WEAK GIBBS MEASURES AS GIBBS MEASURES FOR ASYMPTOTICALLY ADDITIVE SEQUENCES

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Abstract. In this note we prove that every weak Gibbs measure for an asymptotically additive sequences is a Gibbs measure for another asymptotically additive sequence. In particular, a weak Gibbs measure for a continuous potential is a Gibbs measure for an asymptotically additive sequence. This allows, for example, to apply recent results on dimension theory of asymptotically additive sequences to study multifractal analysis for weak Gibbs measure for continuous potentials.

1. Introduction and Preliminaries

Gibbs measures have played a prominent role in ergodic theory since the definition was brought from statistical mechanics into dynamical systems (see for example [Bow, Si]). Existence of Gibbs measures usually requires strong forms of hyperbolicity on the system and of regularity on the potential. In her study of equilibrium measures for intermittent interval maps, Yuri [Y1, Y2, Y3] introduced the notion of weak Gibbs measure. It turns out that these measures, which generalize the classical notion of Gibbs measures, exist under meagre regularity assumptions on the potential and for a wider class of dynamical systems. Technically, the main difference between the two notions is that for Gibbs measures we have a uniform control on the measure of dynamical balls while for weak Gibbs measures this control is not uniform (see Section 1.1 for precise statements). This circle of ideas has been generalized to settings in which instead of considering a single potential we consider a sequence of potentials. This theory, usually called non-additive thermodynamic formalism, was introduced by Falconer [Fa1] with the purpose of studying dimension theory of non-conformal systems. We stress that it is also a well suited theory to study products of matrices. The purpose of the present note is to show that a weak Gibbs measure for a potential is actually a Gibbs measure for a sequence of potentials. That point of view allows for the use of machinery developed to study Gibbs measures for sequences of potentials in the study of weak Gibbs measures for a single potential. As an example of the possibilities that are opened with this viewpoint we prove a variational principle for higher order multifractal analysis of weak Gibbs measures.

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1.1. **Weak Gibbs measures.** This section is devoted to defining the notion of weak Gibbs measure for compact symbolic spaces and to briefly discuss conditions that ensure both, their existence and that they do not have atoms.

Let (Σ, σ) be a one-sided Markov shift defined over a finite alphabet $\mathcal{A}$. This means that there exists a matrix $T = (t_{ij})_{\mathcal{A}\times\mathcal{A}}$ of zeros and ones such that

$$\Sigma = \left\{\omega \in \mathcal{A}^{\mathbb{N}} : t_{\omega_i \omega_{i+1}} = 1 \text{ for every } i \in \mathbb{N}\right\}.$$

The *shift map* $\sigma : \Sigma \to \Sigma$ is defined by $\sigma(\omega_1\omega_2\dots) = (\omega_2\omega_3\dots)$. We assume that (Σ, σ) is topologically mixing, which in this setting means that there exists a positive integer $l \in \mathbb{N}$ such that all the entries of the matrix T^l are strictly positive. The set

$$C_{i_1\cdots i_n} := \{\omega \in \Sigma : \omega_j = i_j \text{ for } 1 \leq j \leq n\}$$

is called *cylinder* of length n. The space Σ endowed with the topology generated by cylinder sets is a compact space. Moreover, the metric $d : \Sigma \times \Sigma \to \mathbb{R}$ defined by $d(\omega, \kappa) := \sum_{i=1}^{\infty} \frac{|\omega_i - \kappa_i|}{2^i}$ induces the same topology as the one generated by the cylinder sets.

In order to understand the geometric or dynamical properties of a probability measure μ on Σ it is of great importance to have good estimates on the measure of the cylinder sets. This simple remark is captured in the definition of Gibbs measure (see [Bow, Chapter 1]) which was later generalized by Yuri [Y1, Y2, Y3] in the following sense.

Definition 1.1. A probability measure μ is called a *weak Gibbs* measure for the potential $\phi : \Sigma \to \mathbb{R}$ if there exists $P \in \mathbb{R}$ and a sequence of positive real numbers $K(n)$ satisfying

$$\lim_{n\to\infty} \frac{\log K(n)}{n} = 0,$$

such that, for every $n \in \mathbb{N}$, every cylinder $C_{i_1\dots i_n}$ and every $\omega \in C_{i_1\dots i_n}$ we have

$$\frac{1}{K(n)} \leq \frac{\mu(C_{i_1\dots i_n})}{\exp(S_n\phi(\omega) - nP)} \leq K(n),$$

where $S_n\phi(\omega) := \sum_{j=0}^{n-1} \phi(\sigma^j\omega)$.

If for every $n \in \mathbb{N}$ we have that $K(n) = K$ we recover the classical notion of a Gibbs measure (see [Bow, Chapter 1]). It turns out that weak Gibbs measures exists under very mild assumptions on the potential and that the number P can be characterized in thermodynamic terms. Let $\phi : \Sigma \to \mathbb{R}$, we define its *n-th variation* by

$$\text{var}_n(\phi) := \sup_{C\in Z_n} \sup_{\omega,\kappa\in C} \{|\phi(\omega) - \phi(\kappa)|\},$$

where Z_n denotes the set of all cylinders of length n. Denote by $\eta_\phi(n) := \text{var}_n(S_n\phi)$. The class of *medium varying functions* is that of bounded, measurable, real-valued functions $\phi : \Sigma \to \mathbb{R}$ such that

$$\lim_{n\to\infty} \frac{\eta_\phi(n)}{n} = 0.$$

Note that every continuous function belongs to this class. Let $\phi : \Sigma \to \mathbb{R}$ be a medium varying function, the *topological pressure* of ϕ is defined by

$$P(\phi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{C\in Z_n} \exp\left(\sup_{\omega\in C} S_n\phi(\omega))\right).$$

Note that the above limit exists and is finite (see [Ke, Lemma 1]). The following result due to Kesseböhmer [Ke, Section 2] establishes the existence of weak Gibbs measure for medium varying functions.

