega}}(\tilde{H}_x > T_U)} \right)^2 \leq (2d)^2 (4d)^2 \sum_{e \in V} |\tilde{\delta}(x,e)|^2 = 64 d^4 \|\tilde{\delta}(x)\|_2^2,$$

so that

$$\mathbb{E}(|C|) \leq 128 d^3 \epsilon \sum_{x \in U} \mathbb{E}\left(\frac{P_{0,\omega}(H_x < T_U)}{P_{x,\bar{\omega}}(\tilde{H}_x > T_U)} \|\tilde{\delta}(x)\|_2^2 \right).$$

Now, since $P_{0,\omega}(H_x < T_U)$ and $P_{x,\bar{\omega}}(\tilde{H}_x > T_U)$ are independent of $\tilde{\delta}(x)$, it follows that

$$\begin{aligned} \mathbb{E}(|C|) &\leq 128 d^3 \epsilon \sigma^2 \sum_{x \in U} \mathbb{E}\left(\frac{P_{0,\omega}(H_x < T_U)}{P_{x,\bar{\omega}}(\tilde{H}_x > T_U)} \right) \\ &\leq 512 d^4 \epsilon \sigma^2 \sum_{x \in U} \mathbb{E}(g_U(0,x)) \\ &= 512 d^4 \epsilon \sigma^2 \mathbb{E}(G_U[\mathbb{1}](0)), \end{aligned}$$

where, to obtain the second inequality, we have used the fact that

$$P_{x,\omega}(\tilde{H}_x > T_U) \leq 4d P_{x,\bar{\omega}}(\tilde{H}_x > T_U) \tag{15}$$

which follows at once from (14). Finally, recalling that $\mathbb{E}(G_U[\mathbb{1}](0)) \leq \frac{4}{3}dL^2$ whenever $\epsilon L < \frac{3}{4}$ by [?, Proposition 2.2], we conclude that

$$\mathbb{E}(|C|) \leq \frac{2048}{3}\epsilon d^5\sigma^2 L^2.$$

Hence, by taking $c_2 > 0$ sufficiently large so that $\frac{1}{5}d\lambda \geq \frac{2048}{3}\epsilon d^5\sigma^2$ when (13) holds, we see that $\mathbb{E}(|C|) \leq \frac{1}{5}d\lambda L^2$ is satisfied as claimed. Thus, in order to conclude the proof it will suffice to show that

$$|\mathbb{E}(B)| \leq \frac{1}{5}d\lambda L^2. \tag{16}$$

To this end, we first notice that, since for all $x \in U$ we have

$$\sum_{e\in V} \tilde{\delta}(x,e) = 0,$$

B remains unchanged if we replace the term $P_{x+e,\omega}(H_x > T_U)$ in the innermost sum with

$$P_{x+e,\omega}(H_x < T_U) - P_{x+e_1,\omega}(H_x < T_U).$$

Since the latter term is independent of $\omega(x)$, we obtain that

$$\mathbb{E}(|B|) \leq \sum_{x\in U, e\in V} \mathbb{E}\left(\frac{P_{0,\omega}(H_x < T_U)}{P^2_{x,\bar{\omega}}(\tilde{H}_x > T_U)}|P_{x+e,\omega}(H_x < T_U) - P_{x+e_1,\omega}(H_x < T_U)||\tilde{d}(x)\cdot e_1||\tilde{\delta}(x,e)|\right)$$

$$= \sum_{x\in U, e\in V} \mathbb{E}\left(\frac{P_{0,\omega}(H_x < T_U)}{P^2_{x,\bar{\omega}}(\tilde{H}_x > T_U)}|P_{x+e,\omega}(H_x < T_U) - P_{x+e_1,\omega}(H_x < T_U)|\right)\mathbb{E}(|\tilde{d}(x)\cdot e_1||\tilde{\delta}(x,e)|). \tag{17}$$

