An integral representation formula of the Schwarzian derivative

Martin Chuaqui* Christian Pommerenke

Abstract

Let f be a conformal map of the unit disk $\mathbb{D}$ onto a domain bounded by a curve C, which is of class $C^{3,\delta}$, except for a finite number of corners. In this paper we derive a representation formula of the Schwarzian derivative Sf, expressed in terms of the integral of the arclength derivative of the curvature of C and a sum of polar terms corresponding to the vertices.

1. Introduction

Let f be a locally univalent analytic map defined on some open set, and let

$$Sf(z) = (\frac{f''}{f'})' - \frac{1}{2}(\frac{f''}{f'})^2$$

be its Schwarzian derivative. The role of this operator in connection with the global univalence of f in the domain and quaisconformal extensions to $\mathbb{C}$ has been studied extensively and is well known. On the other hand, at a local scale, the Schwarzian derivative determines the way the mapping f distorts geodesic curvature, in particular, to what extent curves of constant curvature are preserved under the mapping. To be precise, let $z = z(t)$ be an arclength parametrized curve contained in the domain of f, and let $w(t) = f(z(t))$ be the image curve. The curvatures are given by $k(t) = \frac{d}{dt}\arg\{z'(t)\}$ and

$$\kappa(s) = \frac{d}{ds}\arg\{w'(t)\} = \frac{1}{|f'|}\left(\mathrm{Im}\{(\frac{f''}{f'})z'\} + k(t)\right) . \tag{1.1}$$

Here s denotes the arclength parameter of the image curve and all derivatives of f are evaluated at $z(t)$. Further differentiation yields the important relation

$$\frac{d\kappa}{ds} = \frac{1}{|f'|^2}\left(\mathrm{Im}\{(Sf)(z')^2\} + \frac{dk}{dt}\right) . \tag{1.2}$$

*The author was partially supported by Fondecyt Grant # 1000629.
Key words: Schwarzian derivative, curvature, smooth curve, corners.
2000 AMS Subject Classification #: Primary 30E20, Secondary 30C20, 30C35.

It follows, for example, that a given circle or line γ will be mapped onto a curve of the same type provided the quantity $(Sf)(z')^2$ is real along γ.

Let f be a conformal map of the unit disc $\mathbb{D}$ onto the Jordan domain Ω. First let $\partial\Omega$ be very smooth and let $z(t) = e^{it}$. From (1.1) and (1.2) we obtain that

$$\kappa = \frac{1}{|f'|}\mathrm{Re}\{1 + z\frac{f''}{f'}\},$$

and

$$\frac{d\kappa}{ds} = -\frac{1}{|f'|^2}\mathrm{Im}\{z^2 Sf\}.$$

It follows then from Schwarz's formula that

$$z^2 Sf(z) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}|f'(e^{it})|^2(\frac{d\kappa}{ds})dt. \tag{1.3}$$

One deduces, for instance, that a conformal mapping of the disc onto a domain bounded by a circle must be a Möbius transformation. On the other hand, if $\partial\Omega$ consists of circular arcs forming interior angles $\alpha_k\pi$ at the vertices $w_k = f(z_k)$ then [Ne52, p. 201]

$$z^2 Sf(z) = \sum_{k=1}^{n}\left(\frac{1-\alpha_k^2}{2}\frac{zz_k}{(z-z_k)^2} + ir_k\frac{z+z_k}{z-z_k}\right) + c,$$

with real r_k.

The purpose of the present paper is to derive a similar integral formula for the Schwarzian of a conformal map f onto a domain bounded by a curve that is sufficiently smooth except for a finite number of corners. As it turns out, arbitrary interior angles will not be allowed, for then f' will fail to belong to the Hardy space H^2. The formula will incorporate, in addition to the integral, a sum of polar terms at the points on $\partial\mathbb{D}$ corresponding to the vertices in the image.