Proposition 1.1 (Kesseböhmer)**.** *For every medium varying function $\phi : \Sigma \to \mathbb{R}$ there exists a weak Gibbs measure μ. Moreover, in definition* 1.1*, the constants can be chosen so that*

$$K(n) = \exp(\eta_\phi(n)),$$

and the value of P is equal to $P(\phi)$.

Remark 1.1. Kesseböhmer also showed that a weak Gibbs measure μ for the potential ϕ is fully supported and if in addition we have that $\sup \phi < P(\phi)$ then the measure μ is atom free. Inoquio-Renteria and Rivera-Letelier in [IR-RL, Proposition 3.1] showed that if $\phi : \Sigma \to \mathbb{R}$ is a continuous potential satisfying the following condition: there exists an integer $n \geq 1$ such that

$$\sup_{\omega \in \Sigma} \frac{1}{n} S_n(\phi(\omega)) < P(\phi), \tag{1}$$

then ϕ is cohomologous to a continuous potential $\overline{\phi}$ satisfying

$$\sup_{\omega \in \Sigma} \overline{\phi}(\omega) < P(\overline{\phi}).$$

In particular, if a continuous potential ϕ satisfies equation (1) then the weak Gibbs measure for ϕ is non-atomic. Let us stress that there exists continuous potentials satisfying equation (1) for some $n > 1$ and not for $n = 1$ (see [IR-RL, Remark 3.3]).

Kesseböhmer [Ke] in fact proved that conformal measures corresponding to medium varying function are indeed weak Gibbs measures. In general these are not invariant measures. However, if the potential ϕ is a g-function (see [K] for a precise definition) then the conformal measure is invariant. Also, if we assume more regularity on the potential ϕ, for example potentials belonging to the Bowen class (see [W2, p.329] for a precise definition) then the potential is cohomologous to a g-function (see [W2, Proof of Theorem 4.5]) and thus has an invariant weak Gibbs measure.

1.2. **Non-additive thermodynamic formalism.** Motivated from the study of dimension theory of non-conformal dynamical systems, Falconer [Fa1] introduced a non-additive generalization of thermodynamic formalism. This theory has been considerably developed over the last years (see [B4] for a general account on the theory). In this non-additive setting, the topological pressure of a continuous function is replaced by the topological pressure of a sequence of continuous functions. Different types of additivity assumptions are usually made on the sequence. For example, we say that a sequence $\Phi := (\phi_n)_n$ of continuous functions $\phi_n : \Sigma \to \mathbb{R}$ is *almost additive* if there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$ and every $\omega \in \Sigma$ we have

$$-C + \phi_n(\omega) + \phi_m(\sigma^n \omega) \leq \phi_{n+m}(\omega) \leq C + \phi_n(\omega) + \phi_m(\sigma^n \omega).$$

Under this assumption thermodynamic formalism has been thoroughly studied (see [B2, IY, M]).

Another type of additivity assumption was introduced by Feng and Huang [FeH], a sequence of continuous functions $\Phi := (\phi_n)_n$ is *asymptotically additive* on Σ if for every $\epsilon > 0$ there exists a continuous function ρ_ϵ such that

$$\limsup_{n\to\infty} \frac{1}{n}\|\phi_n - S_n\rho_\epsilon\| < \epsilon, \tag{2}$$

where $\|\cdot\|$ is the supremum norm. Every almost additive sequence is asymptotically additive (see, for example, [ZZC, Proposition 2.1]) but the converse is not true. Examples of almost additive and asymptotically additive sequences will be provided in sub-section 1.3

Building upon the work of Feng and Huang [FeH], the topological pressure in this setting was defined by Cheng, Zhao and Cao [CZC] (see also [CFH, ZZC] for related results). Let $\Phi = (\phi_n)_n$ be an asymptotically additive sequence on a sub-shift Σ. Let $n \in \mathbb{N}$ and $\epsilon > 0$, a subset $E \subset \Sigma$ is an (n, ϵ)*-separated subset* of Σ if $\max_{0\le i\le n-1} d(\sigma^i\omega, \sigma^i\kappa) > \epsilon$ for all $\omega, \kappa \in E, \omega \neq \kappa$. Let

$$P(\Phi, n, \epsilon) := \sup\left\{\sum_{\omega\in E} \exp(\phi_n(\omega)) : E \text{ is an } (n, \epsilon)\text{-separated subset of } \Sigma\right\}.$$

The *topological pressure* for an asymptotically additive sequence Φ on Σ is defined by

$$P(\Phi) := \lim_{\epsilon\to 0}\limsup_{n\to\infty} \frac{1}{n}\log P(\Phi, n, \epsilon). \tag{3}$$

This notion of pressure satisfies the corresponding variational principle (see [FeH]),

$$P(\Phi) = \sup\left\{h(\mu) + \lim_{n\to\infty}\frac{1}{n}\int \phi_n d\mu : \mu \in \mathcal{M}\right\},$$

where $\mathcal{M}$ denotes the set of $\sigma-$invariant probability measures and $h(\mu)$ the entropy of the measure μ (see [W1, Chapter 4]). A measure $\mu \in \mathcal{M}$ attaining the supremum is called *equilibrium state* for Φ.

In what follows we will give another characterization of the pressure using periodic points instead of (n, ϵ)-separated sets. This characterization is well known for the pressure of a single function defined on a topologically mixing sub-shift of finite type. Let $\text{Fix}(\sigma) := \{\omega \in \Sigma : \sigma\omega = \omega\}$. If $\phi : \Sigma \to \mathbb{R}$ is a continuous function and (Σ, σ) is topologically mixing, it was shown by Ruelle (see [Ru, Theorem 7.20]) that

$$P(\phi) = \lim_{n\to\infty}\frac{1}{n}\log\sum_{\omega\in\text{Fix}(\sigma^n)} \exp(S_n(\phi_n(\omega)). \tag{4}$$

We further generalize this result to setting of asymptotically additive sequences.