Now, on the one hand we have $|\tilde{d}(x)\cdot e_1| = |\tilde{\delta}(x,e_1) - \tilde{\delta}(x,-e_1)| \leq |\tilde{\delta}(x,e_1)| + |\tilde{\delta}(x,-e_1)|$ so that by the Cauchy-Schwarz inequality

$$\mathbb{E}(|\tilde{d}(x)\cdot e_1||\tilde{\delta}(x,e)|) \leq 2\sigma^2$$

for all $x \in U, e \in V$. On the other hand, by (15) and the fact that for all $y, x \in U$

$$g_U(y,x,\omega) = \frac{P_{y,\omega}(H_x < T_U)}{P_{x,\omega}(\tilde{H}_x > T_U)}$$

we have that the leftmost expectation in (17) can be bounded from above by

$$(4d)^2 g_U(0,x,\omega)|g_U(x+e,x,\omega) - g_U(x+e_1,x,\omega)|$$

so that in order to show (16) it will suffice to find some constant $c = c(d) > 0$ so that

$$\sup_{e\in V,\omega\in\Omega_\epsilon}\sum_{x\in U} g_U(0,x,\omega)|g_U(x+e,x,\omega) - g_U(x+e_1,x,\omega)| \leq c\left(\epsilon\log L + \frac{1}{L}\right)L^2 \tag{18}$$

and then choose the constant c_2 from the statement of the proposition sufficiently large. But (18) can be shown exactly as in the proof of [?, Proposition 3.1]. We omit the details.

3.2. **Controlling the fluctuations of $G_U[\vec{d}\cdot e_1](0)$.** The next step of the proof is to extend the result in [?, Proposition 3.2], which gives control on the fluctuations of $G_U[\vec{d}\cdot e_1](0)$, to the L^2-setting. The key to this extension is the following lemma, a variation of Azuma-Hoeffding's inequality incorporating the conditional variance of the underlying martingale difference.

Lemma 9. *Let $(F_n)_{n\in\mathbb{N}_0}$ be a martingale with respect to some filtration $(\mathcal{G}_n)_{n\in\mathbb{N}_0}$ satisfying $\mathcal{G}_0 = \{\emptyset, \Omega\}$. If there exist a constant $b > 0$ and a sequence $(v_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_{\geq 0}$ such that for every $n \in \mathbb{N}$*

$$|F_n - F_{n-1}| \leq b \qquad \text{and} \qquad \mathbb{E}((F_n - F_{n-1})^2 | \mathcal{G}_{n-1}) \leq v_n^2,$$

then for any $u > 0$ one has that

$$\max\left\{P\left(\liminf_{n\to+\infty} F_n > F_0 + u\right), P\left(\liminf_{n\to+\infty} F_n < F_0 - u\right)\right\} \leq \exp\left\{-\frac{1}{2}\cdot\frac{u^2}{\sum_{n\in\mathbb{N}} v_n^2 + \frac{1}{3}ub}\right\}.$$

Proof. Call $F_\infty := \liminf_{n\to+\infty} F_n$. Upon noticing that $P(F_\infty - F_0 > u)$ can be bounded from above by

$$\limsup_{n\to+\infty} P\left(F_k - F_0 > u\,,\ \sum_{n=1}^{k} \mathbb{E}((F_n - F_{n-1})^2|\mathcal{G}_n) \leq \sum_{n\in\mathbb{N}} v_n^2 \text{ for some } k \in [1,n]\right),$$

from Theorem 2.1 and Remark 2.1 on [**?**] one obtains the bound

$$P(F_\infty - F_0 > u) \leq \exp\left\{-\frac{1}{2}\cdot\frac{u^2}{\sum_{n\in\mathbb{N}} v_n^2 + \frac{1}{3}ub}\right\}.$$

The remaining bound follows by symmetry, working with $-F_n$ instead of F_n. □

The extension of [**?**, Proposition 3.2] we will require for our purposes is the following.