2. Main Result

Let C be a Jordan curve in $\mathbb{C}$, let $w_1, \ldots, w_n = w_0$ be points on C in cyclic order, and let Γ_k be the closed arc between w_{k-1} and w_k, $k = 1, \ldots, n$. We will assume that the arcs Γ_k are $C^{3,\delta}$ for some $\delta > 0$, and that the curve C forms at w_k an interior angle of $\pi\alpha_k$, $0 \le \alpha_k \le 2$. Then the geodesic curvature $\kappa(s)$ and its arclength derivative $\kappa'(s)$ exist on each open arc and have one-sided limits at each vertex. Let f be a conformal map of $\mathbb{D}$ onto Ω, the interior domain bounded by C, and let $z_k = f^{-1}(w_k)$. It follows from [Po92, Thm.3.6] that $f'''(z)$ is continuous and $f'(z) \neq 0$ for $z \in \overline{\mathbb{D}} \setminus \{z_1, \ldots, z_n\}$.

For $z(t) = e^{it}$ let $s(t)$ be again the arclength parameter on C, and write

$$\lambda(z) = \frac{d\kappa}{ds}(s(t)) = \frac{1}{|f'(z)|}\frac{d\kappa}{dt}.$$

Then $\lambda(z)$ is continuous and bounded on $\partial\mathbb{D} \setminus \{z_1, \ldots, z_n\}$.

Theorem: *If* $\frac{1}{2} < \alpha_k \leq 2$ *for* $k = 1, \ldots, n$ *then, for* $z \in \mathbb{D}$

$$z^2 Sf(z) = \sum_{k=1}^{n} \left(\frac{1-\alpha_k^2}{2} \frac{z_k z}{(z-z_k)^2} + ir_k \frac{z+z_k}{z-z_k} \right) + \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\zeta+z}{\zeta-z} |f'(\zeta)|^2 \lambda(\zeta) |d\zeta| \,. \quad (2.1)$$

Here $f' \in H^2$ *and* $r_k \in \mathbb{R}$.

Remark: If there is some $\alpha_k \leq \frac{1}{2}$ then $f' \notin H^2$ and the integral will not converge unless $\lambda(z)$ tends to zero sufficiently rapidly as $z \to z_k$. It is also interesting to observe that if all the arcs Γ_k are pieces of circles or straight lines, then $Sf(z)$ is meromorphic with poles at the points z_k.

In the proof we will use the following Phragmen-Lindelöf type lemma:

Lemma: *Let* $h : \mathbb{D} \to \mathbb{C}$ *be analytic and continuous on* $\overline{\mathbb{D}} \backslash \{\zeta\}$, *for some* $\zeta \in \partial\mathbb{D}$. *Suppose that for constants* a, b, M_1, M_2:

$$(i) \quad |h(z)| \leq \frac{M_1}{(1-|z|)^a} \,,\ z \in \mathbb{D} \,,$$

$$(ii) \quad |h(z)| \leq \frac{M_2}{|z-\zeta|^b} \,,\ |z| = 1 \,, z \neq \zeta \,.$$

Then

$$|h(z)| \leq \frac{M_2}{|z-\zeta|^b} \,,\ z \in \mathbb{D} \,.$$

Proof: Without loss of generality we may assume that $\zeta = 1$. Let $\epsilon > 0$ be fixed and consider the function $g(z) = (z-\zeta)^b h(z) q(z)$, where

$$q(z) = \exp\left(-\epsilon \sqrt{\frac{1+z}{1-z}} \right) .$$

Since $\mathrm{Re}\sqrt{\frac{1+z}{1-z}} > 0$ it follows that $\limsup_{z \to w} |g(z)| \leq M_2$ if $w \in \partial\mathbb{D} \backslash \{1\}$. On the other hand, $\limsup_{z \to 1} |g(z)| = 0$ because of part (i) and the choice of the function $q(z)$. We conclude from the classical Lindelöf maximum principle that $|g(z)| \leq M_2$ for all $z \in \mathbb{D}$. The lemma now follows by letting $\epsilon \to 0$.

Proof of the Theorem: The proof is long and will be divided into several parts. First we will determine the asymptotic behavior of some functions related to f near the points z_k. This was done in greater generality by Wigley [Wi65]. However, we need more detailed information about the coefficients.

Part 1. Let k be fixed and suppose $\alpha_k \neq 1, 2$. Then the circles of curvature of Γ_k and Γ_{k+1} at w_k have a second point of intersection, w_k^*. Let us assume first that $w_k^* \neq \infty$; the other case is simpler. Let Ω_k be a subdomain of Ω such that $\partial\Omega_k$ consists of arcs of Γ_k and Γ_{k+1} containing w_k, and a third arc of class $C^{3,\delta}$ of high contact with Γ_k and Γ_{k+1}. We may assume also that $w_k^* \notin \overline{\Omega_k}$, and that Ω_k is limited to a small neighborhood of w_k.