Proposition 1.2. *Let $\Phi = (\phi_n)_n$ be an asymptotically additive sequence on a topologically mixing shift of finite type (Σ, σ). Then*

$$P(\Phi) = \lim_{n\to\infty}\frac{1}{n}\log\sum_{\omega\in Fix(\sigma^n)} \exp(\phi_n(\omega)). \tag{5}$$

Remark 1.2. Let us note that Proposition 1.2 has been proved under stronger additivity and regularity assumptions. Indeed, let $\Phi = (\phi_n)_n$ be a sequence of continuous functions on Σ and

$$\gamma_n(\Phi) := \text{var}_n\phi_n = \sup\{|\phi_n(\omega) - \phi_n(\kappa)| : \omega, \kappa \in C_{i_1\ldots i_n}\}. \tag{6}$$

We say that Φ has *tempered variation* if $\lim_{n\to\infty} \gamma_n(\Phi)/n = 0$. Barreira [B2] showed that if $\Phi = (\phi_n)_n$ is an almost additive sequences with tempered variation then the equation (5) holds. It should be noted, however, that a simple modification of the argument in [ZZC, Lemma 2.1] implies that if $\Phi = (\phi_n)_n$ is an asymptotically additive sequence on a topologically mixing sub-shift (Σ, σ) then Φ has tempered variation.

In order to prove proposition 1.2 we require some auxiliary results. A version of the following Lemma was proved in [ZZC, Lemma 2.3].

Lemma 1.1. *Let $\Phi = (\phi_n)_n$ be an asymptotically additive sequence on a topologically mixing sub-shift (Σ, σ) satisfying equation (2). Then $P(\Phi) = \lim_{\epsilon\to 0} P(\rho_\epsilon)$.*

Proof. Let $\bar{\epsilon} > 0$. Since Φ is an asymptotically additive sequence, there exist $\rho_{\bar{\epsilon}} \in C(\Sigma)$ and $N_{\bar{\epsilon}}$ such that for every $\omega \in \Sigma$ and $n \geq N_{\bar{\epsilon}}$

$$\frac{1}{n}(S_n\rho_{\bar{\epsilon}})(\omega) - 2\bar{\epsilon} \leq \frac{1}{n}\phi_n(\omega) \leq \frac{1}{n}(S_n\rho_{\bar{\epsilon}})(\omega) + 2\bar{\epsilon}.$$

Thus, for $n \geq N_{\bar{\epsilon}}$ and $\epsilon > 0$

$$\begin{aligned}
&\exp(-2n\bar{\epsilon}) \sup\left\{\sum_{\omega\in E} \exp((S_n\rho_{\bar{\epsilon}})(\omega)) : E \text{ is an } (n,\epsilon) \text{ separated subset of } \Sigma\right\}\\
&\leq \sup\left\{\sum_{\omega\in E} \exp(\phi_n(\omega)) : E \text{ is an } (n,\epsilon) \text{ separated subset of } \Sigma\right\}\\
&\leq \exp(2n\bar{\epsilon}) \sup\left\{\sum_{\omega\in E} \exp((S_n\rho_{\bar{\epsilon}})(\omega)) : E \text{ is an } (n,\epsilon) \text{ separated subset of } \Sigma\right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&-2\bar{\epsilon} + \frac{1}{n}\log\left(\sup\left\{\sum_{\omega\in E} \exp((S_n\rho_{\bar{\epsilon}})(\omega)) : E \text{ is an } (n,\epsilon) \text{ separated subset of } \Sigma\right\}\right)\\
&\leq \frac{1}{n}\log\left(\sup\left\{\sum_{\omega\in E} \exp(\phi_n(\omega)) : E \text{ is an } (n,\epsilon) \text{ separated subset of } \Sigma\right\}\right)\\
&\leq 2\bar{\epsilon} + \frac{1}{n}\log\left(\sup\left\{\sum_{\omega\in E} \exp((S_n\rho_{\bar{\epsilon}})(\omega)) : E \text{ is an } (n,\epsilon) \text{ separated subset of } \Sigma\right\}\right).
\end{aligned}$$

Hence

$$-2\bar{\epsilon} + P(\rho_{\bar{\epsilon}}) \leq P(\Phi) \leq 2\bar{\epsilon} + P(\rho_{\bar{\epsilon}}).$$

Since for every $\bar{\epsilon} > 0$ we can obtain the above inequality, letting $\bar{\epsilon} \to 0$, we have that $P(\Phi) = \lim_{\bar{\epsilon}\to 0} P(\rho_{\bar{\epsilon}})$. □

Proof of Proposition 1.2. Since Φ is asymptotically additive, for a fixed $\bar{\epsilon} > 0$, there exist a continuous function $\rho_{\bar{\epsilon}}$ and $N_{\bar{\epsilon}} \in \mathbb{N}$ such that for every $n > N_{\bar{\epsilon}}$ we have that

$$(S_n\rho_{\bar{\epsilon}})(\omega) - 2n\bar{\epsilon} \leq \phi_n(\omega) \leq (S_n\rho_{\bar{\epsilon}})(\omega) + 2n\bar{\epsilon}.$$

Therefore, for every $n > N_{\bar{\epsilon}}$ we have that

$$\exp(-2n\bar{\epsilon}) \sum_{\omega \in \mathrm{Fix}(\sigma^n)} \exp((S_n \rho_{\bar{\epsilon}})(\omega)) \leq \sum_{\omega \in \mathrm{Fix}(\sigma^n)} \exp(\phi_n(\omega))$$
$$\leq \exp(2n\bar{\epsilon}) \sum_{\omega \in \mathrm{Fix}(\sigma^n)} \exp((S_n \rho_{\bar{\epsilon}})(\omega)).$$

Thus,

$$-2\bar{\epsilon} + \frac{1}{n} \log \left(\sum_{\omega \in \mathrm{Fix}(\sigma^n)} \exp((S_n \rho_{\bar{\epsilon}})(\omega)) \right) \leq \frac{1}{n} \log \left(\sum_{\omega \in \mathrm{Fix}(\sigma^n)} \exp(\phi_n(\omega)) \right)$$
$$\leq 2\bar{\epsilon} + \frac{1}{n} \log \left(\sum_{\omega \in \mathrm{Fix}(\sigma^n)} \exp((S_n \rho_{\bar{\epsilon}})(\omega)) \right).$$