Proposition 10. *Suppose that $d \geq 3$. Then, for any fixed $\alpha \in [0,1)$ there exist constants $c_3, c_4 > 0$ depending only on d and α such that, for any integer $L \geq 2$ satisfying $\epsilon L < c_3$, one has that*

$$\mathbb{P}\left(\left|G_U[\vec{d}\cdot e_1](0) - \mathbb{E}(G_U[\vec{d}\cdot e_1](0))\right| \geq u\right) \leq 2\exp\left\{-\frac{1}{c_4}\cdot\frac{u^2}{c_{\alpha,L}\sigma^2 + u\epsilon}\right\} \tag{19}$$

for all $u \geq 0$, where

$$c_{\alpha,L} := \begin{cases} L^{1+(2(1-\alpha)/(2-\alpha))} & \text{if } d = 3 \\ L^{(4(1-\alpha)/(2-\alpha))} & \text{if } d = 4 \\ 1 & \text{if } d \geq 5 \text{ and } \alpha \geq \frac{4}{5}. \end{cases}$$

Remark 11. *Since we will be interested in applying* (19) *only to $u \gg \epsilon$, we can assume that the term $\exp\{-\frac{1}{c_4}\cdot\frac{u^2}{c_{\alpha,L}\sigma^2}\}$ is the one dominating the right-hand side of* (19)*. Therefore, since $\sigma \leq \epsilon$, we see that* (19) *is indeed a sharper estimate than the one on* [**?**, Proposition 3.2] *(at least for $u \gg \epsilon$).*

Proof. We follow the proof of [**?**, Proposition 3.2], applying the martingale method introduced therein. First, let us enumerate the elements of U as $U := \{x_n : n \in \mathbb{N}\}$. Now, define the filtration

$$\mathcal{G}_n := \begin{cases} \sigma(\omega(x_1), \ldots, \omega(x_n)) & \text{if } n \geq 1 \\ \{\emptyset, \Omega\} & \text{if } n = 0 \end{cases}$$

and also the bounded (see [**?**, Proposition 2.2] for a justification) $\mathcal{G}_n$-martingale $(F_n)_{n\in\mathbb{N}_0}$ given for each $n \in \mathbb{N}_0$ by

$$F_n := \mathbb{E}(G_U[\vec{d}\cdot e_1](0)|\mathcal{G}_n).$$

Since $F_\infty := \lim_{n\to+\infty} F_n = G_U[\vec{d}\cdot e_1](0)$ and $F_0 = \mathbb{E}(G_U[\vec{d}\cdot e_1](0))$, by Lemma 9 we get

$$\mathbb{P}\left[\left|G_U[\vec{d}(\cdot,\omega)\cdot e_1](0) - \mathbb{E}(G_U[\vec{d}\cdot e_1](0))\right| \geq u\right] \leq 2\exp\left\{-\frac{1}{2}\cdot\frac{u^2}{\sum_{n\in\mathbb{N}} v_n^2 + ub}\right\}$$

for all b and $(v_n)_{n\in\mathbb{N}}$ with $|F_n - F_{n-1}| \leq b$ and $\mathbb{E}((F_n - F_{n-1})^2|\mathcal{G}_{n-1}) \leq v_n^2$ for every n.

Thus, let us find such b and v_n. To this end, for each $n \in \mathbb{N}$ and all environments $\omega, \omega' \in \Omega_\epsilon$ with $\omega \equiv \omega'$ off x_n, i.e. which coincide at every x_i with $i \neq n$, let us define

$$\Gamma_n(\omega,\omega') := G_U[\vec{d}\cdot e_1](0,\omega) - G_U[\vec{d}\cdot e_1](0,\omega').$$

If μ denotes the single site distribution of $\omega_U = (\omega(x_i))_{i\in\mathbb{N}}$ under $\mathbb{P}$, then it is simple to see that

$$F_n(\omega) - F_{n-1}(\omega) = \int \left[\int \Gamma_n(\omega,\omega') d\mu(\omega'(x_n)) \right] d\mu^{\otimes(i>n)}(\omega_{(i>n)}), \tag{20}$$

□

where $\omega_{(i>n)} := (\omega(x_i))_{i>n}$, $\mu^{\otimes(i>n)}$ denotes its distribution under the law $\mathbb{P}$ and $\omega' \equiv \omega$ off x_n. Furthermore, it follows from the proof of [**?**, Proposition 3.2] that for $L \geq 2$ as in the statement of the proposition one has

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