The function

$$\psi_k(w) = \left(a_k \frac{w - w_k}{w - w_k^*} \right)^{\frac{1}{\alpha_k}} \tag{2.2}$$

is analytic and injective for $w \in \Omega_k$, and the image $\psi_k(\partial\Omega_k)$ is a Jordan curve. We may choose the parameter a_k such that the image C_k of $\partial\Omega_k$ under $\psi_k(w)^{\alpha_k} = a_k(w - w_k)/(w - w_k^*)$ has tangents in the directions $e^{\pm i\pi\alpha_k/2}$ at 0. Because of the choice of the Möbius transformation in the definition of ψ_k, the curvatures of C_k are 0 at 0.

If $w_k^* = \infty$ then both circles of curvature are straight lines and the one-sided curvatures of C at w_k are 0. Then it suffices to consider $a_k(w - w_k)$ as the Möbius transformation in the definition (2.2) of ψ_k. In either case, C_k admits parametric representations in the form

$$u(t) = e^{\pm i\pi\alpha_k/2} t + O(t^3)$$

near the origin. It follows that

$$u(t)^{1/\alpha_k} = \pm i t^{1/\alpha_k} \omega(t) \,,$$

where $\omega(t) = 1 + O(t^2)$. Let $\tau = t^{1/\alpha_k}$, that is, $t = \tau^{\alpha_k}$. Then $\psi_k(\partial\Omega_k)$ near 0 can be represented as

$$v(\tau) = \pm i\tau\omega(\tau^{\alpha_k}) \,,$$

and therefore

$$\begin{aligned} \pm i v'(\tau) &= \omega(\tau^{\alpha_k}) + \alpha_k \tau^{\alpha_k} \omega'(\tau^{\alpha_k}) = 1 + O(\tau^{2\alpha_k}) \,, \\ \pm i v''(\tau) &= \alpha_k(1 + \alpha_k)\tau^{\alpha_k - 1}\omega'(\tau^{\alpha_k}) + \alpha_k^2 \tau^{2\alpha_k - 1}\omega''(\tau^{\alpha_k}) = O(\tau^{2\alpha_k - 1}) \,. \end{aligned} \tag{2.3}$$

One further differentiation for $\tau \neq 0$ shows that

$$Sv(\tau) = O(\tau^{2\alpha_k - 2}) + O(\tau^{4\alpha_k - 2}) = O(\tau^{2\alpha_k - 2}) \;,\; \tau \to 0 \,. \tag{2.4}$$

Let φ_k be a conformal mapping of $\mathbb{D}$ onto $f^{-1}(\Omega_k) \subset \mathbb{D}$ such that $\varphi_k(1) = z_k$. Then

$$f_k = \psi_k \circ f \circ \varphi_k \tag{2.5}$$

is a conformal mapping of $\mathbb{D}$ onto $\psi_k(\Omega)$. We want to apply the lemma to $h(z) = Sf_k(z)$ with $\zeta = 1$. Because $\partial f_k(\mathbb{D})$ is of class $C^{3,\delta}$ except at $f_k(1)$, it follows that $h(z)$ is continuous on $\overline{\mathbb{D}}\backslash\{1\}$. Furthermore, in light of the univalence of f_k, $h(z)$ satisfies condition (i) of the lemma with $M_1 = 6$ and $a = 2$. On the other hand, the mapping $f_k \circ \varphi_k^{-1}$ along the boundary $\partial f^{-1}(\Omega_k)$ is of class $C^{3,\delta}$, except at z_k. It follows that $S(f_k \circ \varphi_k^{-1})$ is continuous on $\partial f^{-1}(\Omega_k)\backslash\{z_k\}$, and equation (2.4) implies that near z_k, $|S(f_k \circ \varphi_k^{-1})(z)|$ is $O(|z - z_k|^{2\alpha_k - 2})$. Here we have used that $\tau \sim |z - z_k|$. Since, by the reflection principle, the mapping φ_k is analytic at $1 = \varphi_k^{-1}(z_k)$, it follows that $h(z) = Sf_k(z)$ satisfies condition (ii) of the lemma for some M_2 and $b = 2 - 2\alpha_k$. Because $2 - 2\alpha_k < 1$, the conclusion of the lemma implies the key fact that