Letting $n \to \infty$ and making use of Ruelle's representation of the pressure of a continuous function by means of periodic points as in equation (4) we obtain

$$-2\bar{\epsilon} + P(\rho_{\bar{\epsilon}}) \leq \lim_{n \to \infty} \frac{1}{n} \log \left(\sum_{\omega \in \mathrm{Fix}(\sigma^n)} \exp(\phi_n(\omega)) \right) \leq 2\bar{\epsilon} + P(\rho_{\bar{\epsilon}}).$$

Since for every $\bar{\epsilon} > 0$ we can obtain the above inequality, letting $\bar{\epsilon} \to 0$ and making use of Lemma 1.1 we have that

$$P(\Phi) = \lim_{\bar{\epsilon} \to 0} P(\rho_{\bar{\epsilon}}) \leq \lim_{n \to \infty} \frac{1}{n} \log \left(\sum_{\omega \in \mathrm{Fix}(\sigma^n)} \exp(\phi_n(\omega)) \right) \leq \lim_{\bar{\epsilon} \to 0} P(\rho_{\bar{\epsilon}}) = P(\Phi).$$

□

1.3. **Examples.** Part of the interest in studying non-additive thermodynamic formalism and in particular its almost additive and asymptotically additive versions is the wide range of relevant examples to which it can be applied. Here we provide some examples of sequences satisfying weak additivity properties that arise naturally in certain context.

Product of matrices and Maximal Lyapunov exponents. Let $A = (a_{ij})$ be a $d \times d$ matrix with real coefficients. We define the norm of the matrix by $\|A\| := \max_i \sum_{j=1}^{d} a_{ij}$. Let $\{A_1, \dots A_n\}$ be a finite collection of $d \times d$ real matrices and let (Σ, σ) be the full-shift on n symbols. Furstenberg and Kesten in [FK] studied the following sequence of functions. If $\omega = (i_1, i_2, \dots) \in \Sigma$ define the sequence of functions by $\phi_n(\omega) := \|A_{i_n} A_{i_{n-1}} \cdots A_{i_1}\|$. A fundamental result in [FK] is that if μ is an ergodic measure then μ-almost everywhere the following equality holds

$$\lim_{n \to \infty} \frac{1}{n} \int \log \phi_n d\mu = \lim_{n \to \infty} \frac{1}{n} \log \phi_n(\omega).$$

Since the sequence $(\log \phi_n)_n$ is subadditive the above result now follows form Kingman's subadditive ergodic theorem (proved several years later). The number $\lambda(\omega) := \lim_{n \to \infty} \frac{1}{n} \log \phi_n(\omega)$, when it exists, is called Maximal Lyapunov exponent of ω. This is an important dynamical quantity that appears in a wide range of contexts, from Schrödinger operators to Hausdorff dimension of measures. It turns out that if the matrices in $\{A_1, \dots A_n\}$ are positive (meaning that every entry is positive) then the sequence $(\log \phi_n)_n$ is almost additive (see [Fe1]).

Factor maps. We can find examples of sequences of continuous functions which are almost additive by studying factor maps between sub-shifts (see [YY1, YY2]). Let $(\Sigma_1, \sigma_{\Sigma_1})$ be a topologically mixing sub-shift of finite type and $(\Sigma_2, \sigma_{\Sigma_2})$ a sub-shift. Let $\pi : \Sigma_1 \to \Sigma_2$ be a one-block factor map, i.e., π is a continuous and surjective function from Σ_1 to Σ_2 that satisfies $\pi \circ \sigma_{\Sigma_1} = \sigma_{\Sigma_2} \circ \pi$. For an allowable word $y_1 \dots y_n$ of length n in Σ_2, denote by $|\pi^{-1}[y_1 \dots y_n]|$ the cardinality of the set consisting of exactly one point from each cylinder $C_{x_1 \dots x_n}$ in Σ_1 such that $\pi(C_{x_1 \dots x_n}) \subseteq C_{y_1 \dots y_n}$. For $y = (y_1, \dots, y_n, \dots) \in \Sigma_2$, let $\phi_n(y) = \log |\pi^{-1}[y_1 \dots y_n]|$. Then $\Phi =(\phi_n)_n$ is subadditive. Suppose we have the following condition: for $n, m \in \mathbb{N}$, there exists $0 < D \leq 1$ such that for any allowable word $y_1 \dots y_{n+m}$ of length $(n + m)$ in Σ_2, we have $D|\pi^{-1}[y_1 \dots y_n]||\pi^{-1}[y_{n+1} \dots y_{n+m}]| \leq |\pi^{-1}[y_1 \dots y_{n+m}]|$ (see [YY1] for examples). Then $\Phi = (\phi_n)_n$ is almost additive on Σ_2. We note that Φ appears in the study of dimension of compact invariant sets of some non-conformal expanding maps (see [Fe2, YY1]).

Singular value function and non-conformal Fractal geometry. There is a wide range of techniques in order to compute and estimate the Hausdorff dimension of a conformal repeller, Λ, corresponding to a C^1 map $f : \mathbb{R}^2 \to \mathbb{R}^2$ (see [B2] for precise definitions). The reason behind this is that since the system is conformal the natural (dynamically defined) covers are close enough to balls and can be used to estimate the Hausdorff dimension. In the non conformal setting the situation is much more subtle and the theory is far less developed. Falconer [Fa1] introduced an idea and a technique that has been greatly developed over the last years and that we now recall in a two dimensional setting. The singular values $s_1(A), s_2(A)$ of a 2×2 matrix A are the eigenvalues, counted with multiplicities, of the matrix $(A^*A)^{1/2}$, where A^* denotes the transpose of A. The singular values can be interpreted as the length of the semi-axes of the ellipse which is the image of the unit ball under A. The functions, $\phi_{i,n} : \Lambda \to \mathbb{R}$ be defined by $\phi_{i,n}(x) = \log s_i(d_x f^n)$ and called *singular value functions*. Under certain assumptions it is possible to estimate the dimension of the repeller studying these functions (see for example [Fa2]). It was proved by Barreira and Gelfert [BG, Proposition 4] that if the dynamical system f has dominated splitting (see [B2, p.234] for a precise definition) then the sequences $(\phi_{i,n})_n$ are almost additive.