$$Sf_k \in H^1 \,. \tag{2.6}$$

From (2.2) we deduce that $\psi_k^{-1}(w) = \sigma_k(w^{\alpha_k})$, with σ_k Möbius, hence

$$S\psi_k^{-1}(w) = \frac{1 - \alpha_k^2}{2} \frac{1}{w^2} \,. \tag{2.7}$$

Part 2. Let again k be fixed, and suppose that $\alpha_k = 2$. Then the circles of curvature are tangent and we now define

$$\psi_k(w) = (a_k(w - w_k))^{\frac{1}{2}} \ . \tag{2.8}$$

For a_k properly chosen, the image C_k of $\partial\Omega_k$ under $a_k(w - w_k)$ admits, near the origin, a representation of the form

$$u(t) = -t + O(t^2) \, ,$$

and it follows that

$$u(t)^{\frac{1}{2}} = \pm i t^{\frac{1}{2}} \omega(t) \, ,$$

where $\omega(t) = 1 + O(t)$ only. Let $\tau = t^{\frac{1}{2}}$, that is, $t = \tau^2$. Then C_k can be represented as

$$v(\tau) = \pm i\tau\omega(\tau^2) \, ,$$

and thus

$$\mp i v''(\tau) = 4\tau\omega'(\tau^2) + 4\tau^3\omega''(\tau^2) = O(\tau) \, . \tag{2.9}$$

Thus $Sv(\tau)$ is bounded near $\tau = 0$, and we conclude in this case that Sf_k is actually bounded in $\mathbb{D}$. Hence (2.6) holds. Note also that (2.7) is also valid in this case.

Part 3. Suppose now that $\alpha_k = 1$ for some k. The curve C then has a tangent at w_k but may have different one-sided curvatures κ^+, κ^-. We may assume that $w_k = 0$ and that the tangent line is vertical. The curve C near 0 admits a parametrization of the form

$$u^{\pm}(t) = \pm it + \mu^{\pm}(t) \, , \tag{2.10}$$

where $\mu^{\pm}$ are real valued functions of class $C^{3,\delta}$ and

$$\mu^{\pm}(t) = \frac{1}{2}\kappa^{\pm}t^2 + O(t^3) \ , \ t \to 0 \, . \tag{2.11}$$

This time, let

$$\psi_k(w) = \frac{w}{1 - (ib\log w + c)w} = w + (ib\log w + c)w^2 + O(w^3\log^2 w) \, ,$$

where $b = b_k, c = c_k$ are real constants to be chosen later. Then

$$\psi_k(u^{\pm}(t)) = \pm it + O(t^2\log t) \ , \ t \to 0$$

and

$$\frac{d}{dt}\psi_k(u^{\pm}(t)) = \frac{du^{\pm}}{dt} + (2ib\log u^{\pm} + 2c + ib)u^{\pm}\frac{du^{\pm}}{dt} \, ,$$

$$\frac{d^2}{dt^2}\psi_k(u^{\pm}(t)) = \frac{d^2u^{\pm}}{dt^2} + (2ib\log u^{\pm} + 2c + ib)u^{\pm}\frac{d^2u^{\pm}}{dt^2} + (2ib\log u^{\pm} + 2c + 3ib)(\frac{du^{\pm}}{dt})^2 \, . \tag{2.12}$$

It follows from (2.10) and (2.11) that, as $t \to 0$,

$$\frac{du^{\pm}}{dt} = \pm i + \frac{d\mu^{\pm}}{dt} = \pm i + \kappa^{\pm}t + O(t^2) \, ,$$

and

$$\frac{d^2u^\pm}{dt^2} = \frac{d^2\mu^\pm}{dt^2} = \kappa^\pm + O(t)\,.$$

We therefore deduce from (2.12) that

$$\frac{d^2}{dt^2}\mathrm{Re}\{\psi_k(u^\pm(t))\} = \kappa^\pm \mp \pi b - 2c + O(t\log t)\,, \tag{2.13}$$

because $\log u^\pm(t) = \pm i\frac{\pi}{2} + \log(t \mp i\mu^\pm(t))$. We may choose now b, c such that $\kappa^+ - \pi b - 2c = \kappa^- + \pi b - 2c = 0$. With this, the parameter τ is defined by

$$\tau = \mathrm{Im}\{\psi_k(u^\pm(t))\}\ ,\ -\tau_0 < \tau < \tau_0\,.$$

Then $\tau = \pm t + O(t^2\log t)$, hence $t = |\tau| + O(\tau^2\log\tau)$ as $\tau \to 0$. With this, (2.13) and the choice of b, c, we obtain that

$$\frac{d^2}{d\tau^2}\mathrm{Re}\{\psi_k(u^\pm(\tau))\} = O(\tau\log\tau)\ ,\ \tau \to 0\,. \tag{2.14}$$

Thus the curve C_k has vertical tangent and zero curvature at the origin, and admits a parametrization $v(\tau)$ with the property that $Sv(\tau) = O(\log\tau)$ as $\tau \to 0$. Once more we conclude that Sf_k satisfies (2.6).

As in (2.7), we will need $S\psi_k^{-1}$ in Part 4 of this proof. Let

$$h(w) = \frac{1}{\psi_k(w)} = \frac{1}{w} - ib\log w + c\,,$$

so that $h'(w) = -1/w^2 - ib/w$ and

$$\frac{h''}{h'}(w) = -\frac{1}{w}\frac{2+ibw}{1+ibw} = -\frac{2}{w} + ib + O(w)\,,$$

hence

$$Sh(w) = S\psi_k(w) = \frac{2ib}{w} + O(1)\,.$$

Since $\psi_k'(w)^2(S\psi_k^{-1})(\psi_k(w)) + S\psi_k(w) = 0$ we obtain

$$(S\psi_k^{-1})(\psi_k(w)) = -\frac{2ib}{w}\frac{\left(\dfrac{1}{w} - ib\log w + c\right)^4}{\left(-\dfrac{1}{w^2} - \dfrac{ib}{w}\right)^2}$$

$$= -\frac{2ib}{w}\left(1 - 4ibw\log w + (4c - 2ib)w + O(w^2\log^2 w)\right)$$

$$= -2ib\left(\frac{1}{\psi_k(w)} - 3ib\log\psi_k(w) + O(1)\right)\,.$$

With this,

$$S\psi_k^{-1}(w) = -\frac{2ib}{w} + O(\log w)\,. \tag{2.15}$$

Part 4. Finally, we put together the various individual cases for $k = 1, \ldots, n$. We consider the functions

$$\chi_k = \varphi_k^{-1} : \Omega_k \to \mathbb{D} \quad , \quad g_k = \psi_k \circ f = f_k \circ \chi_k : \Omega_k \to \mathbb{C}\,.$$

If $\alpha_k \neq 1$ then, by (2.7), we have in Ω_k that

$$Sf = S(\psi_k^{-1} \circ g_k) = \frac{1-\alpha_k^2}{2}\left(\frac{g_k'}{g_k}\right)^2 + Sg_k\,, \tag{2.16}$$

where $Sg_k = (\chi_k')^2 (Sf_k) \circ \chi_k + S\chi_k$ has bounded integral over $\{|z| = r\} \cap \partial\Omega_k$ for r near to 1 because $Sf_k \in H^1$ by (2.6).

We see from (2.3) and (2.9) that $\partial f_k(\mathbb{D})$ belongs to the class C^{2,β_k}, where $\beta_k = 2\alpha_k - 1 > 0$ if $\alpha_k < 1$ and β_k is any number with $0 < \beta_k < 1$ if $\alpha_k > 1$. Hence f_k'' satisfies a Hölder condition with exponent $\beta_k = 2\alpha_k - 1 > 0$ by the Kellog-Warschawski theorem [Po92, Thm.3.6]. Since χ_k is conformal near z_k, we conclude that g_k has an expansion of the form

$$g_k(z) = d_k(z - z_k) + c_k(z - z_k)^2 + O((z - z_k)^{2+\beta_k})\ ,\ z \to z_k\,, \tag{2.17}$$

where $d_k \neq 0$. By the Hölder continuity above, corresponding differentiated expansions hold also for g_k' and g_k'', and it follows that

$$1 + z\frac{g_k''(z)}{g_k'(z)} \to 1 + \frac{2c_k z_k}{d_k}\ ,\ z \to z_k\,,$$

and because the curvature of C_k is zero at 0 we conclude that

$$\mathrm{Re}\{1 + \frac{2c_k z_k}{d_k}\} = 0\,.$$

Thus

$$1 + \frac{2c_k z_k}{d_k} = 2ie_k\ ,\ e_k \in \mathbb{R}\,.$$

Further calculations give that

$$\begin{aligned}\left(z\frac{g_k'}{g_k}\right)^2 &= \frac{z^2}{(z-z_k)^2} + \frac{2c_k}{d_k}\frac{z^2}{z-z_k} + O((z-z_k)^{\beta_k - 1}) \\ &= \frac{z_k z}{(z-z_k)^2} + ie_k\frac{z+z_k}{z-z_k} + O((z-z_k)^{\beta_k-1})\,. \end{aligned} \tag{2.18}$$