Average conformal repellers. Let f be a C^1 map defined on a C^∞ manifold and let Λ be a compact, $f-$invariant set. In [BCH] the authors define Λ to be an *average conformal repeller* if for every ergodic $f-$invariant measure supported on Λ all the corresponding Lyapunov exponents are positive and equal. If A is a matrix denote by $m(A) = \|A^{-1}\|^{-1}$. It is shown in [BCH] that the Hausdorff dimension of Λ is the root of the function $s \mapsto P(-s\Phi)$, where $\Phi = (\phi_n)$ and $\phi_n(x) = \log m(Df^n x)$. It is also proven that

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{\|Df^n x\|}{m(Df^n x)} = 0.$$

Zhao, Zhang and Cao [ZZC] remarked that both sequences $(\log \|Df^n x\|)_n$ and $(\log m(Df^n))_n$ are asymptotically additive, while not necessarily almost additive.

2. Weak Gibbs measure for asymptotically additive sequences

The notion of weak Gibbs measure can be extended to the setting of asymptotically additive sequence.

Definition 2.1. A probability measure μ is called a *weak Gibbs measure* for the asymptotically additive sequence $\Phi = (\phi_n)_n$ on Σ if there exists a sequence of positive real numbers $K(n)$ satisfying

$$\lim_{n\to\infty} \frac{\log K(n)}{n} = 0,$$

such that, for every $n \in \mathbb{N}$, every cylinder $C_{i_1\dots i_n}$ and every $\omega \in C_{i_1\dots i_n}$ we have

$$\frac{1}{K(n)} \leq \frac{\mu(C_{i_1\dots i_n})}{\exp(\phi_n(\omega) - nP(\Phi))} \leq K(n),$$

where $P(\Phi)$ is the topological pressure for Φ.

Remark 2.1. It was shown by Barreira [B2] that if Φ is an almost additive sequence of continuous functions with tempered variation, then there exists an ergodic weak Gibbs measure, not necessary invariant though (note that in virtue of Remark 1.2 the regularity assumption is not really needed). This result generalizes that of Kesseböhmer. Indeed, if ϕ is a continuous potential on Σ then $\Phi = (S_n(\phi))_n$ is an almost additive sequence. Moreover, Φ has tempered variation if and only if ϕ is a medium varying function.

In this section we prove that every weak Gibbs measure for an asymptotically additive sequence is a Gibbs measure for, another, asymptotically additive sequence. Simplifying, therefore, the study of such measures.

Theorem 2.1. *If μ is a weak Gibbs measure for an asymptotically additive sequence $\Phi := (\phi_n)_n$ on a topologically mixing Markov shift (Σ, σ) then there exists an asymptotically additive sequence $\Psi := (\psi_n)_n$ on Σ such that the measure μ is Gibbs with respect to Ψ.*

Proof. Since the sequence Φ is asymptotically additive, for every $k \in \mathbb{N}$ there exists a continuous function $\rho_k : \Sigma \to \mathbb{R}$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \|\phi_n - S_n\rho_k\| \leq \frac{1}{k}.$$

Let us consider the following sequence of functions $\Psi = (\psi_n)$, where $\psi_n : \Sigma \to \mathbb{R}$ is defined for $\omega \in C_{i_1\dots i_n}$ by $\psi_n(\omega) := \log \mu(C_{i_1\dots i_n})$. Note that for every $\omega \in \Sigma$ we have that

$$-\log K(n) \leq \psi_n(\omega) - \phi_n(\omega) + nP(\Phi) \leq \log K(n). \tag{7}$$

Indeed, since the measure μ is weak Gibbs for Φ equation (7) simply follows applying logarithm to the following inequalities,

$$\frac{1}{K(n)} \leq \frac{\mu(C_{i_1\dots i_n})}{\exp(-nP(\Phi) + \phi_n(\omega))} \leq K(n).$$

Moreover, the sequence $\Psi = (\psi_n)_n$ is asymptotically additive. Indeed, it follows from equation (7) and the definition of weak Gibbs measure that for every $k \in \mathbb{N}$ the continuous function $(\rho_k + P(\Phi)) : \Sigma \to \mathbb{R}$, defined by $(\rho_k + P(\Phi))(\omega) = \rho_k(\omega) + P(\Phi)$, is such that

$$\limsup_{n\to\infty} \frac{1}{n} \|\psi_n - S_n(\rho_k + P(\Phi))\|$$

$$\leq \limsup_{n\to\infty} \frac{1}{n} \|\psi_n - \phi_n + nP(\Phi)\| + \limsup_{n\to\infty} \frac{1}{n} \|\phi_n - S_n\rho_k\| \leq \limsup_{n\to\infty} \frac{\log K(n)}{n} + \frac{1}{k} = \frac{1}{k}.$$

It now follows from Proposition 1.2 and the definition of weak Gibbs measure that

$$\begin{aligned}
P(\Psi) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in \mathrm{Fix}(\sigma^n)} \exp(\psi_n(\omega)) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in \mathrm{Fix}(\sigma^n)} \mu(C_{i_1\dots i_n}) \\
&\le \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in \mathrm{Fix}(\sigma^n)} \exp(-nP(\Phi)+\phi_n(\omega))K(n) \\
&= \lim_{n\to\infty} \frac{\log K(n)}{n} - P(\Phi) + \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in \mathrm{Fix}(\sigma^n)} \exp(\phi_n(\omega)) \\
&= \lim_{n\to\infty} \frac{\log K(n)}{n} - P(\Phi) + P(\Phi) = 0.
\end{aligned}$$

And on the other hand

$$\begin{aligned}
P(\Psi) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in \mathrm{Fix}(\sigma^n)} \exp(\psi_n(\omega)) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in \mathrm{Fix}(\sigma^n)} \mu(C_{i_1\dots i_n}) \\
&\ge \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in \mathrm{Fix}(\sigma^n)} \exp(-nP(\Phi)+\phi_n(\omega))K(n)^{-1} \\
&= \lim_{n\to\infty} -\frac{\log K(n)}{n} - P(\Phi) + \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in \mathrm{Fix}(\sigma^n)} \exp(\phi_n(\omega)) \\
&= \lim_{n\to\infty} -\frac{\log K(n)}{n} - P(\Phi) + P(\Phi) = 0.
\end{aligned}$$

Therefore $P(\Psi) = 0$. Thus, for every $n \in \mathbb{N}$, every cylinder $C_{i_1\dots i_n}$ and every $\omega \in C_{i_1\dots i_n}$ we have

$$\frac{\mu(C_{i_1\dots i_n})}{\exp(\psi_n(\omega) - nP(\Psi))} = 1.$$

Therefore, the measure μ is Gibbs with respect to Ψ. □

Corollary 2.1. *Every weak Gibbs measure μ for a continuous potential ϕ on Σ is a Gibbs measure for an* asymptotically additive *sequence $\Psi := (\psi_n)_n$ on Σ.*

Proof. The result follows considering $\Phi = (S_n\phi)_n$ in Theorem 2.1. □

Recall that an almost additive sequence Φ is an asymptotically additive sequence and thus it has tempered variation (see Remark 1.2 and [ZZC, Proposition 2.1]). Therefore there always exists a weak Gibbs measure for Φ.

Corollary 2.2. *Let μ be a weak Gibbs measure for an almost additive sequence $\Phi := (\phi_n)_n$. Then there exists an asymptotically additive sequence $\Psi := (\psi_n)_n$ such that the measure μ is Gibbs with respect to Ψ.*

Proof. The result is contained in Theorem 2.1. □

Remark 2.2 (Regularity)**.** Note that the sequence $\Psi = (\psi_n)_n$ constructed in Theorem 2.1 is such that every function $\psi_n : \Sigma \to \mathbb{R}$ is locally constant on cylinders of length n.

Remark 2.3. Let ϕ be a continuous potential for which there exists a Gibbs measure μ. Then the sequence of functions Ψ in Theorem 2.1 is almost additive.

Without loss of generality we can assume that $P(\phi) = 0$. Recall that there exists a constant $C > 0$ such that

$$\frac{1}{C}\exp(S_{n+m}\phi(\omega)) \leq \mu(C_{i_1\dots i_{n+m}}) \leq C\exp(S_{n+m}\phi(\omega)).$$

Note that

$$\psi_{n+m}(\omega) - \psi_n(\omega) - \psi_m(\sigma^n\omega) = \log\frac{\mu(C_{i_1\dots i_{n+m}})}{\mu(C_{i_1\dots i_n})\mu(C_{i_n\dots i_{n+m}})}$$

$$\leq \log\frac{C\exp S_{n+m}\phi(\omega)}{C^{-2}\exp S_n\phi(\omega)\exp S_m\phi(\sigma^n\omega)} = 3\log C.$$

Moreover,

$$\psi_{n+m}(\omega) - \psi_n(\omega) - \psi_m(\sigma^n\omega) = \log\frac{\mu(C_{i_1\dots i_{n+m}})}{\mu(C_{i_1\dots i_n})\mu(C_{i_n\dots i_{n+m}})}$$

$$\geq \log\frac{C^{-1}\exp S_{n+m}\phi(\omega)}{C^{2}\exp S_n\phi(\omega)\exp S_m\phi(\sigma^n\omega)} = -3\log C.$$

Therefore the sequence Ψ is almost additive. Now, if the measure μ is weak Gibbs but not Gibbs then the sequences Ψ is asymptotically additive but not almost additive.

Remark 2.4. It should be stressed that the results in Theorem 2.1 and in its Corollaries the Gibbs condition is satisfied for a constant $C = 1$, thus we obtain equalities instead of inequalities in the Gibbs definition.

3. Applications

3.1. **Multifractal analysis.** In this section we apply several results on dimension theory of non-additive sequences to the study of multifractal analysis of conformal dynamical systems. It is worth stressing that while non-additive thermodynamic formalism was originally developed to study dimension theory on non-conformal settings, it has interesting and powerful applications in the conformal setting. Multifractal analysis is the study of level sets determined by (dynamically defined) local quantities. In general, the geometry of the level sets is rather complicated and in order to quantify its size Hausdorff dimension is used. Combining the results in Section 2 with those recently obtained by Barreira, Cao and Wang [BCW] on multifractal analysis of quotients of asymptotically additive sequences, we obtain new results and recover old ones in multifractal analysis for one-dimensional uniformly hyperbolic dynamical systems. The class of dynamical systems that we will study is the following.

Definition 3.1. Let $I = [0, 1]$ and consider a finite family $\{I_i\}_{i=1}^k$, of closed intervals with disjoint interiors contained in I. A map $T : \cup_{i=1}^k I_i \to I$ is an *expanding Markov map* if

(1) the map is $C^{1+\alpha}$;
(2) the map T is Markov and it can be coded by a topologically mixing sub-shift of finite on the alphabet $\{1, \dots, k\}$;
(3) for every $x \in \cup_{i=1}^k I_i$ we have that $|T'(x)| > 1$.

The *repeller* of such a map is defined by

$$\Lambda := \{x \in \cup_{i=1}^{\infty} I_i : T^n(x) \text{ is well defined for every } n \in \mathbb{N}\}.$$

The Markov structure assumed for expanding Markov maps T, allows for a good symbolic representation. That is, there exists a topologically mixing sub-shift of finite type (Σ, σ) defined on the alphabet $\{1, \dots, k\}$ with the following property: the natural projection, $\pi : \Sigma \to \Lambda$ satisfies $\pi \circ \sigma = T \circ \pi$. Moreover, it is an injective map except on a countable set (which from the point of view of dimension theory is irrelevant).