It follows from (2.16) that

$$z^2 Sf(z) - \frac{1-\alpha_k^2}{2}\frac{z_k z}{(z-z_k)^2} - ir_k\frac{z+z_k}{z-z_k} - z^2 Sg_k(z) = O((z-z_k)^{\beta_k-1})\,,$$

for some $r_k \in \mathbb{R}$, which implies that

$$z^2 Sf(z) - \frac{1-\alpha_k^2}{2}\frac{z_k z}{(z-z_k)^2} - ir_k\frac{z+z_k}{z-z_k}$$

has bounded integral over $\{|z| = r\} \cap \partial\Omega_k$ for r close to 1 because $\beta_k > 0$. Observe that if $|z| = 1$, then $\dfrac{\sqrt{z_k z}}{z - z_k} \in i\mathbb{R}$ and hence

$$\operatorname{Im}\left\{\frac{1-\alpha_k^2}{2}\frac{z_k z}{(z-z_k)^2} + ir_k\frac{z+z_k}{z-z_k}\right\} = 0\ ,\ z \in \partial\mathbb{D}\,. \tag{2.19}$$

Suppose now that $\alpha_k = 1$. By (2.14), we conclude once more that the function f_k satisfies a Hölder condition for some (any) exponent $\beta_k > 0$, and as before, the function g_k admits the expansion (2.17). From equation (2.15) we have that

$$Sf = S(\psi_k^{-1} \circ g_k) = -\frac{2ib_k}{g_k}(g_k')^2 + Sg_k + O(\log(z-z_k))\ ,\ z \to z_k\ ,$$

and hence

$$z^2 Sf(z) = ir_k\frac{z+z_k}{z-z_k} + z^2 Sg_k(z) + O(\log(z-z_k))\,,$$

where we have set $r_k = -b_k z_k g_k'(z_k)$, which is real because the tangent to C_k at $0 = g_k(z_k)$ is vertical. As before, for r close to 1 Sg_k has bounded integral over $\{|z| = r\} \cap \partial\Omega_k$, therefore so does

$$z^2 Sf(z) - ir_k\frac{z+z_k}{z-z_k}\,.$$

If $z \in \partial\mathbb{D}$ then (2.19) holds with $\alpha_k = 1$. We define the function $R(z)$ by

$$R(z) = \sum_{k=1}^{n}\left(\frac{1-\alpha_k^2}{2}\frac{z_k z}{(z-z_k)^2} + ir_k\frac{z+z_k}{z-z_k}\right)\,. \tag{2.20}$$

By construction, if r is near to 1 then $z^2Sf(z) - R(z)$ has bounded integral over $\{|z| = r\} \cap \partial\Omega_k$ for each k, and since $C \backslash \cup\, \partial\Omega_k$ belongs to $C^{3,\delta}$ in $\partial\mathbb{D}\backslash\{z_1, \cdots, z_n\}$, we conclude that $z^2Sf(z) - R(z)$ is in H^1. It follows that for $z \in \mathbb{D}$

$$z^2 Sf(z) - R(z) = \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{\zeta+z}{\zeta-z}\operatorname{Im}\left\{\zeta^2 Sf(\zeta) - R(\zeta)\right\}|d\zeta|\,.$$

This implies (2.1) because $\operatorname{Im}\{R(\zeta)\} = 0$ by (2.19) and (2.20).

Acknowledgment: We thank the referee for his suggestions and careful reading of the manuscript.

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Facultad de Matematicas, P. Universidad Catolica de Chile, Casilla 306, Santiago 22, CHILE, mchuaqui@mat.puc.cl

Technische Universität Berlin, Fachbereich Mathematik, 10623 Berlin, GERMANY, pommeren@math.TU-Berlin.de