When coding a dynamical system with a symbolic one it is often desirable to translate the original metric properties onto the symbolic space. A standard way to do it is to associate to each symbolic cylinder of length n the diameter of its projection. Let $I(i_1, \dots i_n) := \pi(C_{i_1 \dots i_n})$, with a slight abuse of notation we call that set a *cylinder* of length n for T. For every $n \in \mathbb{N}$ we define the function $D_n : \Sigma \to \mathbb{R}$ by $D_n(\omega) := \text{diam}(I(i_1, \dots i_n))$, where $I(i_1, \dots i_n)$ is the cylinder of length n in $[0, 1]$ containing the point $\pi(\omega)$. We denote the sequence of functions obtained in this way by $D := (D_n)_n$. It is a well known, and widely used, result that D_n is comparable with $|(T^{-n})'|$. Indeed, the following results appears in [Pe, Proposition 20.2].

Lemma 3.1. *If T is an expanding Markov map then there exists a constant $K_1 > 0$ such that for every $\omega \in \Sigma$ with $\pi(\omega) = x$ we have*

$$\frac{1}{K_1} \leq \frac{D_n(\omega)}{|(T^{-n})'|(x)} \leq K_1.$$

Remark 3.1. A remarkable fact proved by Urbańksi [U] and also by Jordan and Rams [JR] (using a different language though) is the following. If in the definition of the map T we allow a finite number of points for which the derivative of T is equal to 1, then the sequence $\log D := (\log D_n)_n$ is asymptotically additive. This was used in [XM] when studying multifractal analysis of the set of irregular points.

We now define the local quantities that we will consider. Let μ be a probability measure on $[0, 1]$. The *pointwise dimension* of μ at the point $x \in (0, 1)$ is defined by

$$d_\mu(x) := \lim_{r \to 0} \frac{\log \mu((x - r, x + r))}{\log r},$$

whenever the limit exists. This function describes the power law behaviour of the measure for small intervals and has been extensively studied over the last decade (see for example [B3]). For a weak Gibbs measure μ the pointwise dimension can be computed at a symbolic level (see [JR, Lemma 8]).

Lemma 3.2. *Let T be an expanding Markov map and μ a weak Gibbs measure corresponding to the continuous potential ϕ. Denote by $\overline{\mu} := \mu \circ \pi$, then whenever the pointwise dimension of μ exists we have*

$$\begin{aligned} d_\mu(x) : &= \lim_{r \to 0} \frac{\log \mu((x - r, x + r))}{\log r} = \lim_{n \to \infty} \frac{\log \overline{\mu}(C_{i_1 \dots i_n})}{\log D_n(\omega)} \\ &= - \lim_{n \to \infty} \frac{\log \overline{\mu}(C_{i_1 \dots i_n})}{\sum_{i=0}^{n-1} \log |(T^i)'(x)|}, \end{aligned}$$

where $C_{i_1 \dots i_n}$ is the cylinder of length n containing $\pi^{-1} x = \omega$.

We can now state our result in dimension theory. We denote the Hausdorff dimension of a set by $\dim_H(\cdot)$ (see [Fa3] for definition and properties). Recall from

Section 2 that if μ_i is a weak Gibbs measure then we can define an asymptotically additive sequence $\Psi^i := (\psi_n^i)_n$ by $\psi_n^i(x) = \log \mu_i(C_{i_1 \dots i_n}(x))$. Let $(\mu_i)_{i=1}^r$ be a finite set of non-atomic invariant weak Gibbs measures and $(\Psi^i)_{i=1}^r$ the corresponding asymptotically additive sequences. Denote by $\mathcal{M}_T$ the set of $T-$invariant probability measures and consider the map $\mathcal{P} : \mathcal{M}_T \mapsto \mathbb{R}^r$ defined by

$$\mathcal{P}(\mu) := \left(\frac{-\lim_{n\to\infty} \frac{1}{n} \int_0^1 \psi_n^1(x) d\mu}{\int_0^1 \log |T'| d\mu}, \dots, \frac{-\lim_{n\to\infty} \frac{1}{n} \int_0^1 \psi_n^r(x) d\mu}{\int_0^1 \log |T'| d\mu} \right).$$

Denote by $\Omega \subset \mathbb{R}^r$ the image of $\mathcal{P}$.

Theorem 3.1. *Let T be an expanding Markov map and $(\mu_i)_{i=1}^r$ be non-atomic invariant weak Gibbs measures. Let $\overline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_r) \in \mathbb{R}^r$ and consider the level sets defined by*

$$J(\overline{\alpha}) := \{x \in [0,1] : (d_{\mu_1}(x), d_{\mu_2}(x), \dots, d_{\mu_r}(x)) = \overline{\alpha}\}.$$

Then,

1. *$J(\overline{\alpha}) \neq \emptyset$ if and only if $\overline{\alpha} \in \Omega$.*
2. *If $\overline{\alpha} \in \Omega$ then*

$$\dim_H J(\overline{\alpha}) = \sup \left\{ \frac{h(\nu)}{\int \log |T'| d\nu} : \nu \in \mathcal{M}_T \text{ with } \mathcal{P}(\nu) = \overline{\alpha} \right\}.$$

Proof. Since μ_i is weak Gibbs we have that for every $i \in \{1, \dots, r\}$ the sequences $\Psi^i := (\psi_n^i)_n$ are asymptotically additive (see Theorem 2.1). By Lemma 3.2, we have

$$d_{\mu_i}(x) = -\lim_{n\to\infty} \frac{\psi_n^i(\omega)}{\sum_{i=0}^{n-1} \log |(T^i)'(x)|},$$

where $\pi(\omega) = x$. Thus, the pointwise dimension of μ_i at the point $x \in [0,1]$ is the quotient of two asymptotically additive sequences. Also, since T is an expanding Markov map there exists a constant $H > 0$ such that for every $x \in \Lambda$ and every $n \in \mathbb{N}$ we have $\sum_{i=0}^{n-1} \log |(T^i)'(x)| \geq nH$. The result now follows from [BCW, Theorem 1 part (1)]. □

Remark 3.2. Note that what is relevant in the proof of Theorem 3.1 is that the pointwise dimension of weak Gibbs measures can be computed as the limit of the quotient of two asymptotically additive sequences. The same type of result can be obtained considering, instead of pointwise dimension, local entropies, Birkhoff averages, Lyapunov exponents or quotients of Birkhoff averages.

3.2. **Factors of Gibbs measures.** Pollicott and Kempton [PK] and Chazottes and Ugalde [CU] independently considered problems related to factors of Gibbs measures on full shifts. In particular, they studied conditions under which the factors of invariant Gibbs measures for functions of summable variations are also Gibbs measures for functions having good regularity properties. Their results inculde that images of Gibbs measures for Hölder continuous functions are Gibbs measures for functions of summable variations. Yayama [YY2] considered the factors of invariant Gibbs measures for almost additive sequences of continuous functions, generalizing the theory of factors of Gibbs measures.

Let $\pi : \Sigma_1 \to \Sigma_2$ be a one-block factor map between sub-shifts on finitely many symbols. Denote by $\mathcal{M}_{\Sigma_1}$ the set of σ_{Σ_1}-invariant Borel probability measures and let $\pi\mu \in \mathcal{M}_{\Sigma_2}$ be the projection of $\mu \in \mathcal{M}_{\Sigma_1}$ under π. Denote by $P_{\Sigma_1}(\mathcal{F})$ be the topological pressure of an asymptotically additive sequence of continuous functions $\mathcal{F} = (\log f_n)_n$ on Σ_1. For all $n \in \mathbb{N}, y = (y_1, \dots, y_n, \dots) \in \Sigma_2$, denote by $E_n(y)$ a set consisting of exactly one point from each cylinder $C_{x_1 \dots x_n}$ of length n such that $\pi(C_{x_1 \dots x_n}) \subseteq C_{y_1 \dots y_n}$. Define $g_n(y) := \sup_{E_n(y)} \{\sum_{x \in E_n(y)} f_n(x)\}$ and $\mathcal{G} := (\log g_n)_n$. For $\nu \in M_{\Sigma_2}$ and $y \in C_{i_1 \dots i_n}$ define $h_n(y) := \nu(C_{i_1 \dots i_n})$ and let $\mathcal{H}_\nu := (\log h_n)_n$. The following proposition improves, in the context of full-shifts, on results obtained in more general settings in [YY2].

Proposition 3.1. *Let $\pi : \Sigma_1 \to \Sigma_2$ be a one-block factor map between full shifts. For an almost additive sequence of continuous functions $\mathcal{F}$ on Σ_1 with bounded variation, let $\mu \in \mathcal{M}_{\Sigma_1}$ be the unique equilibrium state which is Gibbs for $\mathcal{F}$. Then $\pi\mu$ is the unique invariant Gibbs measure for the almost additive sequence of continuous functions $\mathcal{G}$ on Σ_2 with the property that $P_{\Sigma_1}(\mathcal{F}) = P_{\Sigma_2}(\mathcal{G})$. Hence $\pi\mu$ is Gibbs for $\mathcal{H}_{\pi\mu}$ which is an asymptotically additive sequence with $P_{\Sigma_2}(\mathcal{H}_{\pi\mu}) = 0$.*

Proof. We make standard arguments to show that $\mathcal{G}$ is almost additive. The first part of the proposition is known from [YY2, Theorem 3.1]. Applying Theorem 2.1, we obtain the second part of the proposition. □

Remark 3.3. We note that this $\mathcal{H}_{\pi\mu}$ is slightly different from $\widetilde{\mathcal{G}} = (\log \tilde{g}_n)_n$ found in [YY2, Theorem 3.7] and there exists $K > 0$ such that $(1/K)\tilde{g}_n(y) \le h_n(y) \le K\tilde{g}_n(y)$ for all $y \in \Sigma_2$. If μ is a weak Gibbs measure for an asymptotically additive sequence $\mathcal{F}$ on a full shift, then, in general, $\pi\mu$ is weak Gibbs for a subadditive sequence $\mathcal{G}$. It would be interesting to study further conditions under which $\mathcal{G}$ can be asymptotically additive in order to apply Theorem 2.1.

Remark 3.4. Studying factors of Gibbs measures has been useful to solve problems on dimensions of non-conformal expanding maps. Measures of full dimension of compact invariant sets of some non-conformal expanding maps have been identified as preimages of Gibbs equilibrium states for subadditive sequences, and in special cases, measures of full dimension are Gibbs measures for almost additive sequences [Fe2, YY1] (see also [KP] for the relation between Hausdorff dimension and subadditive sequences). Applying Theorem 2.1, we immediately obtain a property of measures of full dimension.

3.3. **Convergence to entropy: a result by Varandas ans Zhao.** Let (Σ, σ) be a one-sided Markov shift defined over a finite alphabet and μ be an ergodic, invariant probability measure. The Shannon-McMillan-Breiman Theorem states that the following holds for μ-almost every $\omega \in \Sigma$:

$$h(\mu) := \lim_{n\to\infty} -\frac{1}{n} \log \mu(C_{i_1 \dots i_n}(\omega)). \tag{8}$$

The following result by Varandas and Zhao [VZ2, Theorem B] establishes exponential convergence in equation (8).

Theorem 3.2 (Varandas-Zhao)**.** *Let μ_Φ be a weak Gibbs measure for an asymptotically additive sequence Φ. If*

$$\inf_{\nu\in\mathcal{M}} \lim_{n\to\infty} \int -\frac{1}{n} \log \mu_\Phi(C_{i_1\dots i_n}(\omega)) < \sup_{\nu\in\mathcal{M}} \lim_{n\to\infty} \int -\frac{1}{n} \log \mu_\Phi(C_{i_1\dots i_n}(\omega))$$

then there exists $A^* > 0$ such that for any $0 < A < A^*$ we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_\Phi \left(\left\{ \omega \in \Sigma : \left| -\frac{1}{n} \log \mu_\Phi(C_{i_1\dots i_n}(\omega)) - h(\mu_\Phi) \right| \right\} \right) < 0.$$

The proof of this result is based on a study made by the authors [VZ1] on Large deviations for weak Gibbs measures for non-additive sequences of potentials. Theorem 2.1 shows that, when restricted to the symbolic setting, in order to deduce the large deviations results in [VZ1] or in the proof of Theorem 3.2 it is only necessary to assume that the measure is indeed a Gibbs measure for the appropriate sequence.